High energy production amplitudes in $N = 4$ SUSY and unitarity

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Abstract

The S-channel unitarity of high energy scattering amplitudes in QCD is related to diagrams with an arbitrary number of reggeized gluons. In the leading order the equations for the multi-gluon composite states are integrable at large N_c. Moreover, in the $N = 4$ SUSY the intercept of the BFKL pomeron is constructed at the strong coupling. The non-Fredholm properties of the integral kernels for the color singlet and octet states allow to find their eigenvalues at large anomalous dimensions. The Green function for the BFKL equation in QCD is expressed in terms of non-perturbative phases of eigenfunctions. The spectrum of Pomerons is calculated in various models for the large distance dynamics.

1 Gluon reggeization and Pomeron in QCD

A leading contribution to the elastic cross-section at large energies $\sqrt{s}$ appears from the particle scattering at the fixed momentum transfer $|q| = \sqrt{-t}$. In this region it is convenient to use the t-channel partial wave representation for the scattering amplitude in the form of the Mellin transformation

$$A^p(s,t) = s \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left((-s)^\omega - p s^\omega\right) f^p_\omega(t)\,,\ \omega = j-1\,, \tag{1}$$

where $p = \pm 1$ is the signature of the corresponding contribution. If the leading singularity of the partial wave $f^p_\omega(t)$ is a pole, the amplitude has the so-called Regge behavior [1]. The exchanges of several Regge poles generate the Mandelstam singularities of $f^p_\omega(t)$ in the ω-plane [2, 3, 4].

To describe an approximately constant behavior of the total cross-sections σ a special j-plane singularity - Pomeron with vacuum quantum numbers is introduced

$$\sigma = \frac{1}{s}\,\Im A(s.0)\,,\ A(s,t) \approx is\, s^{\Delta - \alpha'_P q^2}\,, \tag{2}$$

where the intercept Δ and slope α'_P of its trajectory should be rather small.

In the leading logarithmic approximation (LLA), where $\alpha_s \ln s \sim 1$ and $\alpha_s = \frac{g_s^2}{4\pi} \ll 1$, the Born scattering amplitude in QCD is multiplied by the Regge factor

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\, s^{\omega(t)}\,, \tag{3}$$

where the gluon Regge trajectory is given below

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi} \ln \frac{|q^2|}{\lambda^2}\,. \tag{4}$$

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The most essential contribution to the total cross-section appears from production of gluons in multi-Regge kinematics at their large pair energies $\sqrt{s_k}$ and fixed $k_{r\perp}$ corresponding to comparatively small momentum transfers q_i. The production amplitude has the factorized multi-Regge form with effective reggeon-reggeon-gluon vertices [5]

The knowledge of production amplitudes $M_{2\to 2+n}$ gives a possibility to construct the total cross-section and the scattering amplitude with color singlet quantum numbers in the crossing channel. This amplitude can be expressed in terms of the Pomeron wave function Ψ satisfying the BFKL equation in LLA [5]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ ,\ \sigma_t \sim s^{\Delta}\ ,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0\,, \tag{5}$$

where E_0 is the ground stte energy. The operator H_{12} is the BFKL Hamiltonian and Δ is the Pomeron intercept. In the impact parameter representation the hamiltonian can be written as follows [6]

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln|p_1 p_2|^2 - 4\psi(1)\,, \tag{6}$$

where we introduced the complex notations

$$\rho_{12} = \rho_1 - \rho_2\,,\ \rho_r = x_r + iy_r\,,\ p_r = i\frac{\partial}{\partial\rho_r}\,,\ p_r^* = i\frac{\partial}{\partial\rho_r^*}\,. \tag{7}$$

The above Schrödinger equation is invariant under the group of the Möbius transformations [7]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,. \tag{8}$$

As a result, the eigenfunctions and eigenvalues of H_{12} are classified by the conformal weights

$$m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = \frac{1}{2} + i\nu \tag{9}$$

depending on the anomalous dimension γ of twist-2 operators and the integer conformal spin n. Further, the Pomeron wave function is presented in the form [7]

$$\Psi(\vec{\rho}_1,\vec{\rho}_2;\vec{\rho}_0) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^{m}\left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}}\,. \tag{10}$$

Respectively, the hamiltonian has the property of the holomorphic separability and the corresponding holomorphic energies are expressed in terms of the function $\psi(x) = (\ln\Gamma(x))'$

$$E(m,\widetilde{m}) = \epsilon(m) + \epsilon(\widetilde{m}) = 4Re\,\psi\left(\gamma + \frac{|n|}{2}\right) - 4\psi(1)\,. \tag{11}$$

The Pomeron intercept in LLA, obtained at $m = \widetilde{m} = 1/2$, is positive: $\Delta = 4\frac{\alpha N_c}{\pi}\ln 2$, which is not compatible with the s-channel unitarity. We should consider the diagrams with the multi-Reggeon exchanges. The equations for composite states of several reggeized gluons are integrable in the multi-color limit [8, 9].

