Light-Like Wilson Loops in the Generalized Loop Space Setting

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Abstract

Equations of motion for the light-like QCD Wilson loops are studied in the generalized loop space setting. To this end, the classically conformal-invariant non-local variations of the cusped Wilson exponentials lying partially on the light-cone are formulated in terms of the Fréchet derivative. The rapidity and renormalization-group behaviour of the gauge-invariant quantum correlation functions (in particular, the 3D-PDFs) are demonstrated to be connected to certain geometrical properties of the Wilson loops defined in the generalized loop space.

1 Introduction

The QCD factorization approach to the analysis of the semi-inclusive high-energy processes entails the introduction of transverse-momentum dependent parton densities (TMD), which generalise the collinear (integrated) PDFs and contain essential information about three-dimensional intrinsic structure of the nucleon [1]. In Ref. [2] the following factorization scheme (valid in the large Bjorken-x regime) for a generic *transverse-distance dependent* quark distribution function

$$\mathcal{F}\left(x, \boldsymbol{b}_\perp\right) = \int d^2 k_\perp \ \mathrm{e}^{-ib_\perp k_\perp} \mathcal{F}\left(x, \boldsymbol{k}_\perp\right) \tag{1}$$

has been proposed

$$\mathcal{F}\left(x, \boldsymbol{b}_\perp; \eta, \mu^2\right) \approx \mathcal{H}(\mu^2, P^2) \cdot \Phi(x, \boldsymbol{b}_\perp; \eta, \mu^2), \tag{2}$$

where the x-independent jet function $\mathcal{H}$ describes the incoming (collinear) partons and the soft function Φ can be defined as the Fourier transform of an element of the *generalized loop space*

$$\Phi(x, \boldsymbol{b}_\perp; \eta, \mu^2) = \int dz^- \ \mathrm{e}^{-i(1-x)P^+ z^-} \ \mathcal{W}_*(z^-, \boldsymbol{b}_\perp; \eta, \mu^2) \ . \tag{3}$$

Here the η and μ stand for the rapidity and ultra-violet regulators, respectively. The Wilson loop $\mathcal{W}_*$ reads

$$\mathcal{W}_*(z^-, \boldsymbol{b}_\perp; \eta, \mu^2) = \langle 0 | \ \mathcal{U}_P^\dagger[\infty; z] \mathcal{U}_{n^-}^\dagger[z; \infty] \mathcal{U}_{n^-}[\infty; 0] \mathcal{U}_P[0; \infty] \ |0\rangle, \tag{4}$$

and appears to be a cusped configuration consisting of two off-light-cone $\mathcal{U}_P$ with $P^2 \neq 0$, and two light-like $\mathcal{U}_{n^-}$ with $(n^-)^2 = 0$ Wilson lines. The quark jet effects in $\mathcal{H}$, therefore, are separated from the soft function Φ, which accumulates information about the intrinsic $3D$-structure of the nucleon in the large-x domain, the latter being available at the planned EIC and Jefferson Lab, see Ref. [3] and Refs. therein.

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As it follows from the factorization formula, Eq. (2), the soft function Φ contains the *rapidity* as well as *ultraviolet singularities* of the TMD distribution (1). The complex structure of the UV and light-cone (rapidity) divergences and their crucial effects on the evolution of TMDs with the emphasis on the gauge invariance and the properties of the anomalous dimensions has been studied in detail in Refs. [4]. On the other hand, as an element of the GLS, the Wilson loop $\mathcal{W}_*$ obeys the integro-differential equations of motion, which prescribe the behaviour of the quantum correlation functions containing $\mathcal{W}_*$ with respect to the shape variations of the underlying paths [5]. Let us show that the connection between certain (diffeomorphism-invariant) transformations in the GLS and classically conformal invariant shape transformations allows one to simplify the calculation of the evolution kernels for the TMD (1). In particular, the rapidity differential operators can be represented in terms of the *Fréchet differentials* enabling the derivation of the full set of the evolution equations.

