

# 'Buddha's light' of cumulative particles

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April 22, 2015

## Abstract

It is shown analytically that in the cumulative particles production off nuclei multiple interactions lead to a glory-like backward focusing effect. Employing the small phase space method we arrived at a characteristic angular dependence of the production cross section  $d\sigma \sim 1/\sqrt{\pi - \theta}$  near the strictly backward direction. This effect takes place for any number  $n \geq 3$  of interactions of rescattered particle, either elastic or inelastic (with resonance excitations in intermediate states), when the final particle is produced near corresponding kinematical boundary. In the final angles interval including the value  $\theta = \pi$  the angular dependence of the cumulative production cross section can have the crater-like (or funnel-like) form. Such a behaviour of the cross section near the backward direction is in qualitative agreement with some of available data. Rigorous proof of this effect is given for the case of the optimal kinematics, as well as for arbitrary polar scattering angles in the case of the light particle rescattering, but the arguments in favor of the backward azimuthal (axial) focusing are quite general and hold for any kind of the multiple interaction processes.

## 1 Introduction

Intensive studies of the particles production processes in high energy interactions of different projectiles with nuclei, in regions forbidden by kinematics for the interaction with a single free nucleon, began back in the 70th mostly at JINR (Dubna) and ITEP (Moscow). Relatively simple experiments could provide information about such objects as fluctuations of the nucleus density [3] or, discussed much later, few nucleon (or multiquark) clusters probably existing in nuclei. At JINR such processes have been called "cumulative production" [4, 5], at ITEP the variety of properties of such reactions has been called "nuclear scaling" [6]- [8] because certain universality of these properties has been noted, confirmed somewhat later at much higher energy, 400 GeV incident protons [9] and for 40 GeV/c incident pions, kaons and antiprotons [10]. A new wave of interest to this exciting topic appeared lately. New experiment has been performed in ITEP [11] aimed to define the weight of multiquark configurations in the carbon nucleus.

The interpretation of these phenomena as being manifestation of internal structure of nuclei assumes that the secondary interactions, or, more generally, multiple interactions processes (MIP) do not play a crucial role in such reactions [12] - [15].

The development of the Glauber approach [16] to the description of particles scattering off nuclei has been considered many years ago as remarkable progress in understanding the particles-nuclei interactions. Within the Glauber model the amplitude of the particle-nucleus

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scattering is presented in terms of elementary particle-nucleons amplitudes and the nucleus wave function describing the nucleons distribution inside the nucleus. The Glauber screening correction for the total cross section of particle scattering off deuteron allows widely accepted, remarkably simple and transparent interpretation.

Gribov [17] investigated nontrivial peculiarities of the space-time picture of such scattering processes and concluded that the inelastic shadowing corrections play an important role at high enough energy and should be included into consideration. In the case of the large angle particle production the background processes which mask the possible manifestations of nontrivial features of nuclear structure, are subsequent multiple interactions with nucleons inside the nucleus leading to the particles emission in the "kinematically forbidden" region. Leonid Kondratyuk was the first who has noted that rescattering of intermediate particles could lead to the final particles emission in "kinematically forbidden" regions (KFR). The double interaction process in the case of pion production off deuteron has been investigated first in [18]. Later the multiple interaction processes leading to nucleons production in KFR were studied in [19] where the magnitude of the cumulative protons production cross sections was estimated as well.

M.A.Braun and V.V.Vechernin with coauthors made interesting and important observations and investigated processes leading to the particles emission in KFR [20]-[22], including the processes with resonances in intermediate state [20]. They found also that processes with pions in intermediate state lead to the nucleons emission in KFR due to subsequent processes, like  $\pi N \rightarrow N \pi$  [21]. Basic theoretical aspects of MIP leading to the cumulative particles emission and some review of the situation in this field up to 1985 have been presented in [2].

Several authors attempted the cascade calculations of cumulative particles production cross sections relying upon the available computing codes created previously [24] - [27]. The particles production cross section was found to be in reasonable agreement with data. Different kinds of subprocesses play a role in these calculations, and certain work should be performed for detailed comparison. In calculations by NOMAD Collaboration the particles formation time (length) has been considered as a parameter, and results near to the experimental observations have been obtained for this time equal to  $\sim 2 Fm$  [27].

While many authors admitted the important role of the final state interactions (FSI), most of them did not discuss the active role of such interactions, i.e. their contribution to particles production in KFR, see e.g. [28]. Several specific features of the MIP mechanism have been noted previously experimentally and discussed theoretically [19, 23, 2], among them the presence of the recoil nucleons, which amount grows with increasing energy of the cumulative particle, possible large value of the cumulative baryons polarization, and some other, see [2]. The enhancement of the production cross section near the strictly backward direction has been detected in a number of experiments, first at JINR (Dubna) [29, 30] and somewhat later at ITEP (Moscow) [31, 32]. This glory-like effect which can be called also the "Buddha's light" of cumulative particles, has been shortly discussed previously in [19, 2]. More experimental evidence of this effect appeared since that time [33, 34]. We have shown analytically in [1] that presence of the backward focusing effect is an intrinsic property of the multiple interaction mechanism leading to the cumulative particles production.

## 2 Features of kinematics of the processes in KFR

When the particle with 4-momentum  $p_0 = (E_0, \vec{p}_0)$  interacts with the nucleus with the mass  $m_t \simeq Am_N$ , and the final particle of interest has the 4-momentum  $k_f = (\omega_f, \vec{k}_f)$  large enough incident energy,  $E_0 \gg m_t, M_f$ , the restriction takes place

$$\omega_f - zk_f \leq m_t, \tag{2.1}$$

which is the basic restriction for such processes.  $z = \cos \theta < 0$  for particle produced in backward hemisphere. The quantity  $(\omega_f - zk_f)/m_N$  is called the cumulative number (more precise, the

integer part of this ratio plus one).