Generally the BFKL approach can be formulated in terms of an effective field theory for the reggeized gluons similar to the Gribov Pomeron calculus. The gluon trajectory and various reggeon couplings in upper orders of perturbation theory can be calculated from the effective action [10] written for a cluster of gluons and quarks interacting with the reggeized gluons and having their rapidities y in some interval $\eta \ll \ln s$. The Feynman rules for this action are derived in ref. [11]. The effective action approach gives a possibility to construct various reggeon vertices needed to calculate next-to-leading order (NLO) corrections to the BFKL kernel in the color singlet [12, 13] and adjoint [14, 15] representations. The hamiltonian for the composite states of the reggeized gluons in the adjoint representation for the multi-color Yang-Mills theory in LLA coincides with the hamiltonian of the integrable open spin chain [16].

The NLO contributions to the BFKL kernel in QCD were calculated in ref. [12]. In the case of the $N=4$ extended super-symmetric gauge theory the two loop result for its eigenvalues

$$\omega = \frac{g^2 N_c}{4\pi^2}\,\chi(n,\gamma) + \frac{1}{4}\left(\frac{g^2 N_c}{4\pi^2}\right)^2 \Delta(n,\gamma)\,,\;\; \chi(n,\gamma) = 2\psi(1) - 2\Re\psi(\gamma + |n|/2) \tag{12}$$

has the property of the hermitian separability [13]. All expressions entering in $\Delta(n,\gamma)$ contain only the special functions with the maximal transcendentality [17]. The maximal transcendentality property in $N=4$ SUSY is valid in each order of perturbation theory also for the anomalous dimension $\gamma(j)$ of the twist 2 operators [17], which allowed to calculate them up to the fifth order [18, 19, 20]. The singular behavior of $\gamma(j)$ at $j \to 1$, obtained from the BFKL eigenvalue $j-1=\omega(\nu)$ with the substitution $i\nu \to \gamma(j) - j/2$ in Ref. [17], is in an agreement with its direct calculation with the use of the integrability of the theory.

Note, that according to Maldacena the $N=4$ four-dimensional conformal field theory is equivalent to the super-strings living on the anti-de-Sitter 10-dimensional space [21, 22, 23]. As a result, the Pomeron in this model is dual to the reggeized graviton in the anti-de-Sitter space. Because the anomalous dimension of twist-2 operators at large coupling constants is known, one can find the Pomeron intercept in the strong coupling limit [18, 24, 25]

$$j = 2 - \frac{2}{\sqrt{\lambda}} - \frac{1}{\lambda} + \frac{1}{4\lambda^{-3/2}} + \frac{2+6\zeta_3}{\lambda^2} + \ldots\,,\;\; \lambda = g^2 N_c\,. \tag{13}$$

The slope of the anomalous dimension at $j=2$ related to the Pomeron intercept was calculated up to the fifth order of perturbation theory [18, 20]. Moreover, Basso expressed it explicitly in terms of the Bessel functions.

The duality between the BFKL Pomeron and reggeized graviton means, that the Gribov Pomeron calculus can be formulated in the framework of the approach based on an effective theory for the reggeized gravitons derived in Ref. [26]. Further, the graviton-graviton scattering amplitude at high energies in the double-logarithmic approximation was calculated for the usual gravity and its supersymmetric generalizations [27].

2 Multi-gluon production in $N=4$ SUSY

Several years ago Z. Bern, L. Dixon and V. Smirnov suggested an explicit expression for multi-gluon production amplitude in $N=4$ SUSY at $N_c \to \infty$ with the maximal helicity violation [28]. It turned out, that their assumption is violated already in two loops [29]. The reason is that the BDS amplitude does not satisfy the Steinmann constraint about the absence of simultaneous singularities in overlapping channels [30]

$$\Delta_{s_r}\Delta_{s_{r+1}} M_{2\to 2+n} = 0\,. \tag{14}$$

Moreover, the BDS expression does not contain the Mandelstam cut contributions [31]. These terms appear in the planar amplitudes at some physical regions starting from 6 external legs [32]. The correct amplitude differs from the BDS result by a factor - the remainder function R. This function should be conformal invariant in the momentum space [33]. In LLA the factor R can be calculated in the multi-Regge kinematics from the BFKL and BKP equations for gluon states in the adjoint representation and their integral kernels are proportional to local hamiltonians of an open integrable spin chain [16]. Let us consider the analytic properties of the gluon production amplitudes in $N=4$ SUSY at the multi-Regge kinematics. For 5- and 6-point amplitudes such study was done in Ref. [34]. In an accordance with the Steinmann constraint for one gluon production the planar amplitude has the representation (cf. [36])