To this end, let us consider the generic quadrilateral contour[1] with the sides given by the vectors

$$\begin{aligned} \ell_1^\mu &= \ell_1(1^+, 0^-, \mathbf{0}_\perp), \ \ell_2^\mu = \ell_2(0^+, 1^-, \mathbf{0}_\perp), \\ \ell_3^\mu &= -\ell_1(1^+, 0^-, \mathbf{0}_\perp), \ \ell_4^\mu = -\ell_2(0^+, 1^-, \mathbf{0}_\perp). \end{aligned} \tag{5}$$

One can introduce a class of the path transformations generated by the differential operators [7]

$$S_{ij}\frac{\delta}{\delta S_{ij}} = (2\ell_i \cdot \ell_j)\frac{\partial}{\partial(2\ell_i \cdot \ell_j)}, \ S_{ij} = (\ell_i + \ell_j)^2 \ , \tag{6}$$

and

$$\left\langle \frac{\delta}{\delta \ln S} \right\rangle_1 = S_{12}\frac{\delta}{\delta S_{12}} + S_{23}\frac{\delta}{\delta S_{23}}, \ \left\langle \frac{\delta}{\delta \ln S} \right\rangle_2 = S_{23}\frac{\delta}{\delta S_{23}} + S_{34}\frac{\delta}{\delta S_{34}}, \text{etc.} \tag{7}$$

It is easy to see that the rapidities y_i associated with the light-like vectors ℓ_i, being formally infinite, can be regulated as follows:

$$y_1 = \frac{1}{2} \ \ln\frac{\ell_1^+}{\ell_1^-} \sim \pm\frac{1}{2}\lim_{\eta_1\to 0} \ \ln\frac{(\ell_1 \cdot \ell_2)}{\eta_1}. \tag{8}$$

Hence, the differential operators (7) are related to the logarithmic rapidity derivative via

$$\frac{d}{d\ln S_{ij}} \sim \pm\frac{d}{dy_i}. \tag{9}$$

We conjecture then that the rapidity evolution of a correlation function with light-like cusped Wilson loops corresponds to a shape-transformation law of a specific class of elements of the GLS. To reveal this law, we address the shape-transformations in the GLS by means of the so-called *Fréchet derivative* [5].

By definition, the logarithmic Fréchet derivative associated with a given vector V reads

$$D_V[U_\gamma] = U_\gamma \cdot \int_0^1 dt \ U_{\gamma^t} \cdot \mathcal{F}_{\mu\nu}(t) \left[V^\mu(t) \wedge \dot\gamma^\nu(t)\right] \cdot U_{\gamma^t}^{-1}, \tag{10}$$

for the Wilson exponential U_γ evaluated along a given trajectory γ, where

$$U_{\gamma^t} = \mathcal{P}\exp\left[ig\int_0^t \mathcal{A}_\mu(x) \ \dot\gamma^\mu d\sigma\right]_\gamma \ , \ x_\mu(\sigma) = \dot\gamma_\mu\sigma \ , \tag{11}$$

[1]The singularities and renormalization properties of the light-like Wilson polygons have been introduced and extensively studied in Refs. [6].

$$\sigma \in [0,1] \ , \ x_\mu(0) = x_\mu(1) \ , \ U_\gamma = U_{\gamma^1}. \tag{12}$$

Define a vector field

$$V^\mu = V_1^\mu + V_2^\mu = (\ell_1^+ \ , \ \ell_2^- \ , \ \mathbf{0}_\perp) \tag{13}$$

which generates the angle-conserving transformations. Let us show that the operator (7) transforms our $\mathcal{W}_*$ equivalently to this differential if the generating vector V defined in Eq. (13)

$$\left(S_{12} \frac{\delta}{\delta S_{12}} + S_{23} \frac{\delta}{\delta S_{23}} \right) \ \mathcal{W}_* = D_V \ \mathcal{W}_*, \ \mathcal{W}_* = \left\langle 0 \middle| \frac{1}{N_c} \mathrm{Tr} \ U_* \middle| 0 \right\rangle. \tag{14}$$

2 Calculation of the leading-order contributions

Expand Eq. (10) to the leading non-trivial order:

$$\begin{aligned}
&D_V[\mathcal{W}_\gamma]^{(1)} = \\
&\oint_0^1 dt \ \Bigg[\Bigg(\oint_0^t \mathcal{A}_\sigma(x(s)) \ \frac{dx^\sigma}{ds} ds \cdot \{\partial_\mu \mathcal{A}_\nu(y(t)) - \partial_\nu \mathcal{A}_\mu(y(t))\} \{V^\mu(y(t)) \wedge \dot{\gamma}^\nu(y(t))\} \Bigg) \\
&\quad - \Bigg(\{\partial_\mu \mathcal{A}_\nu(y(t)) - \partial_\nu \mathcal{A}_\mu(y(t))\} \{V^\mu(y(t)) \wedge \dot{\gamma}^\nu(y(t))\} \oint_0^t \mathcal{A}_\lambda(x(u)) \ \frac{dx^\lambda}{du} du \Bigg) \Bigg] \\
&\quad + \oint_0^1 \mathcal{A}_\sigma(x) \frac{dx^\sigma}{ds} ds \oint_0^1 dt \, \{\partial_\mu \mathcal{A}_\nu(y(t)) - \partial_\nu \mathcal{A}_\mu(y(t))\} \{V^\mu(y(t)) \wedge \dot{\gamma}^\nu(y(t))\} \, .
\end{aligned} \tag{15}$$

