Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology

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Abstract

We investigate exponential solutions (i.e. the solutions with the scale factors change exponentially over time so that the comoving volume remains the same) in the Einstein-Gauss-Bonnet gravity. We obtain all possible exact vacuum and Λ-term cosmological solution in (4+1) and (5+1) dimensions, generalize some of these solutions to the arbitrary number of spatial dimensions and obtain necessary conditions for exponential solutions to exist in the presence of matter in the form of perfect fluid.

1 Introduction

Exact solutions play important role in any gravitational theory, especially nonlinear. The striking example of the nonlinear theory of gravity is the Lovelock gravity [1] – the most general metric theory of gravity with second order equations of motion (in contrast, for instance, to $f(R)$ gravity which gives fourth order dynamical equations). The Lovelock gravity is a natural generalization of Einstein's General Relativity: it is known [2, 3, 4] that the Einstein tensor is, in any dimension, the only symmetric and divergenceless tensor depending only on the metric and its first and second derivatives with a linear dependence on second derivatives; if one drops the condition of linear dependence on second derivatives, one can obtain the most general tensor which satisfies other mentioned conditions – the Lovelock tensor.

The Lovelock gravity has been intensively studied in the cosmological context (see, e.g., [5, 6, 7, 8, 9, 10, 11]). Many interesting results have been obtained for flat anisotropic metrics due to the fact that its cosmological dynamics is much richer in the Lovelock gravity than in the Einstein one. Since the resulting equations of motion turn out to be complicated enough, researchers usually study some special kind of metric – for example, metrics with with power-law and exponential time dependence of scale factors. The first of them is a generalization of Kasner solution [16, 17]; the latter can be considered as anisotropic generalization of the de Sitter (exponential) expansion. Our study is devoted to the exponential solutions in the Einstein-Gauss-Bonnet gravity.

2 The set-up.

The action under consideration is[1]:

$$S = \frac{1}{2\kappa^2}\int d^{D+1}x\sqrt{-g}\left(R + \alpha\left[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right] + \mathcal{L}_m\right), \qquad (1)$$

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[1] Greek indices run from 0 to D, while Latin one from 1 to D unless otherwise stated.

where D is the number of spatial dimensions, κ^2 is the $(D+1)$-dimensional gravitational constant, $R, R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}$ are the $(D+1)$-dimensional scalar curvature, Ricci tensor and Riemann tensor respectively, α is the coupling constant, $\mathcal{L}_m$ is the Lagrangian of a matter; we consider a perfect fluid with the equation of state $p = \omega\rho$ as a matter source. The action (1) gives the gravitational equations as

$$G_{\mu\nu} + \alpha H_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{2}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad T_{\mu\nu} = -2\,\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m \tag{3}$$

$$H_{\mu\nu} = 2\left(RR_{\mu\nu} - 2R_{\mu\alpha}R^{\alpha}_{\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_{\mu}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}\right) \tag{4}$$

The spacetime metric is

$$ds^2 = -dt^2 + \sum_{k=1}^{N} \mathrm{e}^{2H_k t} dx_k^2, \quad H_k \equiv \text{const} \tag{5}$$

It is easy to show that

$$R^{0i}_{0i} = H_i^2, \quad R^{j_1 j_2}_{j_1 j_2} = H_{j_1}H_{j_2}, \;\; j_1 < j_2, \quad R^{\alpha\beta}_{\mu\nu} = 0, \;\; (\alpha,\beta) \neq (\mu,\nu), \tag{6}$$

the dot denotes derivative w.r.t. t. So, arbitrary component of the Riemann tensor takes the form:

$$R^{\mu\nu}_{\lambda\sigma} = \left\{ \sum_k H_k^2 \delta_0^{[\mu}\delta_k^{\nu]}\delta^0_{[\lambda}\delta^k_{\sigma]} + \sum_{i<j} H_i H_j \delta_i^{[\mu}\delta_j^{\nu]}\delta^i_{[\lambda}\delta^j_{\sigma]} \right\}, \tag{7}$$

square brackets denote the antisymmetric part on the indicated indices. In view of (3)-(7) equations (2) take the form

$$\sum_{i\neq j} H_i^2 + \sum_{\{i>k\}\neq j} H_i H_k + 4\alpha \sum_{i\neq j} H_i^2 \sum_{\{k>l\}\neq\{i,j\}} H_k H_l + 12\alpha \sum_{\{i>k>l>m\}\neq j} H_i H_k H_l H_m = -\omega\varkappa \tag{8}$$

$$\sum_{i>j} H_i H_j + 12\alpha \sum_{i>j>k>l} H_i H_j H_k H_l = \varkappa, \quad \varkappa = \frac{\kappa^2\rho}{2} \tag{9}$$