$$\frac{M_{2\to 3}^{BDS}}{\Gamma(t_1)\Gamma(t_2)|\Gamma_a|} = c_R^a(-\widetilde{s})^{j_2}(-s_1)^{j_1-j_2} + c_L^a(-\widetilde{s})^{j_1}(-s_2)^{j_2-j_1}\,,\ \widetilde{s} = s|k_\perp^a|^2\,,\ |k_\perp^a|^2 = s_1s_2/s\,, \quad (15)$$

where $\Gamma(t_r)$ are reggeon residues, c_R^a and c_L^a are two real vertices for the gluon a with the transverse momentum $k_\perp^a$ produced from the reggeized gluon [29]

$$c_R^a = \frac{\sin\pi\omega_{1a}}{\sin\pi\omega_{12}}\,,\ c_L^a = \frac{\sin\pi\omega_{2a}}{\sin\pi\omega_{21}} \quad (16)$$

and ω_a is the phase of the complex production vertex [29]

$$\Gamma_a(\ln|k_\perp^a|^2 - i\pi) = |\Gamma_a|\,e^{i\pi\omega_a}\,. \quad (17)$$

In a contradiction with the BDS assumption for the production of two particles a and b the expression for the multi-Regge amplitude in a planar approximation should contain five terms fixed from the factorization relations for the Regge poles [36].

To formulate the Steinmann relations in a general case it is needed to introduce the signatures $\tau_r = \pm 1$ for the reggeons in t_r-channels. Then for the planar case the generating function for production amplitudes in all physical regions, related by twists in the corresponding t-channel lines, can be written as follows

$$A_{2\to n+1}^{\tau_1\dots\tau_n} = A + \sum_{r=1}^{n}\tau_r\,A_r + \sum_{r_1<r_2}\tau_{r_1}\tau_{r_2}\,A_{r_1r_2} + \dots + \tau_1\tau_2\dots\tau_n\,A_{1\dots n}\,. \quad (18)$$

The Weis expression for this function has the following factorized form [36]

$$\frac{A_{2\to n+1}^{\tau_1\dots\tau_n}}{s\Gamma(t_1)\Gamma(t_2)|\Gamma_a||\Gamma_b|\dots} = |s_1|^{\omega_1}\xi_1V^{1,2}|s_2|^{\omega_2}\xi_2V^{2,3}\dots V^{n-1,n}|s_n|^{\omega_n}\xi_n \quad (19)$$

where

$$\xi_r = e^{-i\pi\omega_r} - \tau_r\,,\ \xi_{12} = e^{-i\pi\omega_{12}} + \tau_1\tau_2\,,\ V^{1,2} = \frac{\xi_{12}}{\xi_1}c_R + \frac{\xi_{21}}{\xi_2}c_L\,. \quad (20)$$

In particular, for the transition $2 \to 4$ we obtain the following contributions in two physical regions, where the BDS expression is not correct,

$$\frac{A_{2\to 4}^{\tau_1\dots\tau_3}}{s|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\Gamma(t_1)\Gamma(t_2)|\Gamma_a||\Gamma_b|} = \dots + \left(\tau_1\tau_3\,e^{-i\pi\omega_2} + \tau_1\tau_2\tau_3\right)A + \dots\,. \quad (21)$$

Here

$$A = \frac{2\cos\pi\omega_2\,\sin\pi\omega_a\,\sin\pi\omega_b}{i\sin\pi\omega_2} + i\sin\pi(\omega_a+\omega_b) + \cos\pi\omega_{ab} \quad (22)$$

contains the pole $1/\sin\pi\omega_2$ incompatible with perturbation theory. However just in these physical regions there is a contribution of the Mandelstam cuts [29]. Moreover for example at $s, s_2 > 0, s_1, s_3 < 0$ the analytic structure of the cut contribution allows to redefine the Regge pole term by subtracting two first terms from A and including them in the redefined cut contribution [34]. In such a way we obtain the following representation for the remainder function [34, 14]

$$R\,e^{i\pi\delta} = \cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\phi n}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi(\nu,n)\left(\left(\frac{-1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)} - 1\right)\,. \quad (23)$$

Here the phases δ and ω_{ab} are expressed in terms of the cusp anomalous dimension

$$\gamma_K = 4a - 4\zeta(2)\, a^2 + \frac{44}{5}\,\zeta^2(2)\, a^3 + ...\,, \quad a = \frac{g^2 N_c}{8\pi^2}\,, \tag{24}$$

known in all orders of perturbation theory [18, 35]. Further, the variables w and ϕ can be written in terms of the anharmonic ratios in the momentum space [37]

$$u_1 = \frac{s s_2}{s_{012} s_{123}}\,, \quad u_2 = \frac{s_1 t_3}{s_{012} t_2}\,, \quad u_3 = \frac{s_3 t_1}{s_{123} t_2}\,. \tag{25}$$

The impact factor Φ and the BFKL eigenvalue $\omega(\nu, n)$ are known in two loops [37, 32, 14]. The analogous dispersion relations are obtained for the 7-point amplitude in all physical regions having the Mandelstam cut contributions [38].