Let us recall some peculiarities of the multistep processes kinematics established first in [19] and described in details in [2]. It is very selective kinematics, essentially different from the kinematics of the forward scattering off nuclei.

**Rescatterings.** For light particles ( $\pi$ -meson, for example) iteration of the Compton formula

$$\frac{1}{\omega_n} - \frac{1}{\omega_{n-1}} \simeq \frac{1}{m} [1 - \cos(\theta_n)] \quad (2.2)$$

allows to get the final energy in the form

$$\frac{1}{\omega_N} - \frac{1}{\omega_0} = \frac{1}{m} \sum_{n=1}^N [1 - \cos(\theta_n)] \quad (2.3)$$

The maximal energy of final particle is reached for the coplanar process when all scattering processes take place in the same plane and each angle equals to  $\theta_k = \theta/N$ . As a result we obtain

$$\frac{1}{\omega_N^{max}} - \frac{1}{\omega_0} = \frac{N}{m} [1 - \cos(\theta/N)] \quad (2.4)$$

Already at  $N > 2$  and for  $\theta \leq \pi$  the  $1/N$  expansion can be made (it is in fact the  $1/N^2$  expansion):  $1 - \cos(\theta/N) \simeq \theta^2/2N^2 (1 - \theta^2/12N^2)$ , and for large enough incident energy  $\omega_0$  we obtain

$$\omega_N^{max} \simeq N \frac{2m}{\theta^2} + \frac{m}{6N}. \quad (2.5)$$

This expression works quite well beginning with  $N = 2$ . Remarkably, that this rather simple property of rescattering processes has not been even mentioned in the pioneer papers [4] - [8] <sup>1</sup>.

In the case of the nucleon-nucleon scattering (scattering of particles with equal nonzero masses in general case) the following approximate relation has been obtained for the final particle momentum as a result of the  $1/N^2$  expansion at large enough  $N$  and large incident energy

$$k_N^{max} \simeq N \frac{2m}{\theta^2} - \frac{m}{3N}, \quad (2.6)$$

which coincides at large  $N$  with previous result (2.5) for the rescattering of light particles, but preasymptotic corrections are negative in this case and twice greater. The normal Fermi motion of nucleons inside the nucleus makes these boundaries wider [2]:

$$k_N^{max} \simeq N \frac{2m}{\theta^2} \left[ 1 + \frac{p_F^{max}}{2m} \left( \theta + \frac{1}{\theta} \right) \right], \quad (2.7)$$

where it is supposed that the final angle  $\theta$  is large,  $\theta \sim \pi$ . For numerical estimates we took the step function for the distribution in the Fermi momenta of nucleons inside of nuclei, with  $p_F^{max}/m \simeq 0.27$ , see [2] and references there. At large enough  $N$  normal Fermi motion makes the kinematical boundaries for MIP wider by about 40 %.

There is characteristic decrease (down-fall) of the cumulative particle production cross section due to simple rescatterings near the strictly backward direction. However, inelastic processes with excitations of intermediate particles, i.e. with intermediate resonances, are able to fill up the region at  $\theta \sim \pi$ .

**Resonance excitations in intermediate states.** The elastic rescatterings themselves are only the "top of the iceberg". Excitations of the rescattered particles, i.e. production of resonances in intermediate states which go over again into detected particles in subsequent

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<sup>1</sup>This property was well known, however, to V.M.Lobashev, who observed experimentally that the energy of the photon after 2-fold interaction can be substantially greater than the energy of the photon emitted at the same angle in 1-fold interaction.

interactions, provide the dominant contribution to the production cross section. Simplest examples of such processes may be  $NN \rightarrow NN^* \rightarrow NN$ ,  $\pi N \rightarrow \rho N \rightarrow \pi N$ , etc. The important role of resonances excitations in intermediate states for cumulative particles production has been noted first in [20]a and somewhat later in [19]. At incident energy about few GeV the dominant contribution into cumulative protons emission provide the processes with  $\Delta(1232)$  excitation and reabsorption, see [2] and [25]. Experimentally the role of dynamical excitations in cumulative nucleons production at intermediate energies has been established in [?] and, at higher energy, in [?].

When the particles in intermediate states are slightly excited above their ground states, approximate estimates can be made. Such resonances could be  $\Delta(1232)$  isobar, or  $N^*(1470)$ ,  $N^*(1520)$  etc. for nucleons, two-pion state or  $\rho(770)$ , etc for incident pions,  $K^*(880)$  for kaons. This case has been investigated previously with the result for the relative change (increase) of the final momentum  $k_f$  (Eq. (8) of [19])

$$\frac{\Delta k_f}{k_f} \simeq \frac{1}{N} \sum_{l=1}^{N-1} \frac{\Delta M_l^2}{k_l^2}, \quad \Delta k_f^2 \simeq \frac{2}{N^3} \sum_{l=1}^{N-1} l^2 \Delta M_l^2, \quad (2.8)$$

with  $\Delta M_l^2 = M_l^2 - \mu^2$ ,  $k_l$  is the value of 3-momentum in the  $l$ -th intermediate state. This effect can be explained easily: the additional energy stored in the mass of intermediate particle is transferred to the kinetic energy of the final (cumulative) particle.