Continuity equation reads:

$$\dot\rho + (\rho + p)\sum_i H_i = 0 \tag{10}$$

We see that dynamical equations is the system of pure algebraic equations for D Hubble parameters. Left hand sides of these equations does not depend on time, therefore energy density of matter does not depend on time too and continuity equation reduces to the following form:

$$(\rho + p)\sum_i H_i = 0 \iff \left[\begin{array}{rl} \rho = 0 & (\mathbf{a}) \\ p = -\rho & (\mathbf{b}) \\ \sum_k H_k = 0 & (\mathbf{c}) \end{array}\right. \tag{11}$$

This fact impose a number of severe restrictions: exponential solutions exist only if at least one of the conditions (**a**)-(**c**) is satisfied. According to these restrictions all possible exponential solutions can be divided into two large groups: solutions with constant volume ($\sum_k H_k = 0$) and solutions with volume changing in time ($\sum_k H_k \neq 0$); for the latter case we have only two possibilities: vacuum case (**a**) and Λ-term case (**b**); on the contrary, the first case does not impose constraints on choice of matter *a priori*.

3 Solutions with volume changing in time ($\sum_k H_k \neq 0$).

Subtracting i-th dynamical equation from j-th one we obtain:

$$(H_j - H_i)\left(\frac{1}{4\alpha} + \sum_{\{k>l\}\neq\{i,j\}} H_k H_l\right)\sum_k H_k = 0 \tag{12}$$

It follows from (12) that

$$H_i = H_j \quad \vee \quad \sum_{\{k>l\}\neq\{i,j\}} H_k H_l = -\frac{1}{4\alpha} \quad \vee \quad \sum_k H_k = 0 \tag{13}$$

These are necessary conditions for a given set $H_1, \ldots, H_D$ to be a solution of Eqs. (8)–(9). In this section we deal with the following possibilities only:

$$H_i = H_j \quad \vee \quad \sum_{\{k>l\}\neq\{i,j\}} H_k H_l = -\frac{1}{4\alpha} \tag{14}$$

These conditions lead to the fact that solutions with volume changing in time exist only when set of Hubble parameters is divided into subsets with equal values of Hubble parameters belonging to the same subset (in other words, existence of isotropic subspaces is required).

Let us introduce the following notations for the sets of Hubble parameters representing solutions with isotropic subspaces:

$$\begin{gathered}\mathcal{H}_0^{(4)} \equiv \{H,H,H,H\},\ \mathcal{H}_1^{(4)} \equiv \{H,H,H,h\},\ \mathcal{H}_2^{(4)} \equiv \{H,H,h,h\} \\ \mathcal{H}_0^{(5)} \equiv \{H,H,H,H,H\},\ \mathcal{H}_1^{(5)} \equiv \{H,H,H,H,h\},\ \mathcal{H}_{1*}^{(5)} \equiv \{H,H,-H,-H,h\},\end{gathered} \tag{15}$$

As it was mentioned above, there exist solutions with volume changing in time of two types: vacuum solutions and Λ-term solutions (and there are no another varying volume solutions).

1. Vacuum solutions:
   - $D = 4$: only isotropic solution with $H^2 = -\frac{1}{2\alpha}$, $\alpha < 0$ exists.
   - $D = 5$: solutions of two types exist:
     (a) isotropic solution: $H^2 = -\frac{1}{6\alpha}$, $\alpha < 0$;
     (b) solution with (3+2) spatial splitting: $H_1 = H_2 = H_3 \equiv H$, $H_4 = H_5 \equiv \xi H$ and

$$H^2 = \left.\frac{f(\xi)}{4\alpha}\right|_{\xi=\frac{\sqrt[3]{10}}{3}-\frac{\sqrt[3]{100}}{6}-\frac{2}{3}\approx-0.722}, \quad f(\xi) = -\frac{\xi^2+6\xi+3}{3\xi(3\xi+2)} \tag{16}$$

