Isorotating knots and baby Skyrmions

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Abstract

The problem of constructing internally rotating solitons of fixed angular frequency ω in the models of Skyrme family models is reformulated as a variational problem for an energy-like functional, called pseudoenergy, which depends parametrically on the angular frequency ω. Different types of instabilities of the isospinning solitons are investigated.

1 Introduction

Many field theories of interest in fundamental physics support topological solitons – spatially localized, stable lumps of energy whose strongly particle-like characteristics make them natural theoretical models of elementary particles. Perhaps the best developed model from this viewpoint is the Skyrme model, whose solitons are posited to model atomic nuclei. It is of fundamental importance in this context that individual solitons possess both rotational and internal rotational (or *isorotational*) degrees of freedom. In the Skyrme model, the rotational degrees of freedom account, after quantization, for the spin of atomic nuclei, while the isorotational degrees of freedom account, roughly speaking, for their difference in "flavour".

The so-called baby Skyrme model is a modified version of the non-linear $O(3)$ σ-model in $2+1$ dimensions [1], a low-dimensional simplified theory which resembles the conventional Skyrme model in many important respects. This model has a number of applications, e.g. in condensed matter physics where Skyrmion configurations were observed experimentally [2], or in the topological quantum Hall effect [3].

Together with the original Skyrme model in $d = 3+1$ [4] and the Faddeev–Skyrme model [5], the baby Skyrme model can be considered as a member of the Skyrme family. Indeed, the Lagrangian of all these models has similar structure, it includes the usual $O(3)$ sigma model kinetic term, the Skyrme term, which is quartic in derivatives, and the potential term which does not contain the derivatives.

A peculiar feature of the models from the Skyrme family is that the corresponding soliton solutions, Skyrmions and Hopfions, do not saturate the topological bound. In order to attain the topological lower bound and get a linear relation between the masses of the solitons and their topological charges, one has to modify the model, for example drop out the quadratic kinetic term [14, 28] or extend the model by coupling of the Skyrmions to an infinite tower of vector mesons [29]. Thus, the powerful methods of differential geometry cannot be directly applied to describe low-energy dynamics of the Skyrmions and hopfions, one has to analyse the processes of their scattering, radiation and annihilation numerically [30, 13].

Typically, the problem of direct simulation of the soliton dynamics is related with sophisticated numerical methods, the calculations require considerable amount of computational resources, actually this problem is fully investigated only for the low-dimensional baby Skyrme

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model. Even more simple task of full numerical investigation of the spinning solitons beyond rigid body approximation was performed only recently in the FaddeevSkyrme model [16, 18] and in the baby Skyrme model [26, 20], in the case of the original Skyrme model in $d = 3 + 1$ this problem is not investigated yet.

Note that the solitons of the models of the Skyrme family possess both rotational and internal rotational (or isorotational) degrees of freedom. Traditional approach to study the spinning solitons is related with rigid body approximation, both in the context of the Skyrme model [11, 12] and in the baby Skyrme model [13]. The assumption is that the solitons could rotate without changing its shape. It has long been recognized that this is not a very satisfactory approximation, and various attempts have been made to improve on it. This restriction can be weakly relaxed by consideration of the radial deformations which would not violated the rotational symmetry of the hedgehog configuration [13, 15]. Evidently, this approximation is not very satisfactory, a consistent approach is to solve full system of field equation without imposing any spatial symmetries on the isospinning solitons. Furthermore, almost all previous studies of spinning solitons (see e.g. [16, 17]) were concerned with minimization of the total energy functional $E_J[\boldsymbol{\phi}]$ for fixed value of the isospin J. However if we do not assume the spinning soliton will have precisely the same shape as the static soliton, this approach becomes rather involved, it is related with numerical solution of complicated differential-integral equation.

Very recently the isospinning soliton solutions were considered in the FaddeevSkyrme model beyond rigid body approximation [16, 18]. The approach of the paper [18] is to consider the static pseudo-energy minimization problem, where the pseudo-energy functional $F_\omega[\boldsymbol{\phi}]$ depends parametrically on the angular frequency ω. The important conclusion which is general for all models of the Skyrme family, is that there is a new type of instability of the solitons due to the extra nonlinear velocity dependence generated by the Skyrme term [18].