### 3 The small phase space method for the MIP probability calculations

This method, most adequate for analytical and semi-analytical calculations of the MIP probabilities, has been proposed in [19] and developed later in [2]. It is based on the fact that, according to established in [19] and presented in previous section kinematical relations, there is a preferable plane of the whole MIP leading to the production of energetic particle at large angle  $\theta$ , but not strictly backwards. Also, the angles of subsequent rescatterings are close to  $\theta/N$ . Such kinematics has been called optimal, or basic kinematics. The deviations of real angles from the optimal values are small, they are defined mostly by the difference  $k_N^{max} - k$ , where  $k_N^{max}(\theta)$  is the maximal possible momentum reachable for definite MIP, and  $k$  is the final momentum of the detected particle.  $k_N^{max}(\theta)$  should be calculated taking into account normal Fermi motion of nucleons inside the nucleus, and also resonances excitation — deexcitation in the intermediate state. Some high power of the difference  $(k_N^{max} - k)/k_N^{max}$  enters the resulting probability.

Within the quasiclassical treatment adequate for our case, the probability product approximation is valid, and the starting expression for the inclusive cross section of the particle production at large angles contains the product of the elementary subprocesses matrix elements squared, see, e.g., Eq. (4.11) of [2].

After some evaluation, introducing differential cross sections of binary reactions  $d\sigma_l/dt_l(s_l, t_l)$  instead of the matrix elements of binary reactions  $M_l^2(s_l, t_l)$ , we came to the formula for the production cross section due to the  $N$ -fold MIP [19, 2]

$$f_N(\vec{p}_0, \vec{k}) = \pi R_A^2 G_N(R_A, \theta) \int \frac{f_1(\vec{p}_0, \vec{k}_1)(k_1^0)^3 x_1^2 dx_1 d\Omega_1}{\sigma_1^{leav} \omega_1} \prod_{l=2}^N \left( \frac{d\sigma_l(s_l, t_l)}{dt_l} \right) \frac{(s_l - m^2 - \mu_l^2)^2 - 4m^2 \mu_l^2}{4\pi m \sigma_l^{leav} k_{l-1}} \\ \times \prod_{l=2}^{N-1} \frac{k_l^2 d\Omega_l}{k_l(m + \omega_{l-1} - z_l \omega_l k_{l-1})} \frac{1}{\omega'_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N). \quad (3.1)$$

Here  $z_l = \cos \theta_l$ ,  $\sigma_l^{leav}$  is the cross section defining the removal (or leaving) of the rescattered object at the corresponding section of the trajectory, it is smaller than corresponding total cross section.  $G_N(R_A, \theta)$  is the geometrical factor which enters the probability of the  $N$ -fold multiple interaction with definite trajectory of the interacting particles (resonances) inside the nucleus. This trajectory is defined mostly by the final values of  $\vec{k}$  ( $k, \theta$ ), according to the kinematical relations of previous section. Inclusive cross section of the rescattered particle production in the first interaction is  $\omega_1 d^3 \sigma_1 / d^3 k_1 = f_1(\vec{p}_0, \vec{k}_1)$  and  $d^3 k_1 = (k_1^0)^3 x_1^2 dx_1$ ,  $\omega_N = \omega$  — the energy of the observed particle.

To estimate the value of the cross section (3.1) one can extract the product of the cross sections out of the integral (3.1) near the optimal kinematics and multiply by the small phase space available for the whole MIP under consideration [19, 2]. Further details depend on the particular process. For the case of the light particle rescattering,  $\pi$ -meson for example,  $\mu_l^2/m^2 \ll 1$ , we have

$$\frac{1}{\omega'_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N) = \frac{1}{kk_{N-1}} \delta \left[ \frac{m}{k} - \sum_{l=2}^N (1 - z_l) - \frac{1}{x_1} \left( \frac{m}{p_0} + 1 - z_1 \right) \right] \quad (3.2)$$

To get this relation one should use the equality  $\omega'_N = \sqrt{m^2 + k^2 + k_{N-1}^2 - 2kk_{N-1}z_N}$  for the recoil nucleon energy and the well known rules for manipulations with the  $\delta$ -function. When the final angle  $\theta$  is considerably different from  $\pi$ , there is a preferable plane near which the whole multiple interaction process takes place, and only processes near this plane contribute to the final output. At the angle  $\theta = \pi$ , strictly backwards, there is azimuthal symmetry, and the processes from the whole interval of azimuthal angle  $0 < \phi < 2\pi$  provide contribution to the final output (azimuthal focusing, see next section). A necessary step is to introduce azimuthal deviations from this optimal kinematics,  $\varphi_k$ ,  $k = 1, \dots, N-1$ ;  $\varphi_N = 0$  by definition of the plane of the process,  $(\vec{p}_0, \vec{k})$ . Polar deviations from the basic values,  $\theta/N$ , are denoted as  $\vartheta_k$ , obviously,  $\sum_{k=1}^N \vartheta_k = 0$ . The direction of the momentum  $\vec{k}_l$  after  $l$ -th interaction,  $\vec{n}_l$ , is defined by the azimuthal angle  $\varphi_l$  and the polar angle  $\theta_l = (l\theta/N) + \vartheta_1 + \dots + \vartheta_l$ ,  $\theta_N = \theta$ .