2. Λ-term solutions:
   - solution with $(3+2)$ splitting ($H_1 = H_2 = H_3 \equiv H$, $H_4 = H_5 \equiv h$) obeys the following equations:

$$192\,H^6\alpha^3 - 112\,H^4\alpha^2 + (256\,\Lambda\,\pi\,\alpha + 4)\,H^2\alpha - 1 = 0, \quad h = -\frac{4\,H^2\alpha+1}{8H\alpha} \tag{17}$$

     It is easy to check that when $\alpha > 0$ equations (17) has at least one solution for any Λ; when $\alpha < 0$ equations (17) has at least one solution iff $\Lambda > -\frac{5}{48\pi\alpha}$.
   - All the rest Λ-term solutions for $D = 4, 5$ are indicated in the tables 1,2,3.

Table 1: Λ-term solutions, $\alpha > 0$.

	$0 < \Lambda \leqslant \frac{1}{16\pi\alpha}$	$\Lambda > \frac{1}{16\pi\alpha}$
$\mathcal{H}_2^{(4)}$	No	$H^2 = \frac{1}{4}\left(8\pi\Lambda + \frac{1}{2\alpha} \pm \sqrt{\left(8\pi\Lambda + \frac{1}{2\alpha}\right)^2 - \frac{1}{\alpha^2}}\right), h = -\frac{1}{4\alpha H}$
$\mathcal{H}_1^{(4)}$	No	
$\mathcal{H}_0^{(4)}$	$H^2 = \frac{1}{4\alpha}\left(-1 + \sqrt{1 + \frac{16\pi\alpha\Lambda}{3}}\right)$	

Table 2: Λ-term solutions, $\alpha < 0$.

	$\Lambda < -\frac{3}{16\pi\alpha}$	$\Lambda = -\frac{3}{16\pi\alpha}$	$\Lambda > -\frac{3}{16\pi\alpha}$
$\mathcal{H}_2^{(4)}$	No	$H^2 = h^2 = -\frac{1}{4\alpha}$	$H^2 = \frac{1}{4}\left(8\pi\Lambda + \frac{1}{2\alpha}\right) \pm$ $\pm\frac{1}{4}\sqrt{\left(8\pi\Lambda + \frac{1}{2\alpha}\right)^2 - \frac{1}{\alpha^2}},$ $h = -\frac{1}{4\alpha H}$
$\mathcal{H}_1^{(4)}$	No	$H^2 = -\frac{1}{4\alpha},$ $h \in \mathbb{R}$	No
$\mathcal{H}_0^{(4)}$	$H^2 = \frac{1}{4\alpha}\left(-1 \pm \sqrt{1 + \frac{16\pi\alpha\Lambda}{3}}\right)$ solutions with positive square root exist for $\Lambda < 0$ only	$H^2 = -\frac{1}{4\alpha}$	No

Table 3: Λ-term solutions.

	$\alpha > 0$	$\alpha < 0$
$\mathcal{H}_0^{(5)}$	$H^2 = \frac{1}{12\alpha}\left(-1 + \sqrt{1 + \frac{48\pi\alpha\Lambda}{5}}\right), \ \Lambda > 0$	$H^2 = \frac{1}{12\alpha}\left(-1 - \sqrt{1 + \frac{48\pi\alpha\Lambda}{5}}\right), \ \Lambda < -\frac{5}{48\pi\alpha}$ $H^2 = \frac{1}{12\alpha}\left(-1 + \sqrt{1 + \frac{48\pi\alpha\Lambda}{5}}\right), \ \Lambda < 0$
$\mathcal{H}_1^{(5)}$	No	$H^2 = -\frac{1}{12\alpha}, \ h \in \mathbb{R}, \ \Lambda = -\frac{5}{48\pi\alpha}$
$\mathcal{H}_{1^*}^{(5)}$	$H^2 = \frac{1}{4\alpha}, \ h \in \mathbb{R}, \ \Lambda = \frac{1}{16\pi\alpha}$	No

4 Constant volume solutions ($\sum_k H_k = 0$).

In general case of constant volume solution we do not expect any additional relations between Hubble parameters (in contrast to the varying volume case, where only space-times with isotropic subspaces are possible). The full set of solution is rather cumbersome to be written down explicitly, so we restrict ourselves by finding conditions of its existence.