3 Non-Fredholm properties of the BFKL kernel in $N = 4$ SUSY

The Fredholm theory of the integral equations of the type

$$\lambda_n f_n(t) = \int dt'\, K(t,t')\, f_n(t') \tag{26}$$

with the symmetric kernel $K^*(t',t) = K(t,t')$ allows to formulate a simple criterium for the absence of the continuous spectrum for eigenvalues λ. This criterium is the convergency of the integral

$$||K||^2 \equiv \int dt \int dt'\, |K(t,t')|^2 < \infty\,. \tag{27}$$

It is related to the fact, that one can express the norm of K in terms of its eigenvalues

$$||K||^2 = \sum_n |\lambda_n|^2\,. \tag{28}$$

For the case of the BFKL equation in LLA the integral for $||K||^2$ is divergent at large and small $t = \ln k^2$, which is related to the scale invariance at this approximation. But there is also a divergency of this expression at small $t - t'$ and fixed t. This divergency is universal and takes place beyond perturbation theory in all non-abelian gauge models. The reason for it is the presence of the gluon Regge trajectories in the BFKL kernel

$$\Delta K = \delta^2(k - k')\, \left(\omega(\vec{k}) + \omega(\vec{q} - k)\right)\,. \tag{29}$$

Generally the divergency $||K||^2 \sim \delta^2(0)$ leads to a continuous spectrum of eigenvalues of the BFKL kernel at negative ω. But in the case of the $N = 4$ SUSY this property of K allows to predict the dependence of its eigenvalues from the conformal weights m and $\widetilde{m}$ at large negative ω at all orders of the perturbation theory. Indeed, the divergent term of the gluon Regge trajectory in this model is proportional to the cusp anomalous dimensions γ_K

$$\omega(k) = -\frac{1}{4}\,\gamma_K(a)\, \ln\frac{k^2}{\lambda^2}\,, \tag{30}$$

where λ is an infrared cut-off. The BFKL kernel for the Pomeron in $N = 4$ SUSY contains apart from two Regge trajectories also the contribution from the particle production which should compensate the divergency at $\lambda \to 0$ in its integration with a smooth function

$$K(\vec{k}, \vec{k}') \approx \frac{1}{2}\,\gamma_K \left(-\delta^2(k - k')\, \ln\frac{1}{\lambda^2} + \frac{1/\pi}{|k - k'|^2 + \lambda^2}\right)\,. \tag{31}$$

Thus, the eigenvalue of the kernel for Pomeron at large m and $\widetilde{m}$ should have the following asymptotic form in all orders of perturbation theory [39]

$$\lim_{m,\widetilde{m}\to\infty} \omega^P(m,\widetilde{m}) = \int \frac{d^2k'}{|k'|^2}\, K(\vec{k},\vec{k}')\, \left(\frac{k'}{k}\right)^m \left(\frac{k'}{k}\right)^{\widetilde{m}} = -\frac{1}{2}\gamma_K\, \ln|m\widetilde{m}| \tag{32}$$

This prediction is valid in first two loops (see (12)). In an analogous way, the eigenvalue of the BFKL kernel for the composite states in the adjoint representation after subtraction from it the gluon Regge trajectory $\omega(q^2)$ is also proportional to γ_K

$$\lim_{m,\widetilde{m}\to\infty} \omega^A(m,\widetilde{m}) = -\frac{1}{4}\gamma_K\, \ln|m\widetilde{m}| \tag{33}$$

in an agreement with the results found in leading and next-to-leading orders [14]. The above asymptotic expressions for the BFKL eigenvalues in $N=4$ SUSY were obtained also in Ref. [40] with the use of explicit expressions for two gluon production amplitudes in the collinear limit.

4 Green function for the BFKL equation with running α_s

In the next-to-leading approximation the BFKL equation in QCD contains running coupling constant effects. Generally this fact improves the properties of the kernel at large transverse momenta $|k_\perp|$, but as a result the coupling constant rapidly grows near $k_\perp^2 = \Lambda^2$, where $\Lambda \approx 200 Mev$ is the QCD parameter. To calculate the spectrum of Pomerons in this case it is necessary to take into account the non-perturbative effects at least as a boundary condition for wave functions at small $k_\perp^2$ [7, 41]. For this purpose a simple method based on a generalization of the DGLAP-type evolution equation obtained from the BFKL kernel with upper order corrections was developed in Ref.[42]. We use below another approach using for illustration a simplified model, in which the kernel is taken in LLA with the running coupling constant calculated in the same approximation

