Some notes on the phenomenological description of particle creation and its influence on the space-time metrics

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Abstract

The method is proposed for the phenomenological description of particle creation by external fields (in the presence of gravitational field or without it).

The great activity in investigation of particle creation in strong gravitational fields [1] revealed the importance of such processes both in cosmology and in black hole physics. It appeared that the most difficult problem is tat of taking into account the back reaction on the space-time metrics. And it is not only the influence of the created particles, what is rather easy to do, at least in principle, but also the contribution due to the vacuum polarization accompanying necessarily the creation processes (and being, in a sense, its cause). The main obstacle to do this self-consistently is that the construction of the quantum part of the specific model requires the knowledge of the boundary conditions which, in turn, can be formulated only after solving the corresponding Einstein equations with the right hand side (the energy-momentum tensor) with the properly averaged quantum entities. In some special cases when, by definition, the space-time possesses very high symmetry, such a procedure can be fulfilled, at least, in the one loop approximation. For instance, for homogeneous and isotropic cosmological models the quantum normalization demands the modification of the initial classical Einstein-Hilbert action by adding the term quadratic in the scalar curvature. This lead to the violation of the energy dominance - the necessary condition of the well known singularity theorems. The most famous example is the Starobinsky inflationary model [2].

Our idea is the following. The processes of particle creation are essentially nonlocal. But, if the external fields are strong enough, the separation between just created particles becomes of order of their Compton length, and we can safely approximate them by some condensed matter. Since in such an approach the nonlocal processes become, formally, the local ones, there is a hope that the local vacuum polarization ill be automatically incorporated into the formalism as well. The same concerns also the trace anomalies that play essential role in quantum processes of particle creation both in cosmology [3] and in the black hole thermodynamics [4]. One should be rather cautious when constructing the formalism, because it may appear controversial to use the conventional form of the energy-momentum tensor for created particles and just demanding their number non-conservation. The problem is that in deriving the hydrodynamical energy-momentum tensor, as how it is described in the textbooks, one starts from the action for a single particle and obtain the equation of motion by varying its world line, find the expression for the energy and momentum, and then consider the particle ensemble and take the limit of continuous distribution. Therefore, by doing this, one make use of the Lagrangian coordinates for describing

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condensed matter and, implicitly, the conservation of particle number.It follows from this, that we need the more appropriate Euler coordinates from the very beginning, that is, already in the action integral. Such a formalism was developed by J.R.Ray [5], who demonstrated also that the equation of motion for the perfect fluid derived from the proposed action integral is just the famous Euler hydrodynamical equation. The advantage of the Ray's approach is that the particle conservation condition (the continuity equation) enters the action integral explicitly through the corresponding constraint with the Lagrange multiplier.

The first attempt to describe the particle creation phenomenologically was made by the author in 1987 [6]. The proposed recipe was very simple: instead of the continuity equation, considered as one of the constraints, just to equate the number of created particles in unit volume per unit time interval not to zero but to some function of the responsible for this process external fields. Among other things, it was shown that, indeed, it is possible to violate in this way the energy dominance condition. Here we will would like to continue that line of investigation, but making some improvements and obtaining more physically clear and transparent equations.

Let us start with the simplest model: construction of the constraint dynamics for the perfect fluid using the Euler variables.

The dynamical variables in this case are the number density $n(x)$, the four velocity vector of fluid's flow $u^\alpha(x)$ and some auxiliary field $X(x)$ for enumeration of the world-lines. The constraints are the normalization condition $u^\beta u_\beta = 1$, the continuity equation (particle number conservation) $(nu^\beta)_{;\beta} = 0$ and $X_{,\beta}u^\beta = 0 \rightarrow X(x) = const$ on every trajectory (here "comma" denotes the partial derivative, while "semicolon" - covariant derivative with respect to the space-time metrics $g_{\alpha\beta}$ and metric connections). The (invariant) energy density of the fluid equals

$$\varepsilon(n, X) = \mu(X)n + n\Pi(n)\,, \tag{1}$$

where $\Pi(n)$ is the potential energy describing the (self)interaction between the constituent particles, and $\mu(X)$ is their mass distribution. The pressure $p(n)$ is

$$p = n^2\frac{d\Pi}{dn} = -\varepsilon + n\frac{\partial\varepsilon}{\partial n}\,. \tag{2}$$