Then we obtain making the expansion in  $\varphi_l$ ,  $\vartheta_l$  up to quadratic terms in these variables:

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq \cos(\theta/N) (1 - \vartheta_k^2/2) - \sin(\theta/N) \vartheta_k + \sin(k\theta/N) \sin[(k-1)\theta/N] (\varphi_k - \varphi_{k-1})^2/2. \quad (3.3)$$

In the case of the rescattering of light particles the sum enters the phase space of the process

$$\begin{aligned} \sum_{k=1}^N (1 - \cos \vartheta_k) &= N[1 - \cos(\theta/N)] + \cos(\theta/N) \sum_{k=1}^N \left[ -\varphi_k^2 \sin^2(k\theta/N) + \right. \\ &\left. + \frac{\varphi_k \varphi_{k-1}}{\cos(\theta/N)} \sin(k\theta/N) \sin((k-1)\theta/N) \right] - \frac{\cos(\theta/N)}{2} \sum_{k=1}^N \vartheta_k^2 \end{aligned} \quad (3.4)$$

To derive this equality we used that  $\varphi_N = \varphi_0 = 0$  — by definition of the plane of the MIP, and the mentioned relation  $\sum_{k=1}^N \vartheta_k = 0$ . We used also the identity, valid for  $\varphi_N = \varphi_0 = 0$ :

$$\frac{1}{2} \sum_{k=1}^N (\varphi_k^2 + \varphi_{k-1}^2) \sin(k\theta/N) \sin[(k-1)\theta/N] = \cos(\theta/N) \sum_{k=1}^N \varphi_k^2 \sin^2(k\theta/N). \quad (3.5)$$

It is possible to present the quadratic form in angular variables which enters (3.4) in the canonical form and to perform integration easily, see Appendix B and Eq. (4.23) of [2], and also [1]. As a result, we have the integral over angular variables of the following form:

$$I_N(\Delta_N^{ext}) = \int \delta \left[ \Delta_N^{ext} - z_N^\theta \left( \sum_{k=1}^N \varphi_k^2 - \varphi_k \varphi_{k-1} / z_N^\theta + \vartheta_k^2/2 \right) \right] \prod_{l=1}^{N-1} d\varphi_l d\vartheta_l = \frac{(\Delta_N^{ext})^{N-2} (\sqrt{2}\pi)^{N-1}}{J_N(z_N^\theta) \sqrt{N} (N-2)! (z_N^\theta)^{N-1}}, \quad (3.6)$$

$z_N^\theta = \cos(\theta/N)$ . Since the element of a solid angle  $d\Omega_l = \sin(\theta l/N)d\vartheta_l d\varphi_l$ , we made here substitution  $\sin(\theta l/N)d\varphi_l \rightarrow d\varphi_l$  and  $d\Omega_l \rightarrow d\vartheta_l d\varphi_l$ ,  $z_N^\theta = \cos(\theta/N)$ . The whole phase space is defined by the quantity

$$\Delta_N^{ext} \simeq \frac{m}{k} - \frac{m}{p_0} - N(1 - z_N^\theta) - (1 - x_1)\frac{m}{p_0} \quad (3.7)$$

which depends on the effective distance of the final momentum (energy) from the kinematical boundary for the  $N$ -fold process. The Jacobian of the azimuthal variables transformation squared is

$$J_N^2(z) = Det ||a_N||, \quad (3.8)$$

where the matrix  $||a_N||$  defines the quadratic form  $Q_N(z, \varphi_k)$  which enters the argument of the  $\delta$ -function in Eq. (3.6):

$$Q_N(z, \varphi_k) = a_{kl}\varphi_k\varphi_l = \sum_{k=1}^N \varphi_k^2 - \frac{\varphi_k\varphi_{k-1}}{z}. \quad (3.9)$$

For example,

$$\begin{aligned} Q_3(z, \varphi_k) &= \varphi_1^2 + \varphi_2^2 - \varphi_1\varphi_2/z; & Q_4(z, \varphi_k) &= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3)/z, \\ Q_5(z, \varphi_k) &= \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_4)/z, \end{aligned} \quad (3.9a)$$

see next section.

The phase space of the process in (3.1) which depends strongly on  $\Delta_N^{ext}$ , after integration over angular variables can be presented in the form

$$\Phi_N^{pions} = \frac{1}{\omega'_N} \delta(m + \omega_{N-1} - \omega_N - \omega'_N) \prod_{l=1}^N d\Omega_l = \frac{I_N(\Delta_N^{ext})}{kk_{N-1}} = \frac{(\sqrt{2}\pi)^{N-1} (\Delta_N^{ext})^{N-2}}{kk_{N-1} (N-2)! \sqrt{N} J_N(z_N^\theta) (z_N^\theta)^{N-1}} \quad (3.10)$$

The normal Fermi motion of target nucleons inside of the nucleus increases the phase space considerably [19, 2]:

$$\Delta_N^{ext} = \Delta_N^{ext}|_{p_F=0} + \vec{p}_l^F \vec{r}_l / 2m, \quad (3.11)$$

where  $\vec{r}_l = 2m(\vec{k}_l - \vec{k}_{l-1})/k_l k_{l-1}$ . A reasonable approximation is to take vectors  $\vec{r}_l$  according to the optimal kinematics for the whole process, and the Fermi momenta distribution of nucleons inside of the nucleus in the form of the step function. Integration over the Fermi motion leads to increase of the power of  $\Delta_N^{ext}$  and change of numerical coefficients in the expression for the phase space. Details can be found in [19, 2], but they are not important for our mostly qualitative treatment here.

For the case of the nucleons rescattering there are some important differences from the light particle case, but the quadratic form which enters the angular phase space of the process is essentially the same, with additional coefficient:

$$\begin{aligned} \Phi_N^{nucleons} &= \frac{1}{k(m + \omega_{N-1})} \int \delta \left[ \Delta_{N,nucl}^{ext} - (z_N^\theta)^N Q_N(\varphi_k) - \frac{(z_N^\theta)^{N-2}}{2} \sum_{l=1}^N \vartheta_l^2 \right] \prod_{l=1}^N d\Omega_l = \\ &= \left( \frac{\sqrt{2}\pi}{\zeta_0 z_N^{N-1}} \right)^{N-1} \frac{(\Delta_{N,nucl}^{ext})^{N-2}}{(N-2)! \sqrt{N} J_N(z_N^\theta)} \frac{(1 - \zeta_N^2)(1 - \zeta_{N-1}^2)}{4m^2 \zeta_N} \end{aligned} \quad (3.12)$$

where

$$\Delta_{N,nucl}^{ext} = \zeta_N - (1 - x_1)\zeta_N \frac{1 - \zeta_1^2}{1 + \zeta_1^2} - \frac{k}{m + \omega}, \quad (3.13)$$

with  $\zeta_N = \zeta_0 (z_N^\theta)^N$ ,  $\zeta_1 = \zeta_0 z_N^\theta$ . As in the case of the light particle rescattering, the normal Fermi motion of nucleons inside the nucleus can be taken into account.