$$\omega f_\omega(t) = \frac{1}{\beta_0\, t}\, \chi(\hat{\nu})\, f_\omega(t)\,, \tag{34}$$

where

$$t = \ln\frac{k_\perp^2}{\Lambda^2}\,,\ \ \hat{\nu} = -i\,\partial_t\,,\ \ \beta_0 = \frac{11}{12} - \frac{n_f}{18} \tag{35}$$

and the BFKL hamiltonian is hermitian with the wave function normalized as follows

$$||f||^2 = \int_0^\infty t\, dt\, |f(t)|^2\,. \tag{36}$$

The characteristic function $\chi(\nu)$ for $n=0$ is given below

$$\chi(\nu) = 2\psi(1) - 2\Re\psi(\frac{1}{2}+i\nu)\,. \tag{37}$$

The eigenfunction of the BFKL kernel with the running coupling is well known

$$f_\omega(t) = \int_{-\infty}^{\infty} d\nu\, e^{it\nu}\, g_\omega(\nu)\,,\ \ g_\omega(\nu) = \left(\frac{\Gamma(\frac{1}{2}+i\nu)}{\Gamma(\frac{1}{2}-i\nu)}\, e^{-2i\nu\psi(1)}\right)^{\frac{1}{\beta_0\omega}}\,, \tag{38}$$

where the contour of integration over ν is chosen to be along the real axes because the corresponding wave function with $\omega > 0$ falls down at $t\to\infty$.

The Green function satisfies the equation

$$\left(\omega - \frac{1}{\beta_0\, t}\, \chi(\hat{\nu})\, f_\omega(t)\right)\, G_\omega(t,t') = \delta(t-t')\,. \tag{39}$$

Its particular solution is given below (cf. the corresponding semiclassical expression in Ref. [42])

$$G^0_\omega(t,t') = -\frac{it'}{\omega}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,e^{it\nu}\,g_\omega(\nu)\int_{-\infty}^{\infty}\frac{d\nu'}{2}\epsilon(\nu+\nu')\,e^{it'\nu'}\,g_\omega(\nu')\,, \tag{40}$$

where $\epsilon(x) = x/|x|$. Note, that G^0_ω has the following completeness property

$$\lim_{\omega\to\infty} G^0_\omega(t,t') = \frac{1}{\omega}\,\delta(t-t')\,. \tag{41}$$

It provides the correct initial condition at $y=0$

$$G^0(0;t,t') = \delta(t-t') \tag{42}$$

for the operator

$$G^0(y;t,t') = \int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\,e^{-\omega y}\,G^0_\omega(t,t')\,, \tag{43}$$

describing the BFKL evolution in the relative rapidity $y = \ln s/s_0$.

The general solution for the Green function contains a contribution proportional to the solution of the homogeneous BFKL equation

$$G_\omega(t,t') = G^0_\omega(t,t') + \frac{t'}{4\pi\omega}\,c(\omega)\,f_\omega(t)\,f_\omega(t')\,, \tag{44}$$

where the coefficient $c(\omega)$ should be chosen from the condition that the Green function satisfies the physical boundary condition at small t and t'. We assume, that the evolution of the wave function in a semi-classical approximation from the confinement region to large t is described by a non-perturbative phase $\delta^{np}_\omega(t)$ [7, 41]. The perturbative phase $\delta^p_\omega(t)$ for its evolution from large to small t can be calculated with the use of the saddle point approximation for the above integral representation (38) of $f_\omega(t)$

$$\lim_{t_c-t\to\infty} f_\omega(t) \approx \sqrt{2\pi}\,\left(\chi'(-\widetilde{\nu}_\omega)\right)^{-1/2}\,\sin\delta^p_\omega(t)\,, \tag{45}$$

where

$$\delta^p_\omega(t) = \frac{\pi}{4} - t\widetilde{\nu}_\omega(t) - \frac{1}{\beta_0\omega}\left(\Im\ln\frac{\Gamma(1/2+i\widetilde{\nu}_\omega(t))}{\Gamma(1/2-i\widetilde{\nu}_\omega(t))} - 2\psi(1)\widetilde{\nu}_\omega(t)\right)\,. \tag{46}$$

