Scaling violation in logarithmic dimensions in massless scalar quantum field theories.

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Abstract

The method of renormalization group equations is used for the calculation of infrared asymptotic of the propagator in scalar quantum field models with ϕ^3-, ϕ^4- and ϕ^6-interactions in logarithmic dimensions. Most accurate results are based on the 5-loop approximation of the β-function and 4-loop one for self-energy operator.

We investigate the asymptotic behavior of the propagator of the theories ϕ^3, ϕ^4 and ϕ^6 in their logarithmic dimension using the renormalization group technique [1, 2, 3]. The renormalization group equation is the following [1, 2, 3]:

$$\left(-p\frac{\partial}{\partial p}+\beta(g)\frac{\partial}{\partial g}+2\gamma(g)-2\right)D(p,g)=0 \tag{1}$$

where p is a monentum, g is a renormalized coupling constant (or its function), $\beta(g)$ is the beta function, $\gamma(g)$ is the anomalous dimension of field, D is a propagator.

A solution of (1) has the form [3]:

$$\Phi(s,g)=\frac{1}{s^2}\Phi(1,\overline{g}(s,g))\exp\left(2\int\limits_{g}^{\overline{g}(s,g)}\frac{\gamma(x)}{\beta(x)}dx\right) \tag{2}$$

where the following notation is used: $s\equiv\frac{p}{\mu}$; $\Phi\equiv\mu^2 D$; μ is a parameter of renormalization with the dimension of mass; $\overline{g}(s,g)$ is the invariant charge which is defined implicitly by the equation:

$$\ln s=\int\limits_{g}^{\overline{g}}\frac{dx}{\beta(x)}. \tag{3}$$

To find $\Phi(1,g)$ we have to solve the Dyson equation:

$$D^{-1}(p,g)=\Delta^{-1}(p)-\Sigma(p,g) \tag{4}$$

where $\Delta(p)$ is the bar propagator, $\Sigma(p,g)$ is the self-energy operator — the sum of 1-irreducible diagrams.

We make all the calculations within the minimal subtractions (MS) scheme. It holds:

$$\Delta(p)=\frac{1}{p^2}.$$

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We introduce another dimensionless variable: $\Xi \equiv \mu^{-2}\Sigma$. Solving (4) we obtain:

$$\Phi(1,g) = \frac{1}{1-\Xi(1,g)}.$$

Thus, the expression for the propagator (2) takes the form:

$$D(p,g) = \frac{1}{p^2}\frac{1}{1-\Xi(1,\overline{g}(s,g))}\exp\left(2\int\limits_{g}^{\overline{g}(s,g)}\frac{\gamma(x)}{\beta(x)}dx\right). \tag{5}$$

To find $\Xi(1,g)$ we have to calculate some of the Feynman diagrams.
It is convenient to introduce the functions $\rho(g)$ and $V(g)$ defined as:

$$\rho'(g) = \frac{1}{\beta(g)},$$

$$V'(g) = \frac{\gamma(g)}{\beta(g)}.$$

Then we can rewrite the equation (3) as follows:

$$\ln s = \rho(\overline{g}) - \rho(g)$$

or

$$\rho(\overline{g}) = \ln(e^{\rho(g)}s).$$

We denote the combination in the logarithm argument as s_1. Then we obtain:

$$\rho(\overline{g}) = \ln s_1.$$

Thus, the invariant charge $\overline{g}(s,g)$ depends on s and g not separately, but only on the combination $e^{\rho(g)}s$.

The exponent in the formula (5) can be rewritten as follows:

$$\exp\left(2\int\limits_{g}^{\overline{g}(s,g)}\frac{\gamma(x)}{\beta(x)}dx\right) = e^{-2V(g)}e^{2V(\overline{g})},$$

and the expression (5) takes the form:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}\frac{1}{1-\Xi(1,\overline{g}(s_1))}e^{2V(\overline{g}(s_1))}. \tag{6}$$

We will use this formula in the future.