With $\sum_i H_i = 0$ Eqs. (8)-(9) take the form

$$\sum_i H_i^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\varkappa}{2}\right), \quad \sum_i H_i^4 = \frac{1}{2}\left[9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2 + \frac{\omega - 1}{\alpha}\left(\frac{\varkappa}{2}\right)\right] \tag{18}$$

Obviously, for the system (18) to have nontrivial solutions it is necessary that

$$\omega - \frac{1}{3} < 0, \quad \left(\omega - \frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2 + \frac{\omega - 1}{\alpha}\left(\frac{\varkappa}{2}\right) > 0 \tag{19}$$

We see, first of all, that the equation of state parameter w is restricted from above: $w < 1/3$. However, positivity of quadratic and quartic sums is not sufficient for the solution to exist.

Going further we denote:

$$\xi_1 = \frac{\varkappa}{2}, \quad \xi_2 = \frac{1}{\alpha}\left(\frac{\varkappa}{2}\right), \quad \xi = \frac{|\xi_2|}{\xi_1^2} \tag{20}$$

$$a = \xi_1(1-3\omega), \quad r^2 = \frac{1}{2}\left[\xi_1^2(1-3\omega)^2 + \xi_2(\omega-1)\right], \quad \eta_k = H_k^2 \tag{21}$$

Then equations (18) take the form:

$$\sum_i \eta_i = a, \quad \sum_i \eta_i^2 = r^2 \tag{22}$$

Variables $\eta_1, \ldots, \eta_N$ can be considered as Cartesian coordinates in N-dimensional Euclidean space; then the first of the equations (22) specifies $(N-1)$-dimensional hyperplane which intersects each axis of the coordinate system at the point a, the second of the equations (22) describes $(N-1)$-dimensional hypersphere of radius r centred at the origin. Since $a > 0$ and all $\eta_i > 0$ we deal with fragments of the hypersphere and the hyperplane located in the first orthant. These fragments are intersected iff $r \leqslant a \leqslant \sqrt{N}r$.

So, system (18) has nontrivial solutions iff $\frac{r^2}{a^2} \in \left[\frac{1}{N}; 1\right]$. We are concerned in such solutions of the system (18) that satisfy the condition $\sum_i H_i = 0$. It turns out that there is essential difference between even- and odd-dimensional cases. Indeed, let us consider $(4+1)$-dimensional spacetime; Eqs. (22) describe 4-plane and 4-sphere; in the point of contact of these surfaces we have $H_1^2 = H_2^2 = H_3^2 = H_4^2$, therefore, one can choose $H_1, \ldots, H_4$ such that $H_1 = H_2 = -H_3 = -H_4$ and the condition $H_1 + \ldots + H_4 = 0$ is satisfied automatically. Clearly, there is no way one can satisfy the condition $H_1 + \ldots + H_5 = 0$ in the point of tangency of 5-plane and 5-sphere because of one extra positive (or negative) summand. This results can be generalized to the case of arbitrary dimension: for an even-dimensional spacetime there exist solutions of the equations $\sum\limits_i H_i^2 = a, \quad \sum\limits_i H_i^4 = r^2$ such that $\sum_i H_i = 0$ in the vicinity of the point of contact hyperplane and hypersphere specified by Eqs. (22); for an odd-dimensional spacetime there are no solutions in the vicinity of the aforementioned point of contact. In general, there exist a subset $I \subseteq \left[\frac{1}{N}; 1\right]$ such that

$$\sum_i H_i = 0, \quad \sum_i H_i^2 = a, \quad \sum_{i=2}^N H_i^4 = r^2 \quad \text{for all} \quad \frac{r^2}{a^2} \in I \tag{23}$$

We express one of the Hubble parameters from the first of Eqs. (23) and substitute it in the remaining equations:

$$\left(\sum_{i=2}^N H_i\right)^2 + \sum_{i=2}^N H_i^2 = a, \quad \left(\sum_{i=2}^N H_i\right)^4 + \sum_i H_i^4 = r^2 \tag{24}$$

