Post-Newtonian limits for brane-world model

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Abstract

Parametrized post-Newtonian formalism is one of the standard methods of experimental check of gravity theories and limiting the model parameters. In this work we consider the PPN-paramertization of Randall-Sundrum II black hole solutions and show that these solutions are in good agreement with the GR predictions and the observation results.

1 Introduction

Black hole solution is a basic one for any theory of gravity. It describes the compact object that a very massive star at the end of its life cycle collapses into and it also features the curvature of the spacetime produced by the presence of matter specific for the considered gravity model. Any extended theory of gravity should be consistent with the predictions of GR and the observations' results therefore the existence of black holes and their properties are important indicators of the theory's viability. In this work we consider the weak-field limit of the Randall-Sundrum solutions found by Figueras and Wiseman and by Abdolrahimi, Cattoën, Page and Yaghoobpour-Tari to make certain that they agree with GR. We also look for the opportunity to distinguish the Randall-Sundrum model experimentally via these solutions.

2 Parameterized post-Newtonian formalism

Parametrized post-Newtonian limit (PPN) was originally constructed for comparing different metric theories of gravity with each other [1, 2, 3, 4, 5, 6]. For its application several requirements should be fulfilled:

- weak field limit,
- asymptotically flat spacetime,
- small velocities of matter the motion of which obeys the hydrodynamical equations for the prefect fluid.

Such parametrization allows to distinguish metric theories from each other on a number of properties such as the measure of space curvature produced by unit mass, the non-linearity in gravitational superposition, the existence of preferred location and frame effects and the violation of conservation laws of energy, momentum and angular momentum. These properties

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are expressed through the post-Newtonian parameters that are measured experimentally with high precision [6].

Using the parametrized post-Newtonian formalism the metric can be represented as the perturbative expansion around Minkowski spacetime $\eta_{\mu\nu}$ [5]:

$$\begin{aligned} &g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \\ &h_{00} \sim 1/r^2, \quad h_{oj} \sim 1/r^3, \quad h_{ij} \sim 1/r. \end{aligned} \tag{1}$$

In the second post-Newtonian limit the corrections describe the gravitational waves effects and contain the terms of order [7, 8]:

$$h_{00} \sim 1/r^3, \quad h_{0j} \sim 1/r^{2.5}, \quad h_{ij} \sim 1/r^2. \tag{2}$$

To obtain the post-Newtonian parametrization of the considered metric the corresponding field equations should be solved:

$$R_{\mu\nu} = \ 8\pi \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2}\, T \right), \tag{3}$$

where left part can be expressed in terms of perturbations $h_{\mu\nu}$ to Minkowski metric $\eta_{\mu\nu}$ [5]:

$$\begin{aligned} R_{00} &= -\frac{1}{2}\,\nabla^2 h_{00} - \frac{1}{2}\left(h_{jj,00} - 2h_{j0,j0} \right) + \frac{1}{2}\, h_{00,j}\left(h_{jk,k} - \frac{1}{2}\, h_{kk,j} \right) - \frac{1}{4}\,|\nabla h_{00}|^2 + \\ &+ \frac{1}{2}\, h_{jk} h_{00,jk}, \\ R_{0j} &= -\frac{1}{2}\left(\nabla^2 h_{0j} - h_{k0,jk} + h_{kk,0j} - h_{kj,0k}\right), \\ R_{ij} &= -\frac{1}{2}\left(\nabla^2 h_{ij} - h_{00,ij} + h_{kk,ij} - h_{ki,kj} - h_{kj,k,i}\right). \end{aligned} \tag{4}$$

Stress-energy tensor describing the behavior of the matter and appearing in the right part of (3) can be expressed in terms of the post-Newtonian potentials and the post-Newtonan parameter γ parameters [5]:

$$\begin{aligned} T^{00} &= \rho\ \left(1 + \Pi + v^2 + 2U\right), \\ T^{0i} &= \rho\ \left(1 + \Pi + v^2 + 2U + \frac{p}{\rho}\right)\ v^i \\ T^{ij} &= \rho\ \left(1 + \Pi + v^2 + 2U + \frac{p}{\rho}\right)\ v^i v^j + p\delta^{ij}(1 - 2\gamma U), \end{aligned} \tag{5}$$

where ρ, p and v are the density, the pressure and the velocity of matter correspondingly, U is the gravitational potential with the sigh reversed, Π is the density of internal energy including all forms of non-rest-mass, non-gravitational energy, e.g., energy of compression and thermal energy. These post-Newtonian potentials obey the following rules [5]:

$$U \sim v^2 \sim p/\rho \sim \Pi \sim 1/r. \tag{6}$$

The units in which $G = c = 1$ are used here.