We are interested in the asymptotic behavior of the propagator at low or high momentum. Using the renormalization group equation we can calculate only one of the two asymptotics: for some theories we find the infrared (IR) asymptotic, for other - ultraviolet (UV) one. In theories which we will consider the beta function starts in the logarithmic dimension with the quadratic term: $\beta(g) = b_2g^2 + ...$, and the type of the asymptotic is determined by the sings of b_2 and g: if $b_2g > 0$ then it is IR type, if $b_2g < 0$ then it is UV one.

Our aim is to calculate corrections to the main approximation. Suppose, we know 3 terms:

$$\beta(g) = b_2g^2 + b_3g^3 + b_4g^4 + ...$$

Then from (3) we obtain:

$$\overline{g}(s_1) = -\frac{1}{b_2 \ln s_1}\left[1 + \frac{b_3}{b_2^2}\frac{\ln|\ln s_1|}{\ln s_1} + \frac{b_3^2}{b_2^4}\frac{(\ln|\ln s_1|)^2}{(\ln s_1)^2} - \right.$$
$$\left. -\frac{b_3^2}{b_2^4}\frac{\ln|\ln s_1|}{(\ln s_1)^2} + \frac{b_2 b_4 - b_3^2}{b_2^4}\frac{1}{(\ln s_1)^2} + \ldots\right], \tag{7}$$

where $s_1 = e^{\rho(g)}s$ and $\rho(g)$ is defined uniquely by 2 conditions:

$$\begin{cases} \rho'(g) = \frac{1}{\beta(g)} \\ \lim\limits_{g \to 0}\left(\rho(g) + \frac{1}{b_2 g} + \frac{b_3}{b_2^2}\ln|b_2 g|\right) = 0 \end{cases}$$

We use this expression for the calculation of asymptotic behavior of the propagator. Suppose, we know the following approximation for the functions $\beta(g)$, $\gamma(g)$ and $\Xi(1,g)$:

$$\beta(g) = b_2 g^2 + b_3 g^3 + b_4 g^4 + \ldots$$
$$\gamma(g) = c_1 g + c_2 g^2 + c_3 g^3 + \ldots$$
$$\Xi(1,g) = a_1 g + a_2 g^2 + \ldots$$

Then from (6) we receive the following expression for the propagator in terms of the invariant charge:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}|b_2\overline{g}|^{2c_1/b_2}\left[1 + \frac{-2b_3c_1 + 2b_2c_2 + b_2^2a_1}{b_2^2}\overline{g} + \right.$$
$$+ \frac{1}{b_2^4}(b_2b_3^2c_1 - b_2^2b_4c_1 + 2b_3^2c_1^2 - b_2^2b_3c_2 - 4b_2b_3c_1c_2 + 2b_2^2c_2^2 +$$
$$\left. + b_2^3c_3 - 2b_2^2b_3c_1a_1 + 2b_2^3c_2a_1 + b_2^4a_2 + b_2^4a_1^2)\overline{g}^2 + \ldots\right]$$

And substituting (7) in this formula we obtain the propagator in terms of the momentum:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}|\ln s_1|^{-2c_1/b_2}\left[1 + \frac{2b_3c_1}{b_2^3}\frac{\ln|\ln s_1|}{\ln s_1} + \frac{2b_3c_1 - b_2(2c_2 + b_2a_1)}{b_2^3}\frac{1}{\ln s_1} + \right.$$
$$+ \frac{b_3^2c_1(b_2 + 2c_1)}{b_2^6}\frac{(\ln|\ln s_1|)^2}{(\ln s_1)^2} + \frac{b_3(4b_3c_1^2 - b_2(b_2 + 2c_1)(2c_2 + b_2a_1))}{b_2^6}\frac{\ln|\ln s_1|}{(\ln s_1)^2} + \tag{8}$$
$$+ \frac{1}{b_2^6}[2b_3^2c_1^2 - b_2b_3c_1(b_3 + 4c_2) + b_2^3(c_3 + 2c_2a_1) +$$
$$\left. + b_2^2(b_4c_1 + 2c_2^2 - b_3(c_2 + 2c_1a_1)) + b_2^4a_2 + b_2^4a_1^2]\frac{1}{(\ln s_1)^2} + \ldots\right]$$