Hubble parameters can be considered here as a Cartesian coordinates; after reducing the quadratic form $\left(\sum\limits_{i=2}^N H_i\right)^2 + \sum\limits_{i=2}^N H_i^2$ to the canonical form by a coordinate transformation and converting a Cartesian coordinates to spherical $(\varrho, \theta_1, \ldots, \theta_{N-2})$ Eqs. (24) take the form correspondingly:

$$\varrho^2 = a, \quad \varrho^4 f(\theta_1, \ldots, \theta_{N-2}) = r^2 \tag{25}$$

where f is a polynomial in $\sin(\theta_k), \cos(\theta_k)$ for $k = \overline{1, N-2}$. Substituting $\varrho^2 = a$ into the second of Eqs. (25) we obtain

$$F(\theta_1, \ldots, \theta_{N-2}, r, a) \equiv f(\theta_1, \ldots, \theta_{N-2}) - \frac{r^2}{a^2} = 0 \tag{26}$$

For example, for $N=4$ we have:

$$\begin{array}{c}F(\theta_1,\theta_2,r,a)=\frac{1}{4}\sin^4(\theta_1)\left(\frac{1}{2}+\frac{11}{6}\cos^4(2\theta_2)\right)+\frac{1}{2}\left(\cos^2(\theta_1)+\sin^2(\theta_1)\sin^2(2\theta_2)\right)^2+\\+\frac{1}{8}\sin^2(2\theta_1)\cos^2(2\,\theta_2)+\frac{1}{3\sqrt{2}}\sin(2\theta_1)\cos(2\theta_2)\left(\cos^2(\theta_1)-3\sin^2(\theta_1)\sin^2(2\,\theta_2)\right)-\frac{r^2}{a^2}\end{array} \tag{27}$$

So, the problem of the existence of solutions of Eqs. (23) is reduced to the problem of the existence of zeros of function F. We solve this problem numerically. Numerical calculations performed for $N\in[4,\ldots,8]$ shows that functions F has zeros for $\frac{r^2}{a^2}\in[\sigma_+;\sigma_-]$, i.e $I=[\sigma_+;\sigma_-]$, where $\frac{1}{N}\leqslant\sigma_+<\frac{1}{2}<\sigma_-<1$. Using this fact we get:

$$\sigma_+\leqslant\frac{r^2}{a^2}\leqslant\sigma_-\iff\left\{\begin{array}{l}(2\sigma_+-1)\xi_1^2(1-3\omega)^2\leqslant\xi_2(\omega-1)\\(2\sigma_--1)\xi_1^2(1-3\omega)^2\geqslant\xi_2(\omega-1)\end{array}\right. \tag{28}$$

There are two cases, depending on the sign of the parameter α:

I. $\alpha>0$.

$$\left\{\begin{array}{l}(2\sigma_+-1)\xi_1^2(1-3\omega)^2\leqslant\xi_2(\omega-1)\\(2\sigma_--1)\xi_1^2(1-3\omega)^2\geqslant\xi_2(\omega-1)\end{array}\right.\iff\omega\leqslant\frac{1}{3}-\frac{\xi_++\sqrt{\xi_+(\xi_++24)}}{18},\ \xi_+=\frac{\xi}{1-2\sigma_+} \tag{29}$$

II. $\alpha<0$.

$$\left\{\begin{array}{l}(2\sigma_+-1)\xi_1^2(1-3\omega)^2\leqslant\xi_2(\omega-1)\\(2\sigma_--1)\xi_1^2(1-3\omega)^2\geqslant\xi_2(\omega-1)\end{array}\right.\iff\omega\leqslant\frac{1}{3}-\frac{\xi_-+\sqrt{\xi_-(\xi_-+24)}}{18},\ \xi_-=\frac{\xi}{2\sigma_--1} \tag{30}$$

Finally we obtain:

$$\left\{\begin{array}{l}\sum\limits_i H_i=0\\\sum\limits_i H_i^2=-3\left(\omega-\frac{1}{3}\right)\left(\frac{\varkappa}{2}\right)\\\sum\limits_i H_i^4=\frac{1}{2}\left[9\left(\omega-\frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2+\frac{\omega-1}{\alpha}\left(\frac{\varkappa}{2}\right)\right]\end{array}\right.\iff\omega\leqslant\left\{\begin{array}{l}\frac{1}{3}-\frac{\xi_++\sqrt{\xi_+(\xi_++24)}}{18},\ \alpha>0\\\frac{1}{3}-\frac{\xi_-+\sqrt{\xi_-(\xi_-+24)}}{18},\ \alpha<0\end{array}\right. \tag{31}$$