The action integral S can be written in the form ($\sqrt{-g}$ is the determinant of the metric tensor):

$$S = -\int \varepsilon(X, n)\sqrt{-g}dx + \int \lambda_0(x)(u^\beta u_\beta - 1)\sqrt{-g}dx + \int \lambda_1(x)(nu^\beta)_{;\beta}\sqrt{-g}dx + \int \lambda_2(x)X_{,\beta}u^\beta\sqrt{-g}dx\,. \tag{3}$$

Here $\lambda_0(x)$, $\lambda_1(x)$ and $\lambda_2(x)$ are the Lagrange multipliers. Variation of this action integral with respect to the dynamical variables and Lagrange multipliers gives us the following set of equations of motion and constraints:

$$\begin{aligned}
-\frac{\partial\varepsilon}{\partial n} &- \lambda_{,\beta}u^\beta = 0\\
2\lambda_0 u_\alpha &- n\lambda_{1,\alpha} + \lambda_2 X_{,\alpha} = 0\\
-\frac{\partial\varepsilon}{\partial X} &- (\lambda_2 u^\beta)_{;\beta} = 0\\
u^\beta u_\beta &= 1\\
(nu^\beta)_\beta &= 0\\
X_{,\beta}u^\beta &= 0
\end{aligned} \tag{4}$$

It is easy to show, by calculating a convolution of the second equation with the four-velocity vector and making use of the constraints, that $2\lambda_0 = -(\varepsilon + p)$. Also, it is not difficult, by using the integrability conditions ($\lambda_{1;\alpha\beta} = \lambda_{1;\beta\alpha}$ and $X_{;\alpha\beta} = X_{;\beta\alpha}$) and constraints, to obtain the hydrodynamical Euler equation.

$$((\varepsilon + p)u^\beta)_{;\beta}u_\alpha + (\varepsilon + p)u^\beta u_{\alpha;\beta} = p_{,\alpha} \tag{5}$$

Thus, the Lagrange multipliers are, effectively, decoupled, and it is become possible to solve first the equations of motion for dynamical variables and only then to find out the multipliers. In what follows we will also need the expression for the energy-momentum tensor, $T_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g^{\alpha\beta}}$ (L is the Lagrangian). For the hydrodynamical action, considered above, it reads

$$T_{\alpha\beta} = -2\lambda_0 u_\alpha u_\beta + g_{\alpha\beta}(\varepsilon - \lambda_0(u^\gamma u_\gamma - 1) + n\lambda_{,\gamma}u^\gamma - \lambda_2 X_{,\gamma}u^\gamma)\,. \tag{6}$$

By use of the equations of motion and constraints, it can be rewritten in the famous form,

$$T_{\alpha\beta} = (\varepsilon + p)u_\alpha u_\beta - p g_{\alpha\beta}\,. \tag{7}$$

It is noteworthy to say that the Euler equation is just the continuity equation for such a tensor, $T^{\beta}_{\alpha;\beta} = 0$.

The written above action integral is not unique, not only in its form but also in the choice of the independent dynamical variables. One can, for instance, to write the first term under the integral sign as $\varepsilon(n)u^\beta u_\beta$ which is numerically remains the same because of the first constraint (= normalization condition). This will change both the equations of motion and the primary form of the energy-momentum tensor, but eventually, will result only in a redefinition of the Lagrange multiplier λ_0. The only reason to write it in such a way is aesthetic one - just a reminiscence of the conventional derivation of the Lagrange equations for a single relativistic particle. With the invariant particle density n the situation is quite different. The problem is that the total number of particles N is something absolute, it does not depend on the metrics introduced on the given manifold, but the invariant number density does. Namely, it depends on the volume introduced together with the metrics and changes, say, with conformal transformation. So, it is more logical to use the scalar density $\hat{n} = n\sqrt{-g}$ instead of the invariant density n. Of course, when doing this one should still consider the invariant energy density ε as the same function, $\varepsilon(n) = \varepsilon(\hat{n}/\sqrt{-g})$ and take into account this new dependence on the metric tensor when calculating the energy-momentum tensor. It is not very difficult to check that the final results (both the Euler equations and the energy-momentum tensor in terms of the energy density and the pressure) will remain the same.