## 4 The backward focusing effect (Buddha's light of cumulative particles)

This is the sharp enhancement of the production cross section near the strictly backward direction,  $\theta = \pi$ . This effect has been noted first experimentally in Dubna (incident protons, final particles pions, protons and deuterons) [29, 30] and somewhat later by Leksin's group at ITEP (incident protons of 7.5 GeV/c, emitted protons of 0.5 GeV/c) [31].

In the papers [19, 2] where the small phase space method has been developed, it was noted that this effect can appear due to multiple interaction processes (see p.122 of [2]). However, the consideration of this effect was not detailed enough, the explicit angular dependence of the cross section near backward direction,  $\theta = \pi$ , has not been established, estimates and comparison with data have not been made <sup>2</sup>.

The backward focusing effect has been observed and confirmed later in a number of papers for different projectiles and incident energies [32, 33, 34]. It seems to be difficult to explain the backward focusing effect as coming from interaction with dense few nucleon clusters existing inside the nucleus.

Mathematically the focusing effect comes from the consideration of the small phase space of the whole multiple interaction process by the method described in previous section and in [19, 2]. It takes place for any MIP, regardless the particular kind of particles or resonances in the intermediate states. As it was explained in section 2, when the angle of cumulative particle emission is large, but different from  $\theta = \pi$ , there is a preferred plane for the whole process. When the final angle  $\theta = \pi$ , then integration over one of azimuthal angles takes place for the whole interval  $[0, 2\pi]$ , which leads to the rapid increase of the resulting cross section when the final angle  $\theta$  approaches  $\pi$ .

We show first that the azimuthal (axial) focusing takes place for any values of the polar scattering angles  $\theta_k^{opt}$ . For arbitrary angles  $\theta_k$  the cosine of the angle between directions  $\vec{n}_k$  and  $\vec{n}_{k-1}$  is

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq \cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) - \sin(\theta_k - \theta_{k-1})\vartheta_k + \sin(\theta_k)\sin\theta_{k-1}(\varphi_k - \varphi_{k-1})^2/2. \quad (4.1)$$

After substitution  $\sin\theta_k\varphi_k \rightarrow \varphi_k$  we obtain

$$z_k = (\vec{n}_k \vec{n}_{k-1}) \simeq \cos(\theta_k - \theta_{k-1})(1 - \vartheta_k^2/2) - \sin(\theta_k - \theta_{k-1})\vartheta_k + \frac{s_{k-1}}{2s_k}\varphi_k^2 + \frac{s_k}{2s_{k-1}}\varphi_{k-1}^2 - \varphi_{k-1}\varphi_k, \quad (4.2)$$

where we introduced shorter notations  $s_k = \sin\theta_k$ .

It follows from Eq. (4.2) that in general case of arbitrary polar angles  $\theta_k$  the quadratic form depending on the small azimuthal deviations  $\varphi_k$  which enters the sum  $\sum_k(1 - z_k)$  for the  $N$ -fold process is

$$Q_N^{gen}(\varphi_k, \varphi_l) = \frac{s_2}{s_1}\varphi_1^2 + \frac{s_1 + s_3}{s_2}\varphi_2^2 + \frac{s_2 + s_4}{s_3}\varphi_3^2 + \dots + \frac{s_{N-2} + s_N}{s_{N-1}}\varphi_{N-1}^2 - 2\varphi_1\varphi_2 - 2\varphi_2\varphi_3 - \dots - 2\varphi_{N-2}\varphi_{N-1} = ||a||^{gen}(\theta_1, \dots, \theta_{N-1})_{kl}\varphi_k\varphi_l, \quad (4.3)$$

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<sup>2</sup>One of the authors (VBK) discussed the cumulative (backward) particles production off nuclei with professor Ya.A.Smorodinsky who noted its analogy with known optical phenomenon - glory, or "Buddha's light". The glory effect has been mentioned by Leksin and collaborators [33], however, it was not clear to authors of [33], can it be related to cumulative production, or not. In the case of the optical (atmospheric) glory phenomenon the light scatterings take place within droplets of water, or another liquid. A variant of the atmospheric glory theory can be found in [36]. However, the existing explanation of the optical glory is still incomplete, see, e.g. <http://www.atoptics.co.uk/droplets/glofeat.htm>. In nuclear physics the glory-like phenomenon due to Coulomb interaction has been studied in [37] for the case of low energy antiprotons (energy up to few KeV) interacting with heavy nuclei.

with  $s_N = \sin\theta$ . E.g., for  $N = 5$  we have the matrix

$$\|a\|_{N=5}^{gen}(\theta_1, \theta_2, \theta_3, \theta_4) = \begin{bmatrix} s_2/s_1 & -1 & 0 & 0 \\ -1 & (s_1 + s_3)/s_2 & -1 & 0 \\ 0 & -1 & (s_2 + s_4)/s_3 & -1 \\ 0 & 0 & -1 & (s_3 + s_\theta)/s_4 \end{bmatrix}, \quad (4.4)$$

$s_\theta = s_5$ , and generalization to arbitrary  $N$  is straightforward.