Inequalities (31) can be rewritten in terms of the energy density ρ:

$$\rho\geqslant\rho_{\lim}(\omega),\quad\rho_{\lim}(\omega)=\left\{\begin{array}{l}\frac{1}{36\pi\alpha}\frac{1}{2\sigma_+-1}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2},\ \alpha>0\\\frac{1}{36\pi\alpha}\frac{1}{2\sigma_--1}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2},\ \alpha<0\end{array}\right. \tag{32}$$

We see that the above mentioned nonexistence of vacuum solutions has a sharper form: for any ω there exists a low limit for ρ. In the particular case of cosmological constant $\omega=-1$:

$$\rho_{\lim}(-1)=\left\{\begin{array}{l}-\frac{1}{32\pi\alpha}\frac{1}{2\sigma_+-1},\ \alpha>0\\-\frac{1}{32\pi\alpha}\frac{1}{2\sigma_--1},\ \alpha<0\end{array}\right. \tag{33}$$

Since the function $\rho_{\lim}(\omega)$ is growing, for any non-fantom ($\omega\geqslant-1$) matter we have $\rho\geqslant\rho_{\lim}(-1)$.

For $N=4$ numerical calculations give $\sigma_-=0.76\pm0.01$. So, for (4+1)-dimensional spacetime we have:

$$\omega<\left\{\begin{array}{l}\frac{1}{3}-\frac{2\xi+\sqrt{2\xi+(2\xi+24)}}{18},\ \alpha>0\\\frac{1}{3}-\frac{1.92\xi+\sqrt{1.92\xi(1.92\xi+24)}}{18},\ \alpha<0\end{array}\right.,\quad\text{or}\quad\rho\gtrsim\left\{\begin{array}{l}-\frac{1}{18\pi\alpha}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2},\ \alpha>0\\-\frac{1}{1.04}\frac{1}{18\pi\alpha}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2},\ \alpha<0\end{array}\right. \tag{34}$$

where $\xi=\frac{1}{4\pi|\alpha|\rho}$.

For $N = 5$ numerical calculations give $\sigma_+ = 0.23 \pm 0.01$, $\sigma_- = 0.65 \pm 0.01$. So, for $(5+1)$-dimensional spacetime we have:

$$\omega < \begin{cases} \frac{1}{3} - \frac{1.85\xi + \sqrt{1.85\xi + (1.85\xi + 24)}}{18}, & \alpha > 0 \\ \frac{1}{3} - \frac{3.33\xi + \sqrt{3.33\xi(3.33\xi + 24)}}{18}, & \alpha < 0 \end{cases}, \quad \text{or} \quad \rho \gtrsim \begin{cases} \frac{1}{0.54}\frac{1}{36\pi\alpha}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, & \alpha > 0 \\ -\frac{1}{0.3}\frac{1}{36\pi\alpha}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, & \alpha < 0 \end{cases} \tag{35}$$

where $\xi = \frac{1}{4\pi|\alpha|\rho}$.

5 Constant volume solutions with two different Hubble parameters.

1. $([N-1]+1)$-decomposition: $H_1 = \ldots = H_{N-1} \equiv H \in \mathbb{R}$, $H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -(N-1)H$. Substituting these $H_1, \ldots, H_N$ into Eqs. (18) we obtain:

$$\begin{cases} N(N-1)H^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\varkappa}{2}\right) \\ 2N(N-1)(N^2-3N+3)H^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\varkappa}{2}\right) \end{cases} \tag{36}$$

Solution of Eqs. (36) for H^2 and ρ:

$$H^2 = -\frac{1}{3\alpha(N-2)(N-3)}\frac{\omega-1}{\omega-\frac{1}{3}}, \; \rho = \frac{1}{36\pi\alpha}\frac{N(N-1)}{(N-2)(N-3)}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \; \omega < \frac{1}{3}, \; \alpha < 0 \tag{37}$$

2. $\left(\frac{N}{2}+\frac{N}{2}\right)$-decomposition, N is even: $H_1 = \ldots = H_{\frac{N}{2}} \equiv H \in \mathbb{R}$, $H_{\frac{N}{2}+1} = \ldots = H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -H$. Substituting these $H_1, \ldots, H_N$ into Eqs. (18) we obtain:

$$\begin{cases} NH^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\varkappa}{2}\right) \\ 2NH^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\varkappa}{2}\right) \end{cases} \tag{38}$$