The function $V(g)$ is uniquely determined by 2 conditions:

$$\begin{cases} V'(g) = \frac{\gamma(g)}{\beta(g)} \\ \lim\limits_{g \to 0}\left(V(g) - \frac{c_1}{b_2}\ln|b_2 g|\right) = 0 \end{cases}$$

We can try to simplify more the expression (8) by using $s_2 = e^A s_1$.

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}|\ln s_2|^{-2c_1/b_2}\left[1 + \frac{2b_3c_1}{b_2^3}\frac{\ln|\ln s_2|}{\ln s_2} + \right.$$
$$+\left(\frac{2b_3c_1 - b_2(2c_2+b_2a_1)}{b_2^3} - \frac{2c_1}{b_2}A\right)\frac{1}{\ln s_2} + \frac{b_3^2c_1(b_2+2c_1)}{b_2^6}\frac{(\ln|\ln s_2|)^2}{(\ln s_2)^2} +$$
$$+\left(\frac{b_3(4b_3c_1^2 - b_2(b_2+2c_1)(2c_2+b_2a_1))}{b_2^6} - \frac{2b_2b_3c_1 + 4b_3c_1^2}{b_2^4}A\right)\frac{\ln|\ln s_2|}{(\ln s_2)^2} +$$
$$+\left(\frac{2b_3^2c_1^2 - b_2b_3c_1(b_3+4c_2) + b_2^3(c_3+2c_2a_1) + b_2^2(b_4c_1 + 2c_2^2 - b_3(c_2+2c_1a_1)) + b_2^4a_2 + b_2^4a_1^2}{b_2^6} + \right.$$
$$\left.\left.\frac{a_1b_2^3 + 2a_1b_2^2c_1 - 4b_3c_1^2 + 2b_2^2c_2 + 4b_2c_1c_2}{b_2^4}A + \frac{b_2c_1 + 2c_1^2}{b_2^2}A^2\right)\frac{1}{(\ln s_2)^2} + \ldots\right]$$

If it holds $c_1 \neq 0$, then choosing $A = \frac{2b_3c_1 - b_2(2c_2+b_2a_1)}{2b_2^2c_1}$ we receive:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}|\ln s_2|^{-2c_1/b_2}\left[1 + \frac{2b_3c_1}{b_2^3}\frac{\ln|\ln s_2|}{\ln s_2} + \right.$$
$$+\frac{b_3^2c_1(b_2+2c_1)}{b_2^6}\frac{(\ln|\ln s_2|)^2}{(\ln s_2)^2} - \frac{2b_3^2c_1}{b_2^5}\frac{\ln|\ln s_2|}{(\ln s_2)^2} - \tag{9}$$
$$\left.-\frac{a_1^2b_2^2(b_2-2c_1) + 4b_2^2(a_1c_2 - a_2c_1) - 4b_4c_1^2 + 4b_3c_1c_2 + 4b_2(c_2^2 - c_1c_3)}{4b_2^4c_1}\frac{1}{(\ln s_2)^2} + \ldots\right]$$

If $c_1 = 0$, then we have:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}\left[1 - \frac{2c_2+b_2a_1}{b_2^2}\frac{1}{\ln s_2} - \frac{b_3(2c_2+b_2a_1)}{b_2^4}\frac{\ln|\ln s_2|}{(\ln s_2)^2} + \right.$$
$$\left.+\left(\frac{a_1^2b_2^2 + a_2b_2^2 + 2a_1b_2c_2 - b_3c_2 + 2c_2^2 + b_2c_3}{b_2^4} + \frac{a_1b_2 + 2c_2}{b_2^2}A\right)\frac{1}{(\ln s_2)^2} + \ldots\right]$$