3 Randall-Sundrum gravity

The Randall-Sundrum model considers four-dimensional branes with tension embedded into a five-dimensional spacetime (bulk) that is assumed to have an AdS_5 geometry [9, 10]. All the matter and the three fundamental interactions (electromagnetic, strong nuclear and weak nuclear) are localized on the brane except gravity which can propagate into the bulk along the additional dimension. This extra dimension is allowed to be noncompact and even infinite. Randall-Sundrum I (RSI) model considers two branes with different properties helping to solve the hierarchy problem [9]. Moving the second brane to the infinity led to the Randall-Sundrum II (RSII) model with one brane [10].

First black hole solutions were obtained by Chambling, Hawking and Reall [11] and by Dadhich, Maartens, Papadopoulos and Rezania [12] in 2000. A remarkable conjecture was then made that static black holes cannot exist in RSII for a radius much greater than the AdS length ℓ [13, 14, 15]. By using the numerical methods [16, 17] black holes in 5D RSII with a radius up to $\sim 0.2\ell$ and for 6D up to $\sim 2.0\ell$ were constructed [18, 19, 20]. However, by using the same methods, it has subsequently been argued that even very small RSII static black holes do not exist [21, 22]. In 2011 Figueras and Wiseman numerically constructed large RSII static black holes with radius up to $\sim 20\ell$ close to the associated AdS_5-CFT_4 solution [23]. Soon after them in 2013 Abdolrahimi, Cattoën, Page and Yaghoobpour-Tari found the infinite-mass black hole solution [23] by a different numerical method (ACPY solution). This solution agrees well with the Figueras-Wiseman one [23] and thus adds further evidence for the existence of large RSII black holes, despite the doubts expressed by previous works.

4 Results

4.1 Figueras-Wiseman solution

To obtain the post-Newtonian parametrization of the Figueras-Wiseman solution the following field equations were used:

$$G_{\mu\nu} = 8\pi G_4 T_{\mu\nu}^{brane} + \epsilon^2 \left\{16\pi G_4 \langle T_{\mu\nu}^{CFT}[g]\rangle + a_{\mu\nu}[g] + \log\epsilon\, b_{\mu\nu}[g]\right\} + O(\epsilon^4 \log\epsilon), \tag{7}$$

where G_4 is the usual four-dimenstional graviational constant, $T_{\mu\nu}^{brane}$ is the stress-energy tensor of the matter localized on the brane, the additional stress-energy terms $\langle T_{\mu\nu}^{CFT}[g]\rangle$, $a_{\mu\nu}[g]$ and $b_{\mu\nu}[g]$ describe the influence of the extra dimension. They depend on metric only and are expressed via the Fefferman-Graham expansion [25, 26]; ϵ is a small perturbation parameter indicating the deviation of the brane position from the equilibrium state $z = 0$, z is the coordinate along the extra dimension.

The additional term in the post-Newtonian expansion of the Figueras-Wiseman solution calculated in this paper is

$$\delta h_{00}^{\mathrm{FW}} = \frac{121}{27}\frac{\epsilon^2}{\ell^2}\frac{M^2}{r^2}, \tag{8}$$

where M is the mass of the central object. In the considered case it equals the solar mass.