We assume that the gluon propagator in the Feynman gauge reads in the coordinate space

$$\langle 0 | T[A_\mu^a(x) A_\nu^b(y)] | 0 \rangle = D_{\mu\nu}^{ab}(x-y) = K_\epsilon \ \frac{g_{\mu\nu}\delta^{ab}}{[-(x-y)^2]^{1-\epsilon}}, \tag{16}$$

where

$$K_\epsilon = \frac{(\mu^2 \pi)^\epsilon}{4\pi^2} \Gamma(1-\epsilon). \tag{17}$$

Let us consider first the generating vector

$$V_1^\mu = (\ell_1^+ \sigma, 0^-, \mathbf{0}_\perp) \ , \ \sigma \in [0,1]. \tag{18}$$

Computation for the vector

$$V_2^\mu = (0^+, \ell_2^- \sigma', \mathbf{0}_\perp) \tag{19}$$

runs similarly. The contributions from the wedge product

$$V_1^\mu(y(\sigma)) \wedge \dot{\gamma}^\nu(y(\sigma) \tag{20}$$

can be described as follows:

- The sides ℓ_1: $V^\mu \wedge \dot{\gamma}^\nu = 0$ and ℓ_3: $V^\mu \wedge \dot{\gamma}^\nu = 0$, by the asymmetry of the wedge product and the fact that the vectors are parallel;
- The side ℓ_2: $V^\mu \wedge \dot{\gamma}^\nu = -\ell_1^+ \ell_2^- (\partial_+ \wedge \partial_-)$, by the (anti-)linearity of the wedge product;
- The side ℓ_4: $V^\mu \wedge \dot{\gamma}^\nu = 0$, since the vector field equals zero along this part of the path.

We have, therefore, the following combinations of the gluon propagators to be evaluated:

2.1 $\partial_\mu D_{\sigma\nu}(x-y) - \partial_\nu D_{\sigma\mu}(x-y)$ term with $x \in \ell_1$

Given that

$$x = \sigma\ell_1,\ \sigma \in [0,1] \tag{21}$$
$$y = \ell_1 + \sigma'\ell_2,\ \sigma' \in [0,1]\ , \tag{22}$$

one gets

$$\begin{aligned}
dx^\sigma &= \left(\frac{dx^\sigma}{d\sigma}\right) d\sigma = (\ell_1^+, 0^-, \mathbf{0}_\perp) d\sigma \\
dy^\nu &= \left(\frac{dy^\nu}{d\sigma'}\right) d\sigma' = (0^+, \ell_2^-, \mathbf{0}_\perp) d\sigma' = \dot\gamma(\sigma') d\sigma' \\
x - y &= (\sigma - 1)\ell_1 - \sigma'\ell_2 \\
(x-y)^2 &= -2(\sigma-1)\sigma'\ (\ell_1^+ \ell_2^-)\ .
\end{aligned}$$

Straightforward computation yields

$$\begin{aligned}
&\int_0^1 d\sigma'\ d\sigma \frac{dx^\rho}{d\sigma} \left(\frac{\partial}{\partial y^\mu} D_{\rho\nu}(x-y) - \frac{\partial}{\partial y^\nu} D_{\rho\mu}(x-y) \right) [V^\mu(y) \wedge \dot\gamma^\nu(y)] \\
&= \frac{1}{2} K_\epsilon \frac{S_{12}^\epsilon}{\epsilon}.
\end{aligned} \tag{23}$$