and it is convenient to choose $A = -\frac{a_1^2b_2^2 + a_2b_2^2 + 2a_1b_2c_2 - b_3c_2 + 2c_2^2 + b_2c_3}{b_2^2(a_1b_2+2c_2)}$. In this case we obtain:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}\left[1 - \frac{2c_2+b_2a_1}{b_2^2}\frac{1}{\ln s_2} - \frac{b_3(2c_2+b_2a_1)}{b_2^4}\frac{\ln|\ln s_2|}{(\ln s_2)^2} + \ldots\right] \tag{10}$$

The ϕ^3-theory in the Euclidian space has the following lagrangian:

$$L = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!}\phi^3$$

where ϕ is a scalar field, λ is a coupling constant.

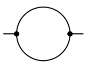
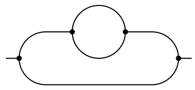
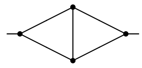
The logarithmic dimension for this theory is $d = 6$. The beta function and the anomalous dimension of field have been calculated in the 3-loop approximation [4]:

$$\beta(g) = -\frac{3}{2}g^2 + -\frac{125}{72}g^3 - \left(\frac{33085}{10368} + \frac{5\zeta(3)}{4}\right)g^4 + \ldots,$$
$$\gamma(g) = \frac{1}{12}g + \frac{13}{432}g^2 + \left(\frac{5195}{62208} - \frac{\zeta(3)}{24}\right)g^3 + \ldots$$

where $\zeta(z)$ is the Riemann's zeta function, $g = \frac{\lambda^2}{64\pi^3}$.

We have $b_2 < 0$. If the coupling constant λ is real then it holds $g > 0$, $b_2g < 0$ and we get the ultraviolet asymptotic. But one usually takes λ to be imaginary that we obtain $g < 0$ and we get the infrared asymptotic.

To find the desired accuracy for the propagator, we have to compute three Feynman diagrams:

$\Sigma = \frac{1}{2}$  $+\frac{1}{2}$  $+\frac{1}{2}$  $+...$

Here, the line is the bar propagator $\frac{1}{p^2}$ and the vertex is the coupling constant λ.

All these diagrams diverge in the logarithmic dimension ($d = 6$). In our calculations we use the dimensional regularization ($d = 6 - 2\varepsilon$) and the R-operation. The result is the following:

$$\Sigma(p, g) = \left[-\frac{8 + 3(\tau - 2\ln s)}{36} g - \frac{1789 + 1116(\tau - 2\ln s) + 180(\tau - 2\ln s)^2}{5184} g^2 + ...\right] p^2$$

and

$$\Xi(1, g) = -\frac{8 + 3\tau}{36} g - \frac{1789 + 1116\tau + 180\tau^2}{5184} g^2 + ...$$

where $g = \frac{\lambda^2}{64\pi^3}$, $\tau = \ln 4\pi - \gamma_E$, and γ_E is the Euler's constant.

From the formula (9) we obtain the infrared asymptotic of the propagator:

$$\Phi(p, g) = e^{-2V(g)} \frac{1}{p^2} |\ln s_2|^{1/9} \left[1 + \frac{125}{1458} \frac{\ln|\ln s_2|}{\ln s_2} - \right.$$
$$\left. - \frac{15625}{531441} \frac{(\ln|\ln s_2|)^2}{(\ln s_2)^2} + \frac{15625}{236196} \frac{\ln|\ln s_2|}{(\ln s_2)^2} - \left(\frac{11291 - 30132\tau + 1296\zeta(3)}{157464}\right) \frac{1}{(\ln s_2)^2} + ...\right]$$

The lagrangian of the O(N)-symmetric ϕ^4-theory in the Euclidian space has the form:

$$L = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}(\phi^2)^2$$

ϕ is an N-component field, λ is a coupling constant, $\lambda > 0$.