In this paper, we review our analysis of the critical behavior of the isospinning solitons of the Skyrme family. We confirm existence of two types of instabilities determined by the relation between the mass parameter of the potential μ and the frequency ω, both in the planar baby-Skyrme model [20] and in the Faddeev–Skyrme model [16, 18]. Interestingly, we observe that the critical behavior of the isospinning baby Skyrmions depends also on the structure of the potential of the model, for example in the case of the "old" model [1] the isospinning configurations of higher degree may become unstable with respect to decay into constituents.

2 Baby Skyrme model

As a starting point we consider the rescaled Lagrangian of the $O(3)$ σ-model with the Skyrme term in $2 + 1$ dimensions [1]

$$L = \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \frac{1}{4}(\partial_\mu \boldsymbol{\phi} \times \partial_\nu \boldsymbol{\phi})^2 - U[\boldsymbol{\phi}] \tag{1}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ denotes a triplet of scalar fields which satisfy the constraint $|\boldsymbol{\phi}|^2 = 1$. Topologically the field is the map $\phi : \mathbb{R}^2 \to S^2$ characterized by the topological charge $B = \pi_2(S^2) = \mathbb{Z}$. Explicitly, $B = \frac{1}{4\pi}\int \boldsymbol{\phi} \cdot \partial_1 \boldsymbol{\phi} \times \partial_2 \boldsymbol{\phi}\, d^2x$.

Note that the first two terms in the functional (1) are invariant under the global $O(3)$ transformations, this symmetry becomes broken via the potential term. The standard choice of the potential of the baby Skyrme model is [1] $U[\boldsymbol{\phi}] = \mu^2[1 - \phi_3]$, thus the symmetry is broken to $SO(2)$ and there is a unique vacuum $\boldsymbol{\phi}_\infty = (0, 0, 1)$. The corresponding solitons of degree $B = 1, 2$ are axially symmetric [1] however the rotational symmetry of the configurations of higher degree becomes broken [13].

The residual symmetry of the configurations with respect to the rotations around the third axis in the internal space allows us to consider the stationary isospinning (i.e. internally rotating) solitons $(\phi_1 + i\phi_2) \mapsto (\phi_1 + i\phi_2)e^{i\omega t}$, , where ω is the angular frequency. The corresponding conserved quantity is the angular momentum $J = \omega\Lambda[\boldsymbol{\phi}]$, where $\Lambda[\boldsymbol{\phi}]$ is the moment of inertia, thus the total energy of the spinning field configuration is $E_J[\boldsymbol{\phi}] = V[\boldsymbol{\phi}] + \frac{J^2}{2\Lambda[\boldsymbol{\phi}]}$.

Evidently, the isorotations of the energy functional of the baby Skyrme model yield the pseudo-energy functional

$$F_\omega[\boldsymbol{\phi}] = V - \frac{1}{2}\omega^2\Lambda(\boldsymbol{\phi}) \tag{2}$$

where the V is the potential energy of the non-rotated configuration and the moment of inertia is $\Lambda(\boldsymbol{\phi}) = \int\limits_{\mathbb{R}_2}\left\{(\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2[1+(\partial_i\boldsymbol{\phi}\cdot\partial_i\boldsymbol{\phi})] - [\boldsymbol{\phi}_\infty\cdot(\boldsymbol{\phi}\times\partial_i\boldsymbol{\phi})]^2\right\}$. The isospinning solitons correspond to the stationary points of the functional (2). However the pseudoenergy is not bounded from below for $\omega > \omega_1 = \sqrt{2}$ independently from the particular choice of the potential $U[\boldsymbol{\phi}]$ [18]. Indeed, the first term in (2) effectively defines the geometry of the deformed sphere S^2 squashed along the direction $\boldsymbol{\phi}_\infty$, the metric on this space becomes singular at $\omega = \omega_1 = \sqrt{2}$.

The second critical frequency is related with condition of positiveness of the effective potential $U_\omega[\boldsymbol{\phi}] = U[\boldsymbol{\phi}] - \omega^2(1-\phi_3^2)$, it approaches zero at some critical value $\omega = \omega_2$. In this limit the isospinning solitons of the baby Skyrme model cease to exist because the vanishing of the potential makes the configuration unstable.