It is worth noticing that the same result can be obtained by applying the derivative $\ell_1 \frac{\partial}{\partial \ell_1}$ to the original integral

$$\ell_1 \frac{\partial}{\partial \ell_1} K_\epsilon \oint \frac{g_{\mu\nu}\ dx^\mu\ dy^\nu}{(-(x-y)^2)^{1-\epsilon}} = \ell_1 \frac{\partial}{\partial \ell_1} K_\epsilon \oint \frac{(\ell_1 \ell_2)\ d\sigma\ d\sigma'}{(-(2\ell_1\ell_2(\sigma-1)\sigma')^2)^{1-\epsilon}} = \frac{1}{2} K_\epsilon \frac{S_{12}^\epsilon}{\epsilon}. \tag{24}$$

2.2 $\partial_\mu D_{\rho\nu}(x-y) - \partial_\nu D_{\rho\mu}(x-y)$ term with $x \in \ell_2$

This term is trivially zero since it represents the self-energy of a light-like Wilson line.

2.3 $\partial_\mu D_{\rho\nu}(x-y) - \partial_\nu D_{\rho\mu}(x-y)$ term with $x \in \ell_3$

One has

$$\int_0^1 d\sigma'\ d\sigma \frac{dx^\rho}{d\sigma} \left(\frac{\partial}{\partial y^\mu} D_{\rho\nu}(x-y) - \frac{\partial}{\partial y^\nu} D_{\rho\mu}(x-y) \right) [V^\mu(y) \wedge \dot\gamma^\nu(y)] = 0. \tag{25}$$

that is equal to the result of the differentiation $\ell_1 \frac{\partial}{\partial \ell_1}$.

2.4 $\partial_\mu D_{\rho\nu}(x-y) - \partial_\nu D_{\rho\mu}(x-y)$ term with $x \in \ell_4$

We introduce the parametrization

$$x = -(1-\sigma)\ell_4,\ \sigma \in [0,1]\ , \tag{26}$$
$$y = \ell_1 + \sigma'\ell_2\ ,\ \sigma' \in [0,1] \tag{27}$$

and split up the calculations into the two terms $\partial_\mu D_{\rho\nu}(x-y)$ and $-\partial_\nu D_{\rho\mu}(x-y)$. The first term $\partial_\mu D_{\rho\nu}(x-y)$ then returns

$$\int_0^1 d\sigma' \ d\sigma \frac{dx^\rho}{d\sigma} \left(\frac{\partial}{\partial y^\mu} D_{\rho\nu}(x-y) \right) [V^\mu(y) \wedge \dot{\gamma}^\nu(y)] =$$

$$-2(\epsilon - 1) \int_0^1 d\sigma' \ d\sigma \ \left[\ell_1 \cdot (\ell_1 + \sigma' \ell_2 + (1-\sigma)\ell_4)\right] \frac{(\ell_2 \cdot \ell_4)}{\left(-(\ell_1 + \sigma'\ell_2 + (1-\sigma)\ell_4)^2\right)^{2-\epsilon}}, \tag{28}$$

while the second term can be shown to give zero [8].

The same procedure applies to the generating vector

$$V_2^\mu = (0^+, \ell_2^-, \mathbf{0}_\perp) \tag{29}$$

with the point y being attached to the side ℓ_3.

Therefore, we obtain

$$\left\langle \frac{\delta}{\delta \ln S} \right\rangle_1 \mathcal{W}_* = \left(S_{12} \frac{\delta}{\delta S_{12}} + S_{23} \frac{\delta}{\delta S_{23}} \right) \mathcal{W}_* = D_V \ \mathcal{W}_*, \tag{30}$$

with

$$V^\mu = V_1^\mu + V_2^\mu = (\ell_1^+ \ , \ \ell_2^- \ , \ \mathbf{0}_\perp). \tag{31}$$

Taking into account the renormalization properties of the light-like Wilson quadrilateral loop [6], we conclude that

$$\mu \frac{d}{d\mu} \ [D_V \ \mathcal{W}_*] = - \sum \Gamma_{\text{cusp}}, \tag{32}$$

where Γ_{cusp} denotes the light-cone cusp anomalous dimension [6, 9] and the summation over the number of cusps is assumed.

3 Conclusions

To summarise:

- The logarithmic Fréchet derivative interpreted as a diffeomorphism-induced differential operator associated with a generating vector field V^μ is shown to be equivalent to the non-local infinitesimal shape-transformation introduced in Ref. [7].

- Therefore, a specific class of the motions in the GLS, referred as the classically conformal-invariant transformation, can be introduced in terms of the Fréchet derivative. Because diffeomorphisms do not produce new cusps, the number of cusps is diffeomorphism-invariant. We conjecture that the light-like cusped Wilson loops possessing different number of cusps correspond to different physical objects obeying different evolution laws.

