Some general properties of U(1) gauged Q-balls

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Abstract

The main properties of (3+1)-dimensional $U(1)$ gauged Q-balls are discussed. A method for obtaining the charge and the energy of such gauged Q-balls using only the nongauged solution for the scalar field in the case, when the back-reaction of the gauge field on the scalar field is small, is presented, as well as the corresponding criteria of its applicability.

It is well known that, according to the well-known Hobart-Derrick theorem [1], there are no static solitons in the (3+1)-dimensional scalar field theory with a nonnegative scalar field potential. Apart from taking potentials which are not nonnegative (moreover, it is not difficult to show that all such solutions are classically unstable), there is a simple way to overcome the restriction put by the Hobart-Derrick theorem. Indeed, one can consider a theory of complex scalar field with global $U(1)$ symmetry, possessing a time-dependent solution of the form

$$\phi(t,\vec{x}) = e^{i\omega t}\varphi(\vec{x}). \tag{1}$$

Such solutions were proposed in [2] (see also [3] for some early papers, in which solitons of this type were discussed). Following [4], now non-topological solitons of this kind are called Q-balls.

A simplest generalization of Q-balls to the gauged case, i.e., from the global $U(1)$ symmetry to the gauge $U(1)$ symmetry, is straightforward. To our knowledge, for the first time this was done in [5]. Later such $U(1)$ gauged Q-balls were discussed in many papers, for the most interesting results see [6, 7]. In the present manuscript we briefly present some new results for these solutions (see [8] for details).

We start with the action

$$S = \int d^4x \left((\partial^\mu\phi^* - ieA^\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right) \tag{2}$$

and take the standard spherically symmetric ansatz for the fields describing a gauged Q-ball:

$$\phi(t,\vec{x}) = \mathrm{e}^{i\omega t} f(r), \qquad A_0(t,\vec{x}) = A_0(r), \qquad A_i(t,\vec{x}) \equiv 0, \tag{3}$$

where $\mu = 0,1,2,3$; $r = \sqrt{\vec{x}^2}$ and $f(r)$, $A_0(r)$ are real functions which vanish at $r \to \infty$. We suppose that the function $f(r)$ has no nodes and $f(0) > 0$. For the scalar field potential, the conditions $V(0) = 0$, $\frac{dV}{df}\Big|_{f=0} = 0$ are supposed to fulfill.

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The corresponding equations of motion take the form

$$2e^2(\omega+g)f^2 = \Delta g, \tag{4}$$

$$2(\omega+g)^2 f + 2\Delta f - \frac{dV}{df} = 0, \tag{5}$$

where $\Delta = \sum\limits_{i=1}^{3}\partial_i\partial_i$, $g = eA_0$, $V(f) = V(\phi^*\phi)$. We define the charge of a gauged Q-ball as

$$Q = 2\int d^3x(\omega+g)f^2. \tag{6}$$

We note that the physical charge is $Q_{phys} = eQ$, but for convenience, below we will use the charge Q, not Q_{phys}. Without loss of generality for simplicity we can consider $\omega \geq 0$, which implies $Q \geq 0$. Indeed, according to [6] the sign of $\omega + g$ always coincides with the sign of ω, whereas $g \equiv 0$ for $\omega = 0$ (see [5]).

The energy of a gauged Q-ball at rest is defined by

$$E = \int d^3x\left((\omega+g)^2f^2 + \partial_i f\partial_i f + V(f) + \frac{1}{2e^2}\partial_i g\partial_i g\right). \tag{7}$$

It is well known that for ordinary (nongauged) Q-balls the relation

$$\frac{dE}{dQ} = \omega \tag{8}$$

holds. It is not difficult to show that the same relation (8) holds for $U(1)$ gauged Q-balls too. The calculation is straightforward and we do not present it here (see [8] for details). Although this relation is very simple, it leads to a rather important consequence. Indeed, there exist well-known estimates for the maximal charge of stable gauged Q-balls, presented in [6] (although these estimates were obtained within the particular model, they are used in many papers concerning gauged Q-balls). It is stated in [6] that for a charge Q, such that for a Q-ball of this charge the inequality $\frac{dE}{dQ} > M$ holds, it is energetically favorable to have a Q-ball with the charge Q_{max} and $Q - Q_{max}$ free scalar particles. The maximal charge Q_{max} is defined as a solution to equation $\frac{dE}{dQ} = M$. But let us recall that the inequality $\omega < M$ should hold for a Q-ball in a theory with $\frac{dV(\phi^*\phi)}{d(\phi^*\phi)}\big|_{\phi^*\phi=0} = M^2 > 0$ (see [5, 6]), otherwise the corresponding solution to equation (5) does not fall off at infinity rapidly enough to ensure the finiteness of the Q-ball charge and energy. So, for any gauged Q-ball in such a theory the inequality $\frac{dE}{dQ} = \omega < M$ holds, and Q-balls with $\frac{dE}{dQ} \geq M$ can never exist. Thus, the procedure used in [6] for estimating the value of the maximal charge of stable gauged Q-balls contradicts the main properties of gauged Q-balls and can not be considered as correct, as well as the consequent statement about the existence of the maximal charge (see also discussion in [8]).