The traditional approach to study the solitons of the model (1) is related with separation of the radial and angular variables [1, 13], thus the consideration becomes restricted to the case of rotationally invariant configurations and the corresponding Euler-Lagrange equations are reduced to a single ordinary differential equation on radial function $f(\rho)$. However more detailed analyse reveal that the higher charge $B \geq 3$ baby Skyrmions may not possess rotational symmetry [1, 25], starting from some critical value of the mass parameter μ the global minimum of the energy functional corresponds to the configurations with discrete symmetries.

The violation of the rotation invariance in the baby Skyrme model attracted a lot of attention recently, it was demonstrated that the effect strongly depends on the particular choice of the potential of the model [8, 9, 10]. Thus, considering the isorotating baby Skyrmions we will consider complete system of coupled partial differential equations on the triplet of functions $\boldsymbol{\phi}(\rho,\theta)$ which follows from the Lagrangian (1).

3 Numerical results

Here we consider evolution of the baby Skyrmions in the model with "old" potential. When the mass parameter is restricted from above as $\mu^2 < 2$, we observe critical behavior of the first type, the effective potential vanishes and both the energy and the angular momentum diverge. When μ^2 increases further, second type of critical behavior is observed, our algorithm ceases to find any critical points when ω is taking the values $\omega > \sqrt{2}$ thought the energy and the angular momentum remain finite. Note that the plots of the energy of the baby Skyrmions as function of isospin look similar with the dependencies $E(J)$ in the Faddeev–Skyrme model [18], up to some value of J the energy remains almost constant, i.e. the configuration spins as a rigid rotator, then the curve $E(J)$ becomes linear up to critical value at which the solution breaks up.

Interestingly, for the rotationally invariant configurations which we can construct using the hedgehog ansatz [1] and considering relatively large values of the mass parameter μ, we observe crossing in both $F_\omega(\omega)$ and $E(\omega)$ curves. Indeed, our numerical simulations confirm that for some (third) critical value of frequency ω_3 the pseudo-energy of the axially symmetric $B \geq 2$ multi-Skyrmion becomes higher than the pseudo-energy of the system of B charge one baby Skyrmions, so the configurations are unstable with respect to decay into constituents as shown in Fig. 1. Typically, increasing the value of the mass parameter μ will increase the stability of the rotationally invariant multisolitons, the critical values of the frequencies which correspond to the crossing between the $F_\omega(\omega)$ curves then increase.