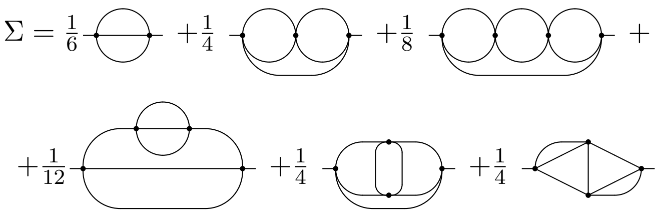
The logarithmic dimension for this theory is $d = 4$. The beta function and the anomalous dimension of field have been computed up to 5 loops [5], see too [3]:

$$\beta(g) = \frac{N + 8}{3} g^2 - \frac{3N + 14}{3} g^3 + \frac{33N^2 + 922N + 2960 + 96\zeta(3)(5N + 22)}{216} g^4 -$$
$$- \frac{1}{3888}[-5N^3 + 6320N^2 + 80456N + 196648 + 96\zeta(3)(63N^2 + 764N + 2332) -$$
$$- 288\zeta(4)(5N^2 + 62N + 176) + 1920\zeta(5)(2N^2 + 55N + 186)]g^5 +$$
$$+ \frac{1}{62208}[13N^4 + 12578N^3 + 808496N^2 + 6646336N + 13177344 +$$
$$+ 16\zeta(3)(-9N^4 + 1248N^3 + 67640N^2 + 552280N + 1314336) -$$
$$- 288\zeta(4)(63N^3 + 1388N^2 + 9532N + 21120) + 256\zeta(5)(305N^3 + 7466N^2 + 66986N + 165084) -$$
$$- 9600\zeta(6)(N + 8)(2N^2 + 55N + 186) + 112896\zeta(7)(14N^2 + 189N + 526) +$$
$$+ 768\zeta(3)^2(-6N^3 - 59N^2 + 446N + 3264)]g^6 + ...,$$

$$\gamma(g) = \frac{N + 2}{36} g^2 - \frac{(N + 2)(N + 8)}{432} g^3 + \frac{5(N + 2)(-N^2 + 18N + 100)}{5184} g^4 -$$
$$- \frac{N + 2}{186624}[39N^3 + 296N^2 + 22752N + 77056 -$$
$$- 48\zeta(3)(N^3 - 6N^2 + 64N + 184) + 1152\zeta(4)(5N + 22)]g^5 + ...$$

where $g = \frac{\lambda}{16\pi^2} > 0$.

We have $b_2 > 0$ and $g > 0$, we therefore get the infrared asymptotic. We need to calculate the self-energy operator ut to 4 loops:



All these graphs are diverged in the dimension $d = 4$. We apply the dimensional regularization $d = 4 - 2\varepsilon$ and the R-operation. The result is the following:

$$
\begin{aligned}
\Xi(1,g) = &-\frac{N+2}{144}(13+4\tau)g^2 - \frac{(N+2)(N+8)}{2592}(167+84\tau+12\tau^2)g^3 - \\
&-\frac{N+2}{41472}[1851N^2+41467N+174518+16(N^2-14N-68)\zeta(3)+ \\
&+24(54N^2+1081N+4466)\tau+16(21N^2+373N+1514)\tau^2+32(N+8)^2\tau^3]g^4+...
\end{aligned}
$$

where $\tau = \ln 4\pi - \gamma_E$, $g = \frac{\lambda}{16\pi^2}$.