Determinant of this matrix can be easily calculated. It can be shown by induction that at arbitrary  $N$  (for details see [1])

$$Det(\|a\|_N^{gen}) = \frac{s_\theta}{s_1}, \quad s_\theta = s_N. \quad (4.5)$$

After integration the delta-function containing the quadratic form over the small azimuthal deviations we obtain

$$\begin{aligned} \int \delta(\Delta - \|a\|_N^{gen}(\theta_1, \dots, \theta_{N-1})_{kl} \varphi_k \varphi_l) d\varphi_1 \dots d\varphi_{N-1} &= \frac{\Delta^{(N-3)/2}}{Det\|a\|_N^{gen}(N-3)!!} (2\pi)^{(N-3)/2} c_{N-3} = \\ &= \sqrt{\frac{s_1}{s_\theta}} \frac{\Delta^{(N-3)/2}}{(N-3)!!} (2\pi)^{(N-3)/2} c_{N-3}, \end{aligned} \quad (4.6)$$

$c_n = \pi$  for odd  $n$ , and  $c_n = \sqrt{2\pi}$  for even  $n$ , and  $N-3 \geq 0$ .

We obtain from above expressions the characteristic angular dependence of the cumulative particles production cross section near  $\theta = \pi$ :

$$d\sigma \sim \sqrt{\frac{s_1}{s_\theta}} \simeq \sqrt{\frac{s_1}{\pi - \theta}}, \quad (4.7)$$

since  $\sin\theta \simeq \pi - \theta$  for  $\pi - \theta \ll 1$ .

This formula does not work at  $\theta = \pi$ , because integration over the azimuthal angle which defines the plane of the whole MIP takes place in the interval  $(0, 2\pi)$ . The result for the cross section is final, of course, as we show in details for the case of the optimal kinematics.

For the optimal kinematics with equal polar scattering angles  $\theta_k = k\theta/N$  (see section 2), and the general quadratic form goes over into quadratic form obtained in [2] with some coefficients:

$$Q^{gen} \rightarrow 2z_N^\theta Q(z_N^\theta, \varphi_k, \varphi_l), \quad z_N^\theta = \cos(\theta/N), \quad (4.8)$$

and

$$Det(\|a\|_N^{gen}) = \left(2z_N^\theta\right)^{N-1} Det(\|a\|_N). \quad (4.8a)$$

It is convenient to present the quadratic form which enters the  $\delta$ -function in (3.6) as

$$\begin{aligned} Q_N(z_N^\theta, \varphi_k, \varphi_l) &= J_2^2 \left( \varphi_1 - \frac{\varphi_2}{2zJ_2^2} \right)^2 + \frac{J_3^2}{J_2^2} \left( \varphi_2 - \frac{J_2^2 \varphi_3}{2zJ_3^2} \right)^2 + \dots \\ &\dots + \frac{J_{N-1}^2}{J_{N-2}^2} \left( \varphi_{N-2} - \frac{J_{N-2}^2 \varphi_{N-1}}{2zJ_{N-1}^2} \right)^2 + \frac{J_N^2}{J_{N-1}^2} \varphi_{N-1}^2. \end{aligned} \quad (4.9)$$

For the sake of brevity we omitted here the dependence of all  $J_k^2$  on their common argument  $z_N^\theta$ . The recurrent relation

$$J_N^2(z) = J_{N-1}^2(z) - \frac{1}{4z^2} J_{N-2}^2(z) \quad (4.10)$$

can be obtained from (4.11), since, as it follows from (3.6) and (3.9)

$$Q_{N+1}(z, \varphi_k, \varphi_l) = Q_N(z, \varphi_k, \varphi_l) + \varphi_N^2 - \varphi_N \varphi_{N-l}/z \quad (4.11)$$



(recall that for the  $N + 1$ -fold process  $\varphi_{N+1} = 0$  by definition of the whole plane of the process),

The following formula for  $J_N^2(z_N^\theta)$  has been obtained in [2]:

$$\begin{aligned} Det||a_{kl}|| &= J_N^2(z_N^\theta) = 1 + \sum_{m=1}^{m < N/2} \left( -\frac{1}{4(z_N^\theta)^2} \right)^m \frac{\prod_{k=1}^m (N - m - k)}{m!} = \\ &= 1 + \sum_{m=1}^{m < N/2} \left( -\frac{1}{4(z_N^\theta)^2} \right)^m C_{N-m-1}^m. \end{aligned} \quad (4.12)$$

Recurrent relations for Jacobians with subsequent values of  $N$  and with same argument  $z$ :

$$J_{N+1}^2(z) = J_N^2(z) - \frac{1}{4z^2} J_{N-1}^2(z) = J_{N-1}^2(z) \left( 1 - \frac{1}{4z^2} \right) - \frac{1}{4z^2} J_{N-2}^2(z) \quad (4.13)$$

can be continued easily to lower values of  $N$  and also used for calculations of  $J_N^2$  at any  $N$  starting from two known values,  $J_2^2(z) = 1$  and  $J_3^2(z) = 1 - 1/(4z^2)$  (see [1]). The Eq. (4.14) can be confirmed in this way.