The obtained value (8) lies within the 1PN limit (1) and points at a potentially observable effect. In Randall-Sundrum model gravity is allowed propagate into the bulk along the extra dimension therefore the effect described by (8) most likely leads to the negative nonlinearity in gravitational superposition. In other words the resulting gravitational field produced by two or more massive objects can be less than the direct vector sum of their contributions. The parameterized post-Newtonian (PPN) parameter β is responsible for such an effect [5, 6, 27]. Therefore the result (8) should be expressed as follows:

$$\beta = 1 - \frac{\epsilon^2}{\ell^2}\frac{121}{108}M^2. \tag{9}$$

The constraint on the PPN parameter β obtained from analysis of the lunar laser ranging data [27] is $|\beta - 1| \leq 1.1 \times 10^{-4}$. The admitted region of the AdS length is limited by the results of the Newton's law test $l < 10^{-5}$ m [29]. Therefore the upper limit on the value of ϵ is:

$$\epsilon \leq 5.7 \times 10^{-47} \ll m_{Pl}. \tag{10}$$

Originally the parameter ϵ was assumed to be negligibly small and the vanishing value found in (10) implies that in fact $\epsilon = 0$. Thus the Figueras-Wiseman four-dimensional black hole solution is not only self-consistent but well consistent with the solar system constraints as well. Therefore this solution is indistinguishable form GR in the 1PN limit after all.

4.2 ACPY solution

The ACPY solution [24] is asymptotically conformal to the Schwarzschild metric and includes a negative five-dimensional cosmological constant Λ_5:

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2M}{r}\right) dt^2 + \left[1 - \frac{1}{-\Lambda_5 r^2} \frac{r - 2M}{r - 1.5M}\left(F(r) - r\frac{dF(r)}{dr}\right)\right]\left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \\ & + \left[r^2 + \frac{F(r)}{-\Lambda_5}\right] d\Omega^2 \\ F(r) = & \, 1 - 1.1241\left(\frac{2M}{r}\right) + 1.956\left(\frac{2M}{r}\right)^2 - 9.961\left(\frac{2M}{r}\right)^3 + \ \ldots \ + 2.900\left(\frac{2M}{r}\right)^{11}. \end{aligned} \tag{11}$$

The function $F(r)$ describes the perturbation caused by the bulk. The best fit for it was obtained in [24].

The field equations induced on the brane were derived by Sasaki, Shiromizu and Maeda [30]:

$$G_{\mu\nu} = -\,\Lambda_4 g_{\mu\nu} + \frac{8\pi}{M_{\text{Pl4}}^2} T_{\mu\nu} + \frac{8\pi}{M_{\text{Pl5}}^3} S_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{12}$$

where Λ_4 is usual four-dimensional cosmological constant, $g_{\mu\nu}$ is the metric on the brane, $T_{\mu\nu}$ is the stress-energy tensor of the matter localized on the brane, $S_{\mu\nu}$ is the local quadratic stress-energy correction, $\mathcal{E}_{\mu\nu}$ is the four-dimensional projection of the five-dimensional Weyl tensor. M_{Pl4} is usual four-dimensional Planck mass and M_{Pl5} is the fundamental five-dimensional Planck mass which is typically much less than the effective Planck mass on the brane.

The induced metric on the brane is asymptotically flat, the bulk is an anti-de-Sitter space-time as in the original Randall-Sundrum scenario [10], then $\mathcal{E}_{\mu\nu} = 0$ [31]. Therefore the correction term due to the contribution from ACPY topology (11) that follows from (12) has the form

$$\delta h_{00}^{\text{AP}} = \frac{l^2 M^2}{96} \frac{1}{r^4} + \mathcal{O}(r^{-5}). \tag{13}$$

According to (1) the expansion term of the 1PN-order should be proportional to r^{-2}. The correction (13) contains the next perturbation order which lies beyond 1PN. Therefore the obtained contribution (13) cannot be observed in the solar system experiments as well. This conclusion on the Randall-Sundrum model predictions confirms the result for the Figueras-Wiseman solution.

5 Conclusions

Consideration of the post-Newtonian expansion of the Figueras-Wiseman solution [23] reveals such possible effect as a negative nonlinearity of gravitational superposition (9). It naturally results from the theory itself because gravity is allowed to propagate to the extra dimension in Randall-Sundrum model. However the breaking of gravitational superposition turns out to depend on a negligibly small parameter (10) thus the predictions of the Figueras-Wiseman solution fully agrees with GR and the present observations. This effect may influence the strong field regime (close binary systems, black holes) as a consequence of curvature growth. So the next step could be the search of such features of the Randall-Sundrum model in the strong field limit. Fortunately this investigation is admissible as the large stable black hole solutions for RSII black holes have been found [23, 24].