The quantity $\widetilde{\nu}_\omega(t) > 0$ is the position of the saddle point satisfying the equation

$$\omega\,\beta_0 t = \chi(\widetilde{\nu}_\omega(t))\,. \tag{47}$$

Accordingly we should fix the coefficient in the additional contribution ΔG_ω as follows

$$c(\omega) = \cot\phi_\omega\,,\ \ \phi_\omega = \delta^{np}_\omega(t) - \delta^p_\omega(t)\,, \tag{48}$$

where ϕ_ω does not depend on t and is similar to the phase of the S-matrix for the one-dimensional scattering problem in quantum mechanics. It contains an important information about the spectrum of the system. Indeed, the physical Green function has the poles at

$$\phi_{\omega_k} = -k\,\pi\,, \tag{49}$$

where k is an integer number. As a result, the Green function satisfies the dispersion representation (cf. Ref. [42])

$$\sqrt{\frac{t}{t'}}\,G_\omega(t,t') = \sum_{k=1}^{\infty}\frac{F_k(t)\,F_k(t')}{\omega-\omega_k}\,,\ \ F_k(t) = \sqrt{\frac{t}{4\pi\omega_k\phi'_{\omega_k}}}\,f_{\omega_k}(t)\,,\ \ \sum_{k=1}^{\infty}F_k(t)\,F_k(t') = \delta(t-t')\,. \tag{50}$$

If we introduce the non-zero gluon mass m using the Higgs mechanism to model non-perturbative effects in QCD, the Pomeron wave function should be taken at large k^2 as follows (see Ref. [43])

$$f_\omega \sim \sin\left(-\nu \ln\left(6.456\,\frac{k^2}{m^2}\right)\right)\,. \tag{51}$$

This function does not contain any additional phase depending on ν and vanishes at $k^2 = m^2/6.456$. At the lattice approach to QCD the gluon propagator can be described by the Higgs model with the vector boson mass $m \approx 540 Mev$ [44]. Therefore for $\Lambda_s \approx 200 Mev$, we have $m^2/6.456 \approx \Lambda_s^2$. The above relations allow us to simplify significantly the boundary condition for the BFKL function (38) imposing the constraint

$$f_\omega(0) = 0\,. \tag{52}$$

As a result, the Pomeron spectrum in the semi-classical approximation (cf. [7])

$$\omega_k \approx \frac{0.40862}{k-\frac{1}{4}}\,,\;\; k = 1, 2, ... \tag{53}$$

is in a good agreement with the exact spectrum obtained from relation $f_{\omega_k}(0) = 0$ for the explicit wave function $f_\omega(t)$.

The Pomeron spectrum in the Higgs model is similar in the form to that obtained from the fit of experimental data [41]. Below we consider another model for the BFKL equation at small momenta.

5 Green function at a non-zero momentum transfer

In the case of an arbitrary momentum transfer $q \neq 0$ it is convenient to use a mixed representation for the Pomeron wave function

$$\Psi(\vec{\rho}, \vec{q}) = \int d^2R\, e^{i\vec{q}\vec{R}}\,\Psi(\vec{\rho}_1, \vec{\rho}_2)\,,\;\; \vec{\rho}_1 = \vec{R} + \frac{\vec{\rho}}{2}\,,\;\; \vec{\rho}_2 = \vec{R} - \frac{\vec{\rho}}{2}\,,\;\; q = p_1 + p_2\,. \tag{54}$$

Due to the Möbius invariance of the BFKL kernel in LLA its eigenfunctions have the simple form [7]

$$\Psi_{\nu,n}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_0) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}}\,,\;\; m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\;\; \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2} \tag{55}$$

and the corresponding eigenvalues are

$$\omega(\nu, n) = \frac{g^2 N_c}{4\pi^2}\,\chi(\nu, n)\,,\;\; \chi(\nu, n) = 2\psi(1) - 2\Re\psi(\frac{1}{2} + i\nu + \frac{|n|}{2})\,. \tag{56}$$

The conformal invariance is broken in the next-to-leading approximation. Providing, that $|\rho|^{-1} \gg |q| \gg \Lambda_s$, the argument of the coupling constant is proportional to $|\rho|^{-2}$ and therefore $t \approx -\ln(|\rho|^2\Lambda_s^2)$. For such small ρ the virtuality of reggeized gluons $|k_\perp|^2$ are much larger than $q_\perp^2$ and therefore one can reduce the BFKL equation to the simple case $q = 0$. For general conformal spins $n \neq 0$ corresponding to higher twist contributions we obtain the following solution omitting the factor containing a linear combination of the azimuthal angle phases $\exp(\pm in\theta)$ (cf. (38))

$$f_\omega(t, n) = \int_{-\infty}^{\infty} d\nu\, e^{it\nu}\, g_\omega(\nu, n)\,,\;\; g_\omega(\nu, n) = \left(\frac{\Gamma(\frac{1}{2} + i\nu + \frac{|n|}{2})}{\Gamma(\frac{1}{2} - i\nu + \frac{|n|}{2})}\, e^{-2i\nu\psi(1)}\right)^{\frac{1}{\beta_0\omega}}\,. \tag{57}$$