The condition  $J_N(\pi/N) = 0$  leads to the equation for  $z_N^\pi$  which solution (one of all possible roots) provides the value of  $\cos(\pi/N)$  in terms of radicals. The following expressions for these jacobians take place [19, 2]

$$J_2^2(z) = 1; \quad J_3^2(z) = 1 - \frac{1}{4z^2}; \quad J_4^2(z) = 1 - \frac{1}{2z^2}, \quad (4.14)$$

$J_3(\pi/3) = J_3(z = 1/2) = 0$ ,  $J_4(\pi/4) = J_4(z = 1/\sqrt{2}) = 0$ . Let us give here less trivial examples. For  $N = 5$

$$J_5^2 = 1 - \frac{3}{4z^2} + \frac{1}{16z^4}, \quad (J_5^2)'_z = \frac{3}{2z^3} - \frac{1}{4z^5} \quad (4.15)$$

and one obtains  $\cos^2(\pi/5) = (3 + \sqrt{5})/8$ ,  $J_5(\pi/5) = 0$ .

At  $N = 6$

$$J_6^2 = 1 - \frac{1}{z^2} + \frac{3}{16z^4} = J_3^2 \left( 1 - \frac{3}{4z^2} \right), \quad (J_6^2)'_z = \frac{2}{z^3} - \frac{3}{4z^5}. \quad (4.16)$$

For  $N = 7$

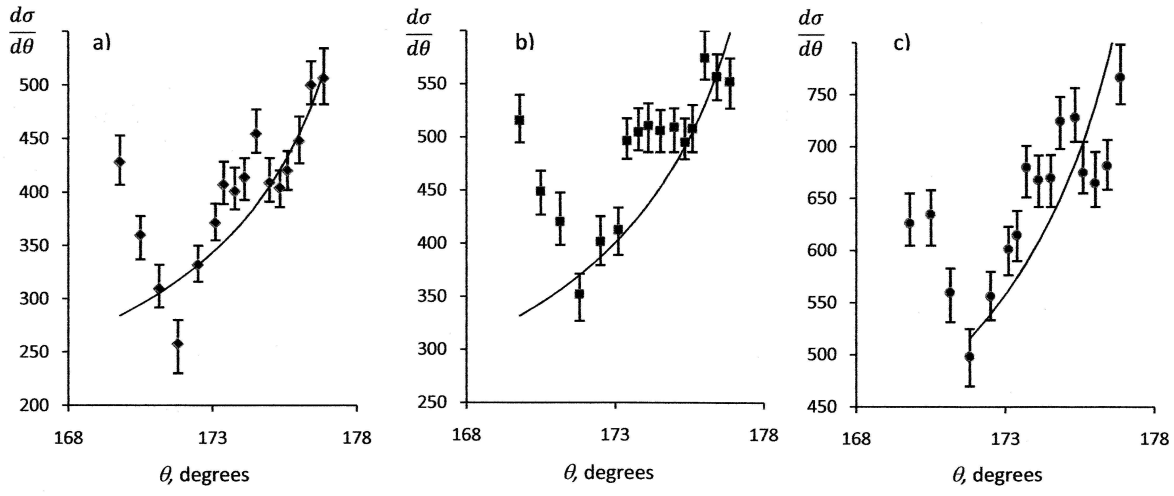
$$J_7^2 = 1 - \frac{5}{4z^2} + \frac{3}{8z^4} - \frac{1}{64z^6}, \quad (J_7^2)'_z = \frac{5}{2z^3} - \frac{3}{2z^5} + \frac{3}{32z^7}, \quad J_7(\pi/7) = 0. \quad (4.17)$$

$$J_8^2 = 1 - \frac{3}{2z^2} + \frac{5}{8z^4} - \frac{1}{16z^6} = J_4^2 \left( 1 - \frac{1}{z^2} + \frac{1}{8z^4} \right), \quad (J_8^2)'_z = \frac{3}{z^3} - \frac{5}{2z^5} + \frac{3}{8z^7}, \quad J_8(\pi/8) = 0 \quad (4.18)$$

For arbitrary  $N$ ,  $J_N^2$  is a polinomial in  $1/4z^2$  of the power  $|(N-1)/2|$  (integer part of  $(N-1)/2$ ), see Eq. (4.14). These equations can be obtained using the elementary mathematics methods as well, see [1]. The case  $N = 2$  is a special one, because  $J_2(z) = 1$  - is a constant. In this case the 2-fold process at  $\theta = \pi$  (strictly backwards) has no advantage in comparison with the direct one, see Eq. (2.5), if we consider the elastic rescatterings.

For particles emitted strictly backwards the phase space has different form, instead of  $J_N(\theta/N)$  enters  $J_{N-1}(\theta/N)$  which is different from zero at  $\theta = \pi$ , and we have instead of Eq. (3.6)

$$I_N(\varphi, \vartheta) = \int \delta \left[ \Delta_N^{ext} - z_N^\pi \left( \sum_{k=1}^N \varphi_k^2 - \varphi_k \varphi_{k-1} / z_N^\pi + \vartheta_k^2 / 2 \right) \right] \left[ \prod_{l=1}^{N-2} d\varphi_l d\vartheta_l \right] 2\pi d\vartheta_{N-1} =$$



**Fig. 4.1.** Angular distributions of secondary protons with kinetic energy between 0.11 and 0.24  $GeV$  emitted from the Pb nucleus, in arbitrary units. The momentum of the projectile protons is 4.5  $GeV/c$ . a) The energy of emitted protons in the interval 0.11 – 0.24  $GeV$ ; b) the energy interval 0.08 – 0.11  $GeV$ ; c) the energy interval 0.06 – 0.08  $GeV$ . Data obtained by G.A.Leksin group at ITEP, taken from Fig. 3 of paper [33].

$$= \frac{(\Delta_N^{ext})^{N-5/2} (2\sqrt{2}\pi)^{N-1}}{J_{N-1}(z_N^\pi) \sqrt{N} (2N-5)!! (z_N^\pi)^{N-3/2}}, \quad (4.19)$$

This follows from Eq. (4.11) where at  $\theta = \pi$  the last term disappears, since  $J_N(\pi/N) = 0$  and integration over  $d\varphi_{N-1}$  takes place over the whole  $2\pi$  interval.