Now, let us start to generalize the scheme in order to include in it the particle creation processes. The simplest (and naive) way to do this is just to replace the continuity equation $(nu^\alpha)_{;\alpha} = 0$ by $(nu^\alpha)_{;\alpha} = \frac{1}{\sqrt{-g}}(\hat{n}u^\alpha)_{,\alpha} = \Phi$ (as was done in [6]), where Φ is some function of the invariants characterizing the field(s) that causes the particle creation. Here we consider two different cases: creation of particles that are quanta of some scalar field and creation of electron-positron pairs by an external electromagnetic field. We postpone the latter to the end of the paper and begin with the scalar field. It should be noted that in this case the creation is caused, actually, by the "external" gravitational field and not by the scalar field itself. The physics of the process is the following. The gravitational field gives rise to the vacuum polarization of the quantum vacuum fluctuations of the scalar field, and it is this polarization that "takes responsibility" for the particle creation. In our phenomenological approach these created scalar quanta are described by the classical energy density, particle number density and four-velocities, so, we don't need to introduce any classical scalar field in the scheme. Thus, the function Φ that describes the law of creation should depend on the gravitational invariants. Calculations made by many groups of scientists [1] show that this function is proportional (at least, in the one-loop approximation) to the square of the Weyl's tensor $\Phi = aC^2$, so, the total action for the matter fields takes the form

$$\begin{aligned} S = -\int \varepsilon(X, \hat{n}/\sqrt{-g})\sqrt{-g}dx \; &+ \int \lambda_0(x)(u^\beta u_\beta - 1)\sqrt{-g}dx + \int \lambda_1(x)((nu^\beta)_{;\beta} - \Phi)\sqrt{-g}dx \\ &+ \int \lambda_2(x)X_{,\beta}u^\beta\sqrt{-g}dx\,. \end{aligned} \tag{8}$$

The equations of motion is almost the same as before, one should only change $2\lambda_0 = -(\varepsilon + p)$ to $2\lambda_0 = \varepsilon - p$ and, replace the continuity equation for the number density by the law of particle creation. Of course, the Euler equation is now modified: there appears and additional term:

$$((\varepsilon + p)u^\beta)_{;\beta} u_\alpha + (\varepsilon + p)u^\beta u_{\alpha;\beta} + \lambda_{1,\alpha}\Phi = p_{,\alpha} \tag{9}$$

Note, that, by the equations of motion, $\lambda_{1,\alpha} = -\frac{\partial\varepsilon}{\partial n} u_\alpha + \lambda_2 X_{,\alpha}$. In the case of the coherent perfect fluid ($p = 0 =$ dust) this modified Euler equation becomes

$$\mu u^\beta u_{\alpha;\beta} + \frac{\Phi}{n}(\mu u_\alpha + \lambda_{1,\alpha}) = 0 \tag{10}$$

or

$$\mu u^\beta u_{\alpha;\beta} + \frac{\Phi}{n}\lambda_2 X_{,\alpha} = 0 \tag{11}$$

which in the absence of particle creation reduces to the geodesics equation as it should be. Surely, the expression for the energy-momentum tensor will also be modified. Here we write it in the implicit form:

$$T_{\alpha\beta} = (\varepsilon + p)u_\alpha u_\beta - p g_{\alpha\beta} + A_{\alpha\beta} + \lambda_1 \Phi g_{\alpha\beta}\,. \tag{12}$$

where $A_{\alpha\beta} = -2\frac{\delta(\lambda_1\Phi)}{\delta g^{\alpha\beta}}$. The continuity equation, $T^\beta_{\alpha;\beta} = 0$, after making use of the equation of motion, gives us the equation for the Lagrange multiplier λ_1,

$$A^\beta_{\alpha;\beta} + \lambda_1 \Phi_{,\alpha} = 0\,. \tag{13}$$

It is in the function λ_1, where the information about the vacuum polarization and trace anomaly is encoded. Thus, together with the modified Euler equation and the Einstein (or some other) gravitational equations this closes the set of dynamical equations.