Solution of Eqs. (38) for H^2 and ρ:

$$H^2 = \frac{1}{3\alpha(N-2)}\frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = -\frac{1}{36\pi\alpha}\frac{N}{N-2}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \quad \omega < \frac{1}{3}, \quad \alpha > 0 \tag{39}$$

3. $([n+1]+n)$-decomposition, $n \equiv \left\lfloor \frac{N}{2} \right\rfloor$, N is odd: $H_1 = \ldots = H_{n+1} \equiv H \in \mathbb{R}$, $H_{n+2} = \ldots = H_N \equiv h \in \mathbb{R}$. It follows from the condition $\sum_i H_i = 0$ that $h = -(1+n^{-1})H$. Substituting these $H_1, \ldots, H_N$ into Eqs. (18) we obtain:

$$\begin{cases} N(1+n^{-1})H^2 = -3\left(\omega - \frac{1}{3}\right)\left(\frac{\varkappa}{2}\right) \\ 2N(1+n^{-1})(1+n^{-1}+n^{-2})H^4 = 9\left(\omega - \frac{1}{3}\right)^2\left(\frac{\varkappa}{2}\right)^2 + \frac{\omega-1}{\alpha}\left(\frac{\varkappa}{2}\right) \end{cases} \tag{40}$$

Taking into account $n = \frac{N-1}{2}$ we get solution of Eqs. (38) for H^2 and ρ:

$$H^2 = \frac{1}{3\alpha}\frac{(N-1)^2}{(N-3)(N^2+N+2)}\frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = -\frac{1}{36\pi\alpha}\frac{N(N-1)(N+1)}{(N-3)(N^2+N+2)}\frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \\ \omega < \frac{1}{3}, \quad \alpha > 0 \tag{41}$$

For $N = 4$ only cases 1 and 2 are realized: it is (3+1)-decomposition and (2+2)-decomposition; for $N = 5$ only cases 1 and 3 are realized: it is (4+1)-decomposition and (3+2)-decomposition and there are no other options for $N = 4, 5$. It is easy to check that for $N = 4, 5$ and $\omega = -1$

solutions (37),(39),(41) turn to solutions derived in our paper [22] with additionally imposed constant volume requirement $\sum_i H_i = 0$.

For $N = 5$ there is one more decomposition, containing 3 different Hubble parameters (see [22]): $H_1 = H_2 \equiv H$, $H_3 = H_4 \equiv -H$, $H_5 \equiv h$; but it follows from $\sum_i H_i = 0$ that $h = 0$ and it reduces to $(2+2)$-decomposition; for the same reasons decomposition $H_1 = \ldots = H_n \equiv H \in \mathbb{R}$, $H_{n+1} = \ldots = H_{2n} \equiv -H$, $H_N \equiv h \in \mathbb{R}$ $\left(n \equiv \left\lfloor \frac{N}{2} \right\rfloor, N \text{ is odd}\right)$ reduces to $(n+n)$-decomposition described above. So, all possible generalization of solutions in $(4+1)$ and $(5+1)$ dimensions found in [22] for $w = -1$ to an arbitrary w are presented in our list.

6 Conclusions

We can write down full classification of exponential solutions in (4+1) and (5+1) dimensions.

- Vacuum solution in a pure Gauss-Bonnet gravity [15]. We have shown that this solution is a particular one and can not be incorporated in other sets of solution of the type considered. It requires absence of both matter and Einstein-Hilbert term.
- Solutions with volume element changing in time. Such solutions require a matter only in the form of cosmological constant. Apart from an isotropic solution, it appears that these solutions exist only when set of Hubble parameters is divided into subsets with equal values of Hubble parameters belonging to the same subset (so, existence of isotropic subspaces is required).
- Solutions with constant volume element. Solutions of this type exist only when matter density exceeds (or equal to) some critical value which depends on the equation of state of the matter. The parameter ω of the matter should be smaller than 1/3. In general, solutions do not have isotropic subspaces, though can have them for special cases.

As space-times with isotropic subspaces represent a particular interest (for example, if multidimensional paradigm is indeed realized in Nature, then our own world belongs to this class and, therefore, the case with three-dimensional subspace is of special interest since it could represent our three-dimensional (spatially) Universe (see [23])) we write down here explicit solutions of constant volume element with isotropic subspaces, generalising those found in [22]. For a general case of constant volume element (without isotropic subspaces) we present the conditions for such solutions to exist.

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