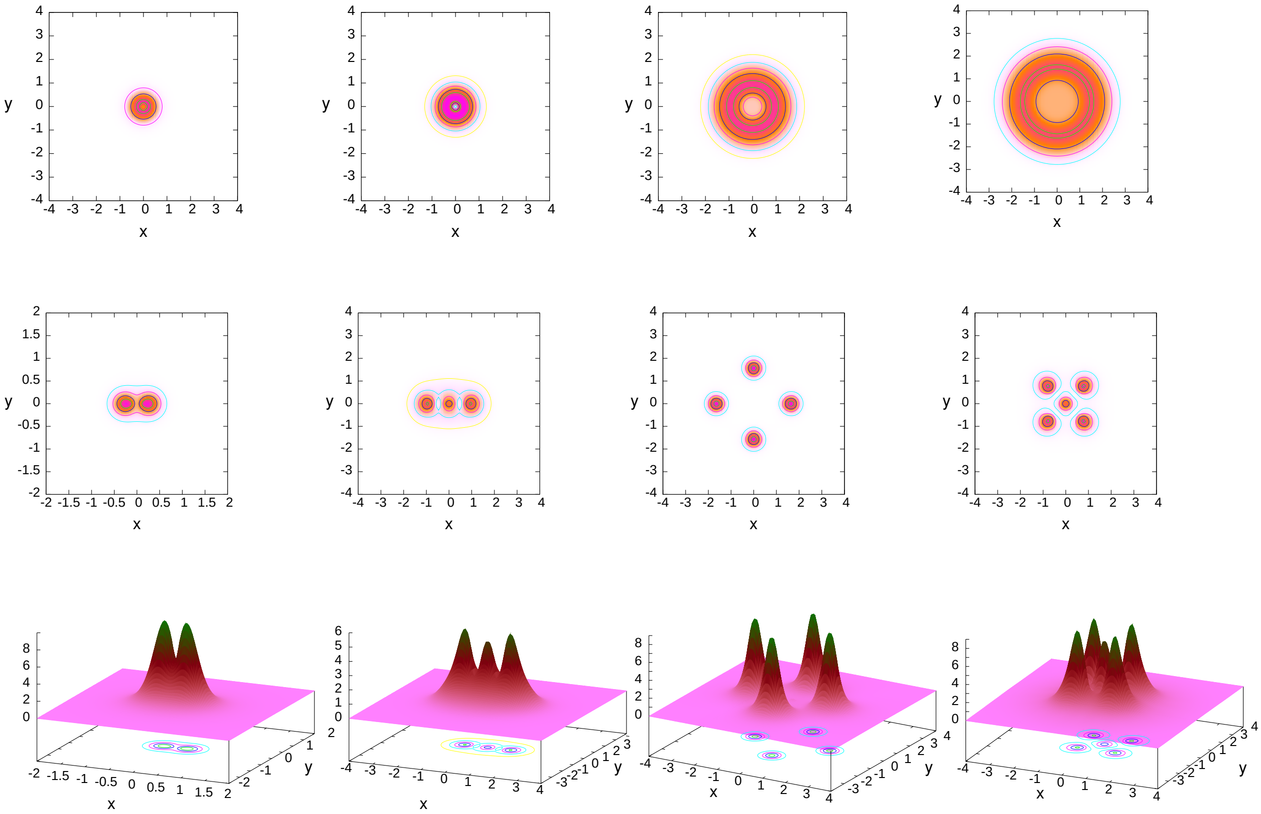


Figure 1: (Color online) Critical behavior of the rotationally invariant soliton solutions of the model with "old" potential. The contour plots of the energy density of the rotationally invariant (upper row) baby Skyrmions with charges $B = 2, 3, 4, 5$ and $\mu^2 = 8$ at $\omega = 0.8$ and their decay into B charge one solitons (2nd and 3rd rows).

4 Faddeev-Skyrme model

Now we consider the internally rotating soliton solutions in the FaddeevSkyrme model [5]. Such solitons are conventionally called hopfions, since they are classified topologically by their Hopf degree (an integer-valued topological invariant). Explicitly, our field is a triplet $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ constrained to the unit sphere S^2 in $3+1$ dimensions. For finite energy solutions the field $\boldsymbol{\phi}$ must tend to a constant value at spatial infinity, which we select to be $\boldsymbol{\phi}_0 = (0, 0, 1)$. This allows a one-point compactification $\mathbb{R}^3 \sim S^3$, thus topologically the field is the map $\boldsymbol{\phi} : \mathbb{R}^3 \to S^2$ characterized by the Hopf invariant $Q = \pi_3(S^2) = \mathbb{Z}$. The energy of the Faddeev-Skyrme model is bound from below by the Vakulenko-Kapitansky inequality [35] $E \geq \text{const}|Q|^{\frac{3}{4}}$.

The FaddeevSkyrme model has the same Lagrangian density (1) as the baby Skyrme model, the potential term now is necessary to stabilize the isospinning solitons. Peculiar feature of the model is that it has a rather rich spectrum of static soliton solutions, which should be understood as (possibly linked or self-knotted) string-like objects, but has only one internal rotational degree of freedom, because the potential breaks the internal rotational symmetry group to $SO(2)$.

For the lowest two values of the Hopf charge $Q = 1, 2$ the hopfion solutions are axially-symmetric [33]. A configuration of this type is labeled as $\mathcal{A}_{n,m}$, it has topological charge $Q = nm$, where $n, m \in \mathbb{Z}$. Here the first subscript labels the number of twists along the loop and the second is the usual $O(3)$ sigma model winding number associated with the map $S^2 \to S^2$ Here, and in the sequel, we are using the notation introduced by Sutcliffe to label Hopf soliton shapes [34].

Briefly, $Q\mathcal{A}_{n,m}$ denotes an axially symmetric hopfion of charge Q, where the position curve (preimage of $-\psi_\infty$) is a circle, and the preimage of a regular value close to $-\psi_\infty$ is a disjoint union of m closed curves, each winding n times around the circle. In fact $Q = nm$, so including Q in the label is redundant, but convenient. A soliton of type $Q\widetilde{\mathcal{A}}_{n,m}$ has the same qualitative form, but with axial symmetry weakly broken, so the position curve is not exactly circular. Later we will encounter solitons whose position curves are links of two components. These will be denoted $Q\mathcal{L}^{a,b}_{p,q}$ where the subscripts denote the Hopf charges of each component, and the superscripts denote the extra Hopf degree of each component due to its linking with the other component. We will also encounter hopfions whose position curves are torus knots of type (a, b) (where a and b denote the windings of the curve around the S^1 factors in T^2). We denote these $Q\mathcal{K}_{a,b}$.