From the formula (10) we receive the following result for the infrared asymptotic of the propagator:

$$
\begin{aligned}
D(p,g) = &e^{-2V(g)}\frac{1}{p^2}\Bigg[1 - \frac{N+2}{2(N+8)^2}\frac{1}{\ln s_2} + \frac{3(N+2)(3N+14)}{2(N+8)^4}\frac{\ln|\ln s_2|}{(\ln s_2)^2} - \\
&-\frac{9(N+2)(3N+14)^2}{2(N+8)^6}\frac{(\ln|\ln s_2|)^2}{(\ln s_2)^3} + \frac{9(N+2)(3N+14)^2}{2(N+8)^6}\frac{\ln|\ln s_2|}{(\ln s_2)^3} + \\
&+\frac{(N+2)}{48(N+8)^6}[319N^4+9942N^3+116469N^2+607364N+1204452+ \\
&+168(N+8)^4\tau+24(N+8)^4\tau^2-384\zeta(3)(N+8)(5N+22)]\frac{1}{(\ln s_2)^3} + \\
&+\frac{27(N+2)(3N+14)^3}{2(N+8)^8}\frac{(\ln|\ln s_1|)^3}{(\ln s_1)^4} - \frac{135(N+2)(3N+14)^3}{4(N+8)^8}\frac{(\ln|\ln s_2|)^2}{(\ln s_2)^4} - \\
&-\frac{3(N+2)(3N+14)}{32(N+8)^8}[319N^4+9942N^3+115173N^2+595268N+1176228+168(N+8)^4\tau+ \\
&+24(N+8)^4\tau^2-384\zeta(3)(N+8)(5N+22)]\frac{\ln|\ln s_2|}{(\ln s_2)^4} - \\
&-\frac{N+2}{384(N+8)^8}[7068N^6+322295N^5+6183232N^4+63882945N^3+ \\
&+374808430N^2+1182947372N+1567304328+ \\
&+96\zeta(3)(N+8)(25N^3+1096N^2+9052N+21984)+1920\zeta(5)(N+8)^2(2N^2+55N+186)+ \\
&+12(N+8)^4(461N^2+6606N+25948)\tau+ \\
&+24(N+8)^4(63N^2+953N+3778)\tau^2+144(N+8)^6\tau^3]\;\frac{1}{(\ln s_2)^4}+...\Bigg]
\end{aligned}
$$

The lagrangian of the ϕ^6-theory is the following:

$$
L = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{6!}\phi^6
$$

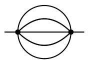
where ϕ is a scalar field, λ is a coupling constant, $\lambda > 0$.

The logarithmic dimension is $d = 3$. For this model two terms of the beta function and the anomalous dimension of field have been calculated [3]:

$$\beta(g) = \frac{20}{3}g^2 - \left(\frac{1124}{15} + \frac{15\pi^2}{2}\right)g^3 + ...,$$
$$\gamma(g) = \frac{1}{90}g^2 - \frac{2}{81}g^3 + ...,$$

where $g = \frac{\lambda}{64\pi^2} > 0$.

Using the renormalization group equation we get the IR-asymptotic. We need only 1 diagram for Σ:

$\Sigma = \frac{1}{120}$  $+...$

We obtain:

$$\Xi(1,g) = -\frac{13+3\tau}{270}g^2 + ...$$

where $\tau = \ln 4\pi - \gamma_E$.

Using the formula (10) we get:

$$D(p,g) = e^{-2V(g)}\frac{1}{p^2}\left(1 - \frac{1}{2000}\frac{1}{\ln s_2} + \frac{3(2248+225\pi^2)}{8000000}\frac{\ln|\ln s_2|}{(\ln s_2)^2} + ...\right).$$

Conclusion.

Using the renormalization group equation we have calculated the infrared asymptotic of the propagator for the ϕ^3-, ϕ^4- and ϕ^6-theories. The equation includes a beta function and an anomalous dimension of field. These data are not enough for calculation of the asymptotic behavior of the propagator. One needs also to know a self-energy operator as a function of a coupling constant with a fixed value of momentum. To find this function it required to calculate Feynman diagrams of self-energy operator.

For the ϕ^3-theory the propagator in the main approximation is power with logarithm (scaling is violated), and in the ϕ^4- and ϕ^6-theories in the main approximation scaling is not violated. Corrections in all these cases are expressed in terms of the logarithm and the logarithm logarithm of the momentum. Asymptotic contains both universal and non-universal terms with respect to renormalization scheme.

References

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