- Application of the developed formalism to the derivation of the evolution equations for the TMDs on the light-cone is a subject of ongoing investigation and will be reported elsewhere.

References

[1] J. C. Collins and D. E. Soper, *Nucl. Phys.* **B193**, 381 (1981); *Nucl. Phys.* **B194**, 445 (1982); J. C. Collins, D. E. Soper and G. F. Sterman, *Nucl. Phys.* **B223**, 381 (1983); *Phys. Lett.* **B109**, 388 (1982); *Nucl. Phys.* **B250**, 199 (1985); J. Collins, *"Foundations of perturbative QCD,"* Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (2011) A. V. Belitsky and A. V. Radyushkin, *Phys. Rept.* **418**, 1 (2005); U. D'Alesio, F. Murgia, *Prog. Part. Nucl. Phys.* **61**, 394 (2008); X-d Ji and F. Yuan, *Phys. Lett.* **B543**, 66 (2002); A. V. Belitsky, X. Ji and F. Yuan, *Nucl. Phys.* **B656**, 165 (2003); D. Boer, P.J. Mulders and F. Pijlman, *Nucl. Phys.* **B667**, 201 (2003)

[2] I. O. Cherednikov, T. Mertens, P. Taels and F. F. Van der Veken, *Int. J. Mod. Phys. Conf. Ser.* **25**, 1460006 (2014); I. O. Cherednikov, *Few Body Syst.* **55**, 303 (2014)

[3] D. Boer, M. Diehl, R. Milner, R. Venugopalan, W. Vogelsang, D. Kaplan, H. Montgomery and S. Vigdor *et al.*, arXiv:1108.1713 [nucl-th]; A. Accardi, V. Guzey, A. Prokudin and C. Weiss, *Eur. Phys. J.* **A48**, 92 (2012); J. Dudek, *et al.*, *Eur. Phys. J.* **A48**, 187 (2012)

[4] I. O. Cherednikov and N. G. Stefanis, *Phys. Rev.* **D77**, 094001 (2008); *Nucl. Phys.* **B802**, 146 (2008); *Phys. Rev.* **D80**, 054008 (2009); N. G. Stefanis, I. O. Cherednikov, *Mod. Phys. Lett.* **A24**, 2913 (2009); I. O. Cherednikov, A. I. Karanikas and N. G. Stefanis, *Nucl. Phys.* **B840**, 379 (2010)

[5] J. N. Tavares, *Int. J. Mod. Phys.* **A9**, 4511 (1994); R. A. Brandt, F. Neri and M. A. Sato, *Phys. Rev.* **D24**, 879 (1981); R. A. Brandt, A. Gocksch, M. A. Sato and F. Neri, *Phys. Rev.* **D26**, 3611 (1982); Y. Y. Makeenko and A. A. Migdal, *Phys. Lett.* **B88**, 135 (1979); *Nucl. Phys.* **B188**, 269 (1981); Y. Y. Makeenko, *"Methods of Contemporary Gauge Theory"*, Cambridge Monographs on Mathematical Physics (2002); I.O. Cherednikov, T. Mertens and F.F. Van der Veken, *"Wilson Lines in Quantum Field Theory"*, De Gruyter Studies in Mathematical Physics, Berlin (2014)

[6] I. A. Korchemskaya and G. P. Korchemsky, *Phys. Lett.* **B287**, 169 (1992); A. Bassetto, I. A. Korchemskaya, G. P. Korchemsky and G. Nardelli, *Nucl. Phys.* **B408**, 62 (1993); G. P. Korchemsky, J. M. Drummond and E. Sokatchev, *Nucl. Phys.* **B795**, 385 (2008)

[7] I. O. Cherednikov, T. Mertens and F. F. Van der Veken, *Phys. Rev.* **D86**, 085035 (2012); *Phys. Part. Nucl.* **44**, 250 (2013); T. Mertens and P. Taels, *Phys. Lett.* **B727**, 563 (2013)

[8] I. O. Cherednikov and T. Mertens, *Phys. Lett.* **B**, in print; arXiv:1401.2721 [hep-th]; *Phys. Lett.* **B734**, 198 (2014)

[9] G. P. Korchemsky and A. V. Radyushkin, *Nucl. Phys.* **B 283** (1987) 342; G. P. Korchemsky and G. Marchesini, *Nucl. Phys.* **B406**, 225 (1993)