The charge $Q = 1$ $\mathcal{A}_{1,1}$ configuration possesses the maximum of the energy density at the origin, the energy density isosurfaces are squashed spheres as seen in Fig.**??**. The charge $Q = 2$ $\mathcal{A}_{2,1}$ solutions have toroidal structure(see Fig.**??**). Inclusion of the mass term increases the attraction in the system, the total energy of the massive hopfion increases monotonically as mass parameter μ increases [28].

However as the Hopf charge Q increases, the landscape of local energy minima becomes much more complicated, there are a number of local energy minima of various types and geometries which are slightly different in energy (cf **??**). Furthermore, the number of local minima increases with the charge.

Hence one can ask unambiguously "what is the degree Q isospinning hopfion of given angular frequency $\omega \in \mathbb{R}$?" It is interesting to discover how the energy of such solitons varies as their angular frequency (or their conserved isospin) changes. We will find several examples where these energy curves cross, so that the most energetically favourable shape of the hopfion for a given degree Q changes when one isospins them fast enough [18].

Since isorotation involves only rotational symmetry of the target space S^2, it is convenient to consider the Faddeev-Skyrme model on a general oriented Riemannian manifold M. This allows one to treat in unified fashion the case of principal interest, $M = \mathbb{R}^3$, and the cases of soliton chains or strings, $M = \mathbb{R}^2 \times S^1$, sheets $M = \mathbb{R} \times T^2$, or geometrically nontrivial domains (of potential interest for cosmological applications, for example).

Given a time-dependent field $\phi : \mathbb{R} \times M \to S^2$, we have at each fixed time t a mapping $\phi(t, \cdot) : M \to S^2$ which we shall, in a slight abuse of notation, again denote ϕ, and a time derivative $\dot{\phi}$, which is a section of the bundle $\phi^{-1}TS^2$ over M. Using these, we define, at time t, the kinetic and potential energy functionals to be

$$T = \int_M \frac{1}{2}|\dot{\phi}|^2 + \frac{1}{2}|\phi^*(\iota_{\dot{\phi}}\Omega)|^2, \quad V = \int_M \frac{1}{2}|\mathrm{d}\phi|^2 + \frac{1}{2}|\phi^*\Omega|^2 + U(\phi), \tag{3}$$

where Ω is the area form on S^2, $\phi^*\Omega$ its pullback to M, ι denotes interior product, and $U : S^2 \to [0, \infty)$ is a smooth potential function which we assume attains its minimum value 0 at some point $\psi_\infty \in S^2$, and is invariant under rotations about ψ_∞.

Let $\omega > 0$ be a fixed constant. We seek time-periodic solutions of period $2\pi/\omega$. By definition, these are critical points $\phi : S^1_\omega \times M \to S^2$ of the action functional $S(\phi) = \int_{S^1_\omega}(T - V)$, where $S^1_\omega = \mathbb{R}/(2\pi/\omega)\mathbb{Z}$ is the circle of length $2\pi/\omega$. Denote by X_ω the completion in C^1 of the set of smooth maps $\phi : S^1_\omega \times M \to S^2$ of finite action. We define an action of the group $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on X_ω as follows: $([\alpha], \phi) \mapsto \phi_{[\alpha]}$, $\quad \phi_{[\alpha]}(t, x) = R(\alpha)\phi(t - \alpha/\omega, x)$, where $R(\alpha)$ denotes the $SO(3)$ matrix generating rotation through angle α about the axis ψ_∞. Clearly, $S(\phi_{[\alpha]}) = S(\phi)$ for all $([\alpha], \phi)$, since the action is separately invariant under both time translation and isorotation about ψ_∞. Denote by $X^{S^1}_\omega$ the set of fixed points of this action. Then $\phi \in X^{S^1}_\omega$ if and only if $\phi(t,x) = R(\omega t)\psi(x)$ for some map $\psi : M \to S^2$. We may think of ψ as the stationary field ϕ when viewed in an internally corotating frame. Since S^1 is compact, it follows from the Principle of Symmetric Criticality [32] that $\phi \in X_\omega^{S^1}$ is a critical point of $S : X_\omega \to \mathbb{R}$ if and only if it is a critical point of the restricted action $S : X_\omega^{S^1} \to \mathbb{R}$. Now