To illustrate the azimuthal, or axial focusing which takes place near  $\theta = \pi$  the ratio is useful of the phase spaces near the backward direction and strictly at  $\theta = \pi$ . The ratio of the observed cross sections in the interval of several degrees slightly depends on the elementary cross sections and is defined mainly by this ratio of phase spaces. It is

$$R_N(\theta) = \frac{\Phi(z)}{\Phi(\theta = \pi)} = \sqrt{\frac{\Delta_N^{ext}}{z_N^\pi}} \frac{(2n-5)!!}{2^{N-1}(N-2)!} \frac{J_{N-1}(z_N^\pi)}{\sin(\pi/N) J_N(z_N^\theta)} \quad (4.20)$$

Near  $\theta = \pi$  we use that

$$J_N(z_N^\theta) \simeq \sqrt{\frac{\pi - \theta}{N}} [J_N^2]'(z_N^\pi) \sin \frac{\pi}{N} \quad (4.21)$$

and thus we get

$$R_N(\theta) = C_N \sqrt{\frac{\Delta_N^{ext}}{\pi - \theta}} \quad (4.22)$$

with

$$C_N = \frac{J_{N-1}(z_N^\pi) \sqrt{N}}{[(J_N^2)'(z_N^\pi)]^{1/2} [\sin(\pi/N)]^{3/2}} \frac{(2N-5)!!}{\sqrt{z_N^\pi} (N-2)! 2^{N-1}} \quad (4.23)$$

We need also values of  $J_{N-1}[\pi/N]$  to estimate the behaviour of the cross section near  $\theta = \pi$ . Integration over variable  $x_1$  leads to multiplication  $C_N$  by factor  $(2N-3)/(2N-2)$ , i.e. it makes it smaller, increasing the effect under consideration. The constant  $C_N$  is between 0.38 and 0.26 for  $N$  between 3 and 7, slightly decreasing with increasing  $N$  [1].

Inclusion of resonance excitation in one (or several) intermediate states leads to the increase of the quantity  $\Delta_N^{ext}$  according to formulas of section 2, and to the increase of the phase space

of the whole MIP, but the effect of azimuthal focusing persists. Quite similar results can be obtained for the case of nucleons, only some technical details are different, see section 3. The inclusion of the normal Fermi motion of nucleons inside the nucleus increases the values of  $\Delta_N^{ext}$ , but numerical coefficient in  $C_N$  becomes smaller. The behaviour given by Eq. (4.15) is in good agreement with available data, the value of the constants  $C_N$  is not important for our semiquantitative treatment.

There are other data besides shown at Fig. 4.1 (incident protons of  $4.5\text{ GeV}/c$ , detected cumulative particle also proton), where the agreement of the  $1/\sqrt{(\pi - \theta)}$  law with data is quite good, see [1].

## 5 Discussion and conclusions

The nature of the cumulative particles is complicated and not well understood so far. There are different possible sources of their origin, one of them are the multiple collisions inside the nucleus, i.e. elastic or inelastic rescatterings. We have shown that the enhancement of the particles production cross section off nuclei near the backward direction, the glory-like backward focusing effect, is a natural property of the multiple interaction mechanism for the cumulative particles production. It takes place for any multiplicity of the process,  $N \geq 3$ , when the momentum of the emitted particle is close to the corresponding kinematical boundary. The universal dependence of the cross section,  $d\sigma \sim 1/\sqrt{\pi - \theta}$  near the final angle  $\theta \sim \pi$ , takes place regardless the multiplicity of the process. This statement by itself is quite rigorous and presented for the first time in the literature. The competition of the processes of different multiplicities can make this effect difficult for observation in some cases. Presently we can speak only about qualitative, in some cases semiquantitative agreement with data. It is not clear yet how the transition to strictly backward direction proceeds. The angular distribution of emitted particles near  $\theta = \pi$  can have a narrow dip, i.e. it may be of a crater (funnel)-like form. Further studies are necessary for better understanding.

This effect, observed in a number of experiments at JINR and ITEP, is a clear manifestations of the fact that multiple interactions make important contribution to the cumulative particles production probability, although it does not exclude the contribution of interaction of the projectile with few-nucleon, or multiquark clusters possibly existing in nuclei. We have proved rigorously the existence of the azimuthal focusing for arbitrary polar angles in case of the light particles rescattering, and for the case of the optimal (basic) configuration of the MIP, also for nucleons rescattering. Obviously, the azimuthal focusing, discussed e.g. in [36] for the optical glory phenomenon, takes place for any kind of MIP; only some technical details are different.

It would be important to detect the backward focusing effect for different types of produced particles, baryons and mesons. This effect can be considered as a "smoking gun" of the MIP mechanism. If this nuclear glory-like phenomenon is observed for all kinds of cumulative particles, its universality would be a strong argument in favor of importance of MIP. The role of the multiple interaction processes leading to the large angle particles production off nuclei is certainly underestimated, still, by many authors. Further efforts are necessary to settle this extremely difficult and important challenge of disentangling between the nontrivial effects of the nuclear structure and the MIP contributions.

We are indebted to V.M.Lobashev who supported strongly the main idea that the background multiple interaction processes should be investigated and their contribution should be subtracted from measured cross sections to determine the weight of few-nucleon or multiquark clusters in nuclei. We thank S.S.Shimansky, B.Z.Kopeliovich, A.P.Krutenkova, A.B.Kurepin, V.L.Matushko for useful remarks and discussions.

This work is supported in part by Fondecyt (Chile), grant number 1130549.

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