The perturbative phase $\delta^p_\omega(t,n)$ of the wave function below the turning point t_c can be calculated with the use of the saddle point approximation for this integral

$$\lim_{t_c - t\to\infty} f_\omega(t,n) \approx \sqrt{2\pi}\,\left|\chi'(\widetilde{\nu}_\omega(t,n))\right|^{-1/2}\,\sin\delta^p_\omega(t,n)\,, \tag{58}$$

where

$$\delta^p_\omega(t,n) = \frac{\pi}{4} - t\widetilde{\nu}_\omega(t,n) - \frac{1}{\beta_0\omega}\left(\Im\ln\frac{\Gamma(1/2+i\widetilde{\nu}_\omega(t,n)+|n|/2)}{\Gamma(1/2-i\widetilde{\nu}_\omega(t,n)+|n|/2)} - 2\psi(1)\widetilde{\nu}_\omega(t,n)\right). \tag{59}$$

The quantity $\widetilde{\nu}_\omega(t,n)$ is the position of the saddle point satisfying the equation

$$\omega\,\beta_0 t = \chi(\widetilde{\nu}_\omega(t,n),n)\,. \tag{60}$$

Note, that the phase $\delta^p_\omega(t,n)$ contains the pole $1/\omega$, which leads to the Pomeron spectrum of the type of after quantization (65).

A particular solution of the BFKL equation for the Green function in the region of small ρ and ρ' is given below (cf. eq. (40))

$$G^0_\omega(t,\theta;t',\theta') = -\frac{it'}{\omega}\sum_{n=-\infty}^{\infty}\frac{e^{in(\theta-\theta')}}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,e^{it\nu}\,g_\omega(\nu,n)\int_{-\infty}^{\infty}\frac{d\nu'}{2}\epsilon(\nu+\nu')\,e^{it'\nu'}\,g_\omega(\nu',n)\,, \tag{61}$$

where θ and θ' are azimuthal angles of vectors $\vec{\rho}$ and $\vec{\rho}'$. It satisfies the relation

$$\lim_{\omega\to\infty} G^0_\omega(t,\theta;t',\theta') = \frac{1}{\omega}\,\delta(t-t')\,\delta(\theta-\theta')\,, \tag{62}$$

leading to a correct asymptotic behavior for the evolution operator at the small relative rapidity y.

The physical Green function having the oscillatory behavior at $t_c - t\to\infty$ (or at $t_c - t'\to\infty$) with the phase compatible to the boundary condition at small t (t') is given by the expression

$$G_\omega(t,\theta;t',\theta') = G^0_\omega(t,\theta;t',\theta') + \frac{t'}{4\pi\omega}\sum_{n=-\infty}^{\infty}\frac{e^{in(\theta-\theta')}}{2\pi}\cot\phi_\omega(n)\,f_\omega(t,n)\,f_\omega(t',n)\,, \tag{63}$$

where the parameter $\phi_\omega(n)$ is equal to the difference of the non-perturbative and perturbative phases of the wave functions for various values of n

$$\phi_\omega(n) = \delta^{np}_\omega(t,n) - \delta^p_\omega(t,n)\,. \tag{64}$$

The Green function contains the poles at $\omega = \omega_k(n)$ being solutions of the equation.

$$\phi_{\omega_k(n)}(n) = k\,\pi \tag{65}$$

at integer k and therefore it can be presented in the form (cf. eq. (50))

$$\sqrt{\frac{t'}{t}}\,G_\omega(t,\theta;t',\theta') = \sum_{n=-\infty}^{\infty}\frac{e^{in(\theta-\theta')}}{2\pi}\sum_{k=1}^{\infty}\frac{F_k(t,n)\,F_k(t',n)}{\omega-\omega_k(n)}\,, \tag{66}$$

where

$$F_k(t,n) = \sqrt{\frac{t\,\phi'_{\omega_k(n)}}{4\pi\omega_k(n)}}\,f_{\omega_k(n)}(t,n)\,. \tag{67}$$

Let us consider the region $q^2 \gg \Lambda_s^2$, where we can calculate $\phi_\omega(n)$ in the perturbation theory due to the asymptotic freedom. The asymptotic behavior of the eigenfunction $\Psi(\vec{\rho},\vec{q})$ (54) at small $|\rho||q| \ll 1$ corresponding to the Möbius invariant solution (55) is given below [7]

$$\lim_{|\rho|\to 0}\Psi_{\nu,n}(\vec{\rho},\vec{q}) \sim \cos\left(\gamma_{\nu,n}(\vec{\rho},\vec{q}) + \beta_n(\vec{\rho},\vec{q})\right)\,, \tag{68}$$

where

$$\gamma_{\nu,n}(\vec{\rho},\vec{q}) = 2\nu \ln \frac{|q\rho|}{4} + \frac{1}{2i} \ln \frac{\Gamma(1-i\nu+\frac{n}{2})\Gamma(1-i\nu-\frac{n}{2})}{\Gamma(1+i\nu+\frac{n}{2})\Gamma(1+i\nu-\frac{n}{2})}\,,\ \beta_n(\vec{\rho},\vec{q}) = \frac{n}{2i}\ln(\frac{-\rho q^*}{\rho^* q}). \tag{69}$$