$$S(R(\omega t)\psi(x)) = \frac{2\pi}{\omega}\left\{\frac{1}{2}\omega^2\int_M (|\psi_\infty \times \psi|^2 + |\mathrm{d}(\psi_\infty \cdot \psi)|^2) - V(\psi)\right\},$$

so a uniformly isorotating field is a critical point of S if and only if the static field $\psi : M \to S^2$ is a critical point of the functional

$$F_\omega(\psi) = \int_M \left\{\frac{1}{2}(|\mathrm{d}\psi|^2 - \omega^2|\mathrm{d}(\psi_\infty \cdot \psi)|^2) + \frac{1}{2}|\psi^*\Omega|^2 + (U(\psi) - \frac{1}{2}\omega^2|\psi_\infty \times \psi|^2)\right\}. \qquad (4)$$

This pseudoenergy functional is counterpart of the (2) in the planar baby Skyrme model.

The first two terms of (4), taken together, can be interpreted as the Dirichlet energy of the map $\psi : M \to S^2$, where S^2 is given the deformed metric $\langle X, Y\rangle_\omega = X\cdot Y - \omega^2(\psi_\infty\cdot X)(\psi_\infty\cdot Y)$ for all $X, Y \in T_\psi S^2$. For $0 < \omega < 1$ this metric gives S^2 the geometry of an oblate sphere, squashed along the direction of ψ_∞. For $\omega > 1$, the metric is singular, changing from Riemannian to Lorentzian in a strip around the equator (orthogonal to ψ_∞). Consequently, the pseudoenergy F_ω is no longer bounded below for $\omega > \omega_1 = 1$ which, as we will see, has strong phenomenological consequences.

The third term of F_ω is just the usual Faddeev-Skyrme term (quartic in spatial derivatives). The fourth and fifth terms together can be interpreted as a deformed potential $U_\omega(\psi) = U(\psi) - \frac{1}{2}\omega^2|\psi_\infty \times \psi|^2$. Hence, if $\omega > \omega_2 = \mu$, F_ω is again unbounded below. A particularly convenient choice for U is $U(\psi) = \frac{1}{2}\mu^2(1-\psi_3^2)^2$. Then $\psi_\infty = (0,0,1)$ and the deformed potential is

$$U_\omega(\psi) = \frac{1}{2}(\mu^2 - \omega^2)(1-\psi_3^2)^2.$$

This is the potential we use in all our numerical simulations.

Since the model is invariant under global rotations of ϕ about ψ_∞, it has an associated conserved Noether charge, called *isospin* $J = \int_M \left\{\dot{\phi}\cdot(\psi_\infty \times \phi) + \langle \mathrm{d}(\psi_\infty\cdot\phi), \phi^*(\iota_{\dot{\phi}}\Omega)\rangle\right\}$. For uniformly isorotating fields of the form (4), this equals $J = \Lambda(\psi)\omega$, where $\Lambda(\psi)$ is the moment of inertia. Hence $F_\omega(\psi) = V(\psi) - \frac{1}{2}\omega^2\Lambda(\psi)$, while the total energy of the field (4) is $V + T = V(\psi) + J^2/(2\Lambda(\psi))$.

Similar to the case of the baby Skyrme model there are two natural variational problems for ψ:

1. For fixed ω, extremize $F_\omega(\psi) = V(\psi) - \frac{1}{2}\omega^2\Lambda(\psi)$;
2. For fixed J, extremize $E_J(\psi) = V(\psi) + J^2/(2\Lambda(\psi))$.