The consideration of the other black hole solution by Abdolrahimi, Page et al. [24] shows that the terms describing the bulk influence (13) greatly exceed the limits of the post-Newtonian approximation. As a result both recent large Randall-Sundrum black holes solutions seems to be well consistent with GR at the solar system scale. For obtaining the limits on its parameters other tests are needed.

References

[1] K. S. Thorn, C. M. Will, *Astrophys. J.* **163**, 595 (1971).

[2] C. M. Will, *Astrophys. J.* **163**, 611 (1971).

[3] C. M. Will, K. Jr. Nordtvedt, *Astrophys. J.* **177**, 757 (1972).

[4] K. Jr. Nordtvedt, C. M. Will, *Astrophys. J.* **177**, 775 (1972).

[5] C. M. Will, *Theory and experiment in gravitational physics* (Cambridge University Press, Cambridge, 1981).

[6] C. M. Will, *Living Rev. Relativity* **17**, 4 (2014).

[7] T. Damour, G. Exposito-Farèse, *Phys. Rev.* **D 53**, 5541 (1996).

[8] T. Damour, G. Exposito-Farèse, *Phys. Rev.* **D 54**, 1474 (1996).

[9] L. Randall, R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999).

[10] L. Randall, R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999).

[11] A. Chamblin, S. W. Hawking, H. S. Reall, *Phys. Rev.* **D61**, 065007 (2000).

[12] N. Dadhich, R. Maartens, P. Papadopoulos, V. Rezania, *Phys. Rev. Lett.* **B487**, 1 (2000).

[13] T. Tanaka, *Prog. Theor. Phys. Suppl.* **148**, 307 (2002).

[14] R. Emparan, A. Fabbri, N. Kaloper, *J. High Energy Phys.* **08**, 043 (2002).

[15] R. Emparan, J. Garcia-Bellido, N. Kaloper, *JHEP* **01**, 079 (2003).

[16] T. Wiseman, *Phys. Rev.* **D65**, 124007 (2003).

[17] T. Wiseman, *Class. Quant. Grav.* **20**, 1137 (2003).

[18] H. Kudoh, T. Tanaka, T. Nakamura, *Phys. Rev.* **D68**, 024035 (2003).

[19] H. Kudoh, *Prog. Theor. Phys.* **110**, 1059 (2003).

[20] H. Kudoh, *Phys. Rev.* **D69**, 104019 (2004).

[21] H. Yoshino, *J. High Energy Phys.* **01**, 068 (2009).

[22] B. Kleihaus, J. Kunz, E. Radu, D. Senkbeil, *Phys. Rev.* **D83**, 104050 (2011).

[23] P. Figueras, T. Wiseman, *Phys. Rev. Lett.* **107**, 081101 (2011).

[24] S. Abdolrahimi, C. Cattoën, D. N. Page, S. Yaghoobpour-Tari, *JCAP* **06**, 039 (2013).

[25] S. de Haro, K. Skenderis, S. N. Solodukhin, *Commun. Math. Phys.* **217**, 595 (2001).

[26] S. de Haro, K. Skenderis, S. N. Solodukhin, *Class. Quant. Grav.* **18**, 3171 (2001).

[27] S. G. Turyshev, *Phys. Usp.* **52**, 1 (2009).

[28] J. G. Williams, S. G. Turyshev, D. H. Boggs, *Phys. Rev. Lett.* **93**, 261101 (2004).

[29] D. J. Kapner, T. S. Cook, E. G. Adelberger, J. H. Gundlach, B. R. Heckel, C. D. Hoyle, H. E. Swanson, *Phys. Rev. Lett.* **98**, 021101 (2007).

[30] M. Sasaki, T. Shiromizu and K. I. Maeda, *Phys. Rev.* **D62**, 024008 (2000).

[31] R. Maartens, *Phys. Rev.* **D62** 084023 (2000).