Let us turn now to case of the external electromagnetic field. Particles are created in pairs, say, electrons and positrons. Separately, they cannot form the perfect fluid, instead - the electron-positron plasma in the external field. The problem is too complex. For the sake of simplicity (and brevity) we will consider all the particles as noninteracting directly with each other, i.e., the hydrodynamical pressure is absent, $p = 0$, and the energy density equals $\varepsilon = m_0 n$, where m_o is the electron mass, so, we assume that there are no other particles from the very beginning except the created ones. And, again, for the sake of simplicity we will consider in this paper only the case of the external electric field. Thus, the total action integral contains two (actually) identical hydrodynamical parts (we will distinguish them by the "tilde" sign), the conventional electromagnetic action, the part describing the particle's electromagnetic interaction and, at last, that one, responsible for the pair creation. Namely,

$$\begin{aligned} S_{tot} &= S_{hydro} + \tilde{S}_{hydro} + S_{em} + S_{int} + S_{cr} \\ S_{hydro}(\tilde{S}_{hydro}) &= -\int m_0 \hat{n} dx + \int \lambda_0 (u^\alpha u_\alpha - 1)\sqrt{-g}dx + \lambda_2 X_{,\alpha} u^\alpha \sqrt{-g} dx \\ S_{em} &= -\frac{1}{16\pi}\int F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}dx\,, \end{aligned} \tag{14}$$

where $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$ - the electromagnetic field tensor, and A_α - its vector-potential. To go further, we need to introduce the electric current four-vector. In our case of identical particles (antiparticles) it is simply $j^\alpha = enu^\alpha$ ($\tilde{j}^\alpha = -e\tilde{n}\tilde{u}^\alpha$), where e is the elementary electric charge. The action integral for their interaction with the electromagnetic field reads as follows

$$S_{int} = -\int A_\alpha(\hat{j}^\alpha + \hat{\tilde{j}}^\alpha)dx + \int \lambda_3 (j^\alpha + \tilde{j}^\alpha)_{;\alpha}\sqrt{-g}dx + \int \lambda_\alpha(\hat{j}^\alpha - e\hat{n}u^\alpha)dx + \int \tilde{\lambda}_\alpha(\hat{\tilde{j}}^\alpha + e\hat{\tilde{n}}\tilde{u}^\alpha)dx\,. \tag{15}$$

The definition of the electric current four-vector j^α with the corresponding vectorial Lagrange multiplier λ_α is added to the conventional $A_\alpha j^\alpha$-term for further convenience, while the continuity constraint $(j^\alpha + \tilde{j}^\alpha)_{;\alpha}$ is really necessary here, because due to the change in the set of dynamical variables (the four-velocity u^α instead of the world-line trajectory $x(\tau)$ in the conventional description) the gauge invariance is not automatically incorporated into the formalism. The last term in the total action integral, S_{cr}, is responsible for the particle creation,

$$S_{cr} = \int \lambda_1((nu^\alpha)_{;\alpha} - \Phi(L_{em}))\sqrt{-g}dx + \int \tilde{\lambda}_1((\tilde{n}\tilde{u}^\alpha)_{;\alpha} - \Phi(L_{em}))\sqrt{-g}dx\,. \tag{16}$$

where $L_{em} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}$. Now the dynamical variables are $\hat{n} = n\sqrt{-g}$, $\hat{\tilde{n}} = \tilde{n}\sqrt{-g}$, u^α, $\tilde{u}^\alpha$, X, $\tilde{X}$, A_α, $\hat{j}^\alpha = j^\alpha\sqrt{-g}$, $\hat{\tilde{j}}^\alpha = \tilde{j}^\alpha\sqrt{-g}$. By varying the total action integral with respect to these variables (except the vector-potential A_α) and Lagrange multipliers one gets the following set of equations of motion and constraints