It is clear that these two problems are precisely equivalent: if ψ solves 1, then it solves 2 with $J = \Lambda(\psi)\omega$, and if ψ' solves 2, it solves 1 with $\omega = J/\Lambda(\psi')$. Previous studies [17] of isorotating[1] solitons have used formulation 2, however similar to case of the isospinning solitons of the planar baby Skyrme model, we will mainly use formulation 1. This has several advantages. First, it is directly clear, as we have shown (using the Principle of Symmetric Criticality), that solutions of 1 correspond via (4) to genuine solutions of the field theory. Second, the Euler-Lagrange equation corresponding to 1 is a PDE, similar in structure to the static field equation of the Faddeev-Skyrme model, whereas the equation corresponding to 2 is a rather more complicated differential-integral equation. Consequently, it is a fairly simple

[1]Actually, [17] concerns spatially rotating Skyrmions, but within the axially symmetric ansatz used therein, rotation is equivalent to isorotation.

matter to adapt existing numerical techniques, developed for the static FS model, to deal with problem 1. Third, formulation 1 makes it clear that, in the case $\mu > 1$, there is no reason why isospinning solitons should persist for frequencies $\omega \in (1, \mu)$, since F_ω is unbounded below when $\omega > \min\{1, \mu\}$. Hence, we have the possibility that isospinning hopfions are destabilized by nonlinear velocity terms in the field equation *before* they reach the upper limit $\omega = \mu$.

4.1 Numerical results

In this section we briefly present the results of numerical simulations of the Faddeev-Skyrme model [18]. Local minima of the pseudo-energy functional $F_\omega(\psi)$ have been found for a range of values of the Hopf degree Q and of ω. The majority of simulations were carried out with $\mu = 2$. This choice guarantees that $\omega_1 < \omega_2$, and thus allows us to investigate the possibility that solitons become unstable to processes other than pion decay.

The numerical algorithm we employed is significantly different and it is much more complex than the approach we used in the above considered case of the baby Skyrme model. To construct the solitons of the Faddeev-Skyrme model one needs a serious amount of computational power which, in our case was available on the computational cluster ARC1 (Leeds, UK).

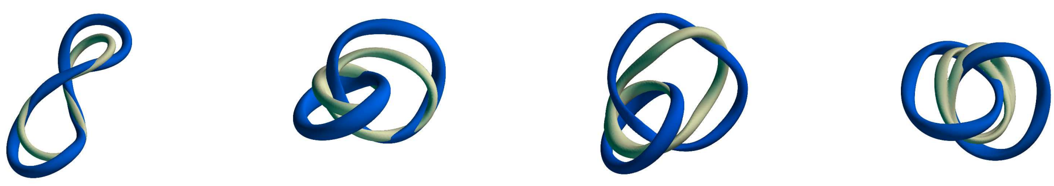
When $\mu < 1$ our simulations terminate at $\omega = \mu$, as expected. When $\mu > 1$ our algorithm ceases to find any critical points when $\omega = 1$. This is again consistent with our expectations, as the pseudo-energy is not bounded from below when $\omega > 1$. However, the existence of solutions with $\omega > 1$ is not ruled out, as they may continue to exist as saddle points of the pseudo-energy. The borderline case $\mu = 1$ is particularly interesting: the graphs of $E(\omega)$ grow rapidly as ω approaches 1.

At degree 4 we have been able to find configurations of types $4\mathcal{A}_{2,2}$, $4\widetilde{\mathcal{A}}_{4,1}$ and $4\mathcal{L}_{1,1}^{1,1}$. The $4\mathcal{A}_{2,2}$ is axially symmetric and may be thought of as two adjacent $2\mathcal{A}_{2,1}$ solitons. The position curve of the $4\mathcal{A}_{2,2}$ in our model consists of two adjacent circles, whereas in the massless Faddeev-Skyrme model it is a single circle.

Interestingly, we observe that the $4\mathcal{A}_{2,2}$ configuration at $\omega = 0.58$ undergoes a bifurcation. We were also able to find a configuration which seems to be a saddle point of the pseudo-energy, suggesting that the bifurcation is a pitchfork bifurcation.

When $\omega > 0.65$ our algorithm is unable to find the $4\mathcal{A}_{2,2}$ and instead converges to the $4\mathcal{L}_{1,1}^{1,1}$. Since the $4\mathcal{A}_{2,2}$ configuration is axially symmetric it is likely to continue to exist as a critical point of the pseudo-energy beyond $\omega = 0.65$.

At degree five we found two distinct local minima of the pseudo-energy, namely a $5\mathcal{L}_{1,1}^{1,2}$ link and a $5\widetilde{\mathcal{A}}_{5,1}$ buckled ring. When $\omega = 0$ the link has the lower energy. The two curves for $E(\omega)$ cross when $\omega = 0.82$ and thereafter the ring has the lower energy. In contrast, the $E(J)$ curves for these configurations do not cross and the link has the lower energy for any given value of J.