Now we turn to the particular case when the back-reaction of the gauge field is supposed to be small ($|g(r)| \ll \omega$, $|f(r) - f_0(r)| \ll f_0(r)$, where $f_0(r) = f_0(r,\omega)$ is a nongauged Q-ball solution in the case $e = 0$). In this case one can use the linear approximation in $g(r)$ and $\varphi(r) = f(r) - f_0(r)$ above the nongauged background solution, which simplifies the analysis. Equations (4) and (5) can be reduced to the form

$$\Delta g - 2e^2\omega f_0^2 = 0, \tag{9}$$

$$\Delta\varphi + \omega^2\varphi + 2\omega g f_0 - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0}\varphi = 0, \tag{10}$$

where f_0 is defined as a solution to the equation

$$\omega^2 f_0 + \Delta f_0 - \frac{1}{2}\frac{dV}{df}\bigg|_{f=f_0} = 0. \tag{11}$$

In this approximation, the charge and the energy take the form

$$Q = Q_0 + \triangle Q = Q_0 + 4\pi \int\limits_0^\infty drr^2(2gf_0^2 + 4\omega f_0\varphi), \tag{12}$$

$$E = E_0 + \triangle E = E_0 + 4\pi\omega \int\limits_0^\infty drr^2(gf_0^2 + 4\omega f_0\varphi) \tag{13}$$

where Q_0 and E_0 are defined by Eqs. (6), (7) with $f_0(r)$ instead of $f(r)$ and with $g \equiv 0$.

Now we are ready to calculate $\triangle Q$ and $\triangle E$. We will not present the detailed calculations, which can be found in [8], but briefly present the main steps of the derivation. First, we take equation (10), multiply it by $\frac{df_0}{d\omega}$, integrate over the spatial volume and perform integration by parts. Then, we take equation (11) and differentiate it with respect to ω. Using the latter equation, we arrive at

$$\omega \int \left(gf_0 \frac{df_0}{d\omega} - \varphi f_0 \right) d^3x = 0. \tag{14}$$

Now, let us consider the charge (12). According to (14),

$$\triangle Q = 4\pi \int\limits_0^\infty drr^2 g \frac{dq}{d\omega}, \tag{15}$$

where $q = 2\omega f_0^2$. Using the relation $\Delta \frac{dg}{d\omega} = e^2 \frac{dq}{d\omega}$, which follows from (9), we can show that

$$\int d^3x g \frac{dq}{d\omega} = \frac{1}{2} \frac{d}{d\omega} \int d^3x gq. \tag{16}$$

Let us define

$$I = \frac{1}{2} \int d^3x gq. \tag{17}$$

Then, for (12) and (13) we get

$$\triangle Q = \frac{dI}{d\omega}, \qquad \triangle E = \omega \frac{dI}{d\omega} - I. \tag{18}$$

It is possible to calculate the integral I explicitly [8]. Indeed, for a given background solution f_0, the spherically symmetric solution to (9), which vanishes at infinity, takes the form [5]

$$g = g(r) = -e^2 \int\limits_r^\infty q(y) y dy - e^2 \frac{1}{r} \int\limits_0^r q(y) y^2 dy. \tag{19}$$

Substituting it into (17), after some algebra we get

$$I = -16\pi e^2 \omega^2 \int\limits_0^\infty f_0^2(r) r \int\limits_0^r f_0^2(y) y^2 dy \, dr. \tag{20}$$

Thus, to examine the main properties of gauged Q-balls in a theory with a small back-reaction of the gauge field, it is not necessary to solve explicitly the corresponding linearized differential equations. Instead of this, one can simply take the corresponding nongauged background solution $f_0(r, \omega)$ to obtain I and calculate the corresponding $E(Q)$ dependence.

Of course, the result presented above can be used only if the corrections are much smaller than the background solution itself. In order to check the validity of the linear approximation, we propose the parameter (we present it without the detailed derivation, it can be found in [8])

$$\alpha(\omega)=\max_i\left\{\frac{|g(0)|}{\omega},\frac{|2\triangle E-\omega\triangle Q|}{2\omega Q_0},\left|\frac{g(0)}{f_0(r_i)}\frac{df_0(r_i)}{d\omega}\right|\right\}. \qquad (21)$$

One should calculate (21) at several different points r_i for a given ω to obtain better estimates. The fulfillment of the condition $\alpha(\omega)\ll 1$ suggests, although it does not ensure, that the linear approximation is valid. Similarly to the case of (20), only the background solution $f_0(r,\omega)$ is necessary for calculating $\alpha(\omega)$. It is necessary to note that although the factor $\alpha(\omega)$ is proportional to e^2, the smallness of e^2 does not guarantee the fulfillment of $\alpha(\omega)\ll 1$ in the general case. A simple justification of this fact can be found in Appendix B of [8].

Finally, we compare the energies of gauged (obtained in the linear approximation in $\alpha(\omega)$) and nongauged Q-balls at a given charge Q. Equality of charges of gauged (with $\omega=\omega_1$) and nongauged (with $\omega=\omega_2$) Q-balls leads to

$$\triangle Q(\omega_1)\approx(\omega_2-\omega_1)\left.\frac{dQ_0}{d\omega}\right|_{\omega=\omega_1}. \qquad (22)$$

Thus, for the energies of gauged and nongauged Q-balls we can get

$$E(\omega_1)-E_0(\omega_2)\approx\triangle E(\omega_1)-(\omega_2-\omega_1)\left.\frac{dE_0}{d\omega}\right|_{\omega=\omega_1}\approx\triangle E(\omega_1)-\omega_1\triangle Q(\omega_1)=-I(\omega_1)>0,$$

where we have used (8). Thus, at least for small $\alpha(\omega)$, for any charge $Q>0$ the energy of a gauged Q-ball is larger than the energy of the corresponding nongauged Q-ball with the same charge.

In [9] one can find explicit examples demonstrating how the presented technique can be used for calculations.

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