$$\begin{aligned}
\hat{n} &: \quad -m_0u^\beta u_\beta - \lambda_{1,\beta}u^\beta - e\lambda_\beta u^\beta = 0\\
u^\alpha &: \quad -2m_0nu_\alpha + 2\lambda_0u_\alpha - n\lambda_{1,\alpha} + \lambda_2X_{,\alpha} - en\lambda_\alpha = 0\\
X &: \quad (\lambda_2u\beta)_{;\beta} = 0\\
\lambda_0 &: \quad u^\beta u_\beta = 1\\
\lambda_1 &: \quad (nu^\beta)_{;\beta} = \Phi(L_{em})\\
\lambda_2 &: \quad X_{,\beta}u^\beta = 0\\
j^\alpha &: \quad -A_\alpha - \lambda_{3,\alpha} + \lambda_\alpha = 0\\
\lambda_3 &: \quad (j^\beta + \tilde{j}^\beta)_{;\beta} = 0\\
\lambda_\alpha &: \quad j^\alpha = enu^\alpha
\end{aligned} \tag{17}$$

(for the "tilde" equations one should change $e \to -e$). In the same way as before we can easily find, that $2\lambda_0 = m_0n\,;\qquad 2\tilde{\lambda}_0 = m_0\tilde{n}$. Also, $\lambda_\alpha = \tilde{\lambda}_\alpha$. By varying the vector-potential A_α we obtain the following modified Maxwell equations

$$\left(\left(1 - (\lambda_1 + \tilde{\lambda}_1)\frac{\partial\Phi}{\partial L_{em}}\right)F^{\alpha\beta}\right)_{;\beta} = -4\pi\left(j^\alpha + \tilde{\jmath}^\alpha\right)\,. \tag{18}$$

This reminds the Maxwell equations inside a condensed matter with the "dielectric constant" $\left(1 - (\lambda_1 + \tilde{\lambda}_1)\frac{\partial\Phi}{\partial L_{em}}\right)$.

Making use of all the equations of motion as well as the integrability conditions and constraints, we derive the following expression for the modified Lorentz force:

$$\begin{aligned}
m_0u_{\alpha;\beta}u^\beta &= eF_{\alpha\beta}u^\beta - \frac{\lambda_2}{n}\Phi(L_{em})X_{,\alpha}\\
\tilde{m}_0\tilde{u}_{\alpha;\beta}\tilde{u}^\beta &= -eF_{\alpha\beta}\tilde{u}^\beta - \frac{\tilde{\lambda}_2}{\tilde{n}}\Phi(L_{em})\tilde{X}_{,\alpha}\,.
\end{aligned} \tag{19}$$

At last, let us find the energy-momentum tensor. After some calculations one gets the result

$$\begin{aligned}
T_{\alpha\beta} &= m_0nu_\alpha u_\beta + m_0\tilde{n}\tilde{u}_\alpha\tilde{u}_\beta\\
&- \frac{1}{4\pi}D_{\alpha\gamma}F^\gamma_\beta + \frac{1}{16\pi}g_{\alpha\beta}D_{\gamma\delta}F^{\gamma\delta}\\
&+ (\lambda_1 + \tilde{\lambda}_1)\left(\Phi - L_{em}\frac{\partial\Phi}{\partial L_{em}}\right)g_{\alpha\beta}
\end{aligned} \tag{20}$$

where

$$D_{\alpha\gamma} = \left(1 - (\lambda_1 + \tilde{\lambda}_1)\frac{\partial\Phi}{\partial L_{em}}\right)F_{\alpha\gamma} \tag{21}$$

It seems very interesting that in this case the continuity equation, $T^{\beta}_{\alpha;\beta} = 0$ is automatically satisfied despite of the presence of the Lagrangian multipliers which are not the dynamical variables.

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