$5\widetilde{\mathcal{A}}_{5,1}$ $5\mathcal{L}_{1,2}^{1,1}$ $6\mathcal{L}_{3,1}^{1,1}$ $6\mathcal{L}_{2,2}^{1,1}$

Figure 2: Solitons with $Q = 5, 6$ and $\mu = 2$: position curves for the $5\widetilde{\mathcal{A}}_{5,1}$, $5\mathcal{L}_{1,2}^{1,1}$, $6\mathcal{L}_{2,2}^{1,1}$ and $6\mathcal{L}_{3,1}^{1,1}$ configurations, with light green and blue position curves corresponding to $\omega = 0$ and $\omega = 1$ respectively.

At degree six we again found two distinct local minima of the pseudo-energy. These were links of type $6\mathcal{L}^{1,1}_{2,2}$ and $6\mathcal{L}^{1,1}_{3,1}$, shown in figure 2. Unlike the study [28], we did not find a $6\mathcal{A}_{3,2}$ configuration; this could be due to a different choice of potential function or a different choice of the mass parameter μ. The $6\mathcal{L}^{1,1}_{2,2}$ link has a lower energy than the $6\mathcal{L}^{1,1}_{3,1}$ link for all values of ω and for all values of J.

At degree seven the only minimum of the pseudo-energy found was a $7\mathcal{K}_{3,2}$ knot.

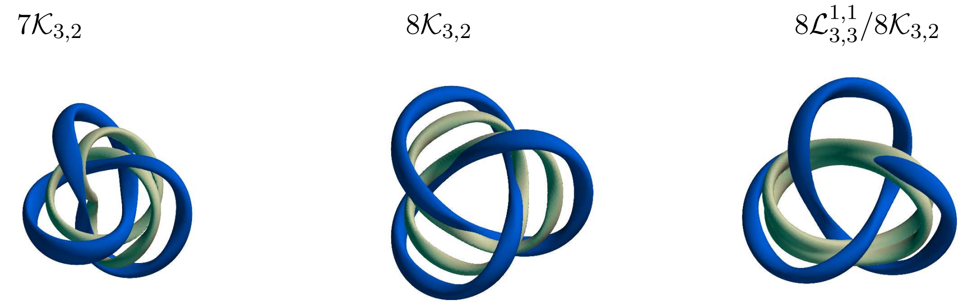


Figure 3: Solitons with $Q = 7, 8$ and $\mu = 2$. The top row shows position curves at $\omega = 0$ (light green) and $\omega = 1$ (blue).

At degree eight the three energy minima found were a link of type $8\mathcal{L}^{1,1}_{3,3}$ and a knot of type $8\mathcal{K}_{3,2}$, see figure 3. We did not find a soliton corresponding to the $8\mathcal{A}_{4,2}$ configuration in [28]. Within the limits of numerical accuracy the $8\mathcal{L}^{1,1}_{3,3}$ and $8\mathcal{K}_{3,2}$ configurations are degenerate in energy when $\omega = 0$. As ω increases the knot energy grows faster than that of the link, so that the link has the lower energy. When ω reaches the value 0.38 the link collapses to the knot, which has a smaller pseudo-energy. It is likely that when $\omega \geq 0.38$ the link is an unstable critical point of the pseudo-energy, and that it continues to have a lower energy than the knot.

5 Conclusions

We have studied isospinning soliton solutions of the Skyrme family, the low-dimensional baby Skyrme model and the 3+1 dimensional Faddeev-Skyrme model [18, 16, 20, 26]. Here we used reformulation of the minimization problem considering the stationary points of the pseudoenergy functional $F_\omega(\omega)$ which we found numerically without imposing any assumptions about the spatial symmetries. Our results confirm that the solitons persist for all range of values of $\omega \leq \min\{\sqrt{2}, \mu\}$, where μ is the mass of the scalar excitations, and their qualitative shape is independent of the frequency ω. Thus, there are two types of instabilities of the solitons from the Skyrme family, one is due to radiation of the scalar field and another one is related with destabilization of the rotating solitons by nonlinear velocity terms.

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