We should consider two functions $\Psi^{\pm}_{\nu,n}$ being respectively even $(+)$ and odd $(-)$ under the substitution $n \to -n$ to obtain the factorized expressions

$$\lim_{|\rho|\to 0} \Psi^{+}_{\nu,n}(\vec{\rho},\vec{q}) \sim \cos\gamma_{\nu,n}(\vec{\rho},\vec{q})\,\cos\beta_n(\vec{\rho},\vec{q})\,,\ \lim_{|\rho|\to 0} \Psi^{-}_{\nu,n}(\vec{\rho},\vec{q}) \sim \sin\gamma_{\nu,n}(\vec{\rho},\vec{q})\,\sin\beta_n(\vec{\rho},\vec{q})\,. \tag{70}$$

Thus, by matching these results at $t = \ln|q|^2/\Lambda_s^2$ with eigenfunction (57) multiplied by corresponding factors depending on $\beta_n(\vec{\rho},\vec{q})$ we obtain the following equations describing the spectrum of the Regge trajectories $\omega = \omega_k^{\pm}(n,|q|^2)$ (cf. Ref. [7])

$$\frac{\pi}{2} - \gamma_{\widetilde{\nu}_\omega,n}(\vec{\rho},\vec{q}) - \delta^p_{\omega_k^+(n,|q|^2)}(t,n) = k\pi\,,\ \gamma_{\widetilde{\nu}_\omega,n}(\vec{\rho},\vec{q}) - \delta^p_{\omega_k^-(n,|q|^2)}(t,n) = k\pi. \tag{71}$$

Here $\widetilde{\nu}_\omega$ is determined from eq. (60).

6 BFKL Pomeron at a thermostat

One of possible models for the non-perturbative BFKL dynamics is related to the introduction of a t-channel temperature $T \neq 0$ [45]. It is natural to expect, that the Pomeron melts in such a thermostat because the confining potential existing between two gluons disappears at large T. But the situation turns out to be more complicated. Namely, in Ref. [46] it was discovered, that sometimes the non-zero temperature causes an opposite effect. Namely, the positions of the Regge poles $\omega(t)$ can increase with the temperature [46]. The explanation of this phenomenon is related to the fact, that the temperature effects for the correlation functions in the t-channel are introduced by imposing the periodicity of them in the euclidian time. After their analytic continuation to the s-channel this constraint leads to the periodicity of the Pomeron wave functions in one of two transverse gluon coordinates $\vec{\rho}$. If this compactification appears in the direction orthogonal to the momentum transfer $\vec{q}$, topologically the impact parameter space $\vec{\rho}$ becomes a cylinder with its axes situated along the line connecting two gluons. In this case the color electric field is compressed to a string between the gluons, which leads to a confining force increasing the Pomeron intercept [46].

Let us consider this phenomenon in more details. According to Ref. [45] the wave function of the Pomeron at the thermostat in LLA can be obtained from expression (55) by the conformal transformation of gluon coordinates

$$\rho \to \exp(2\pi T\,\rho)\,,\ \rho^* \to \exp(2\pi T\,\rho^*)\,, \tag{72}$$

where T is the t-channel temperature. In complex coordinates the modified expression is periodic in $\Im\rho$ and its region of definition is restricted to the strip

$$0 < \Im\rho < T^{-1}\,. \tag{73}$$

With the use of the mixed representation (54) in the limit $1/|\rho| \gg |q|$ one can obtain the asymptotic result similar to (68) at $T = 0$ [46]. After its symmetrization (or anti-symmetrization) under the transformation $n \to -n$ we have (cf. (70) for $T = 0$)

$$\lim_{|\rho|\to 0} \Psi^{+}_{\nu,n}(\vec{\rho},\vec{q}) \sim \cos\gamma^T_{\nu,n}(\vec{\rho},\vec{q})\,\cos\beta^T_n(\vec{\rho},\vec{q})\,,\ \lim_{|\rho|\to 0} \Psi^{-}_{\nu,n}(\vec{\rho},\vec{q}) \sim \sin\gamma^T_{\nu,n}(\vec{\rho},\vec{q})\,\sin\beta^T_n(\vec{\rho},\vec{q})\,, \tag{74}$$

where

$$\gamma^T_{\nu,n}(\vec{\rho},\vec{q}) = 2\nu\ln\frac{2\pi T|\rho|}{4} + \gamma^{\vec{q};T}_{\nu,n}\,,\ \beta^T_n(\vec{\rho},\vec{q}) = \frac{n}{2i}\ln\frac{\rho}{\rho^*} \tag{75}$$

and

$$\gamma_{\nu,n}^{\vec{q},T}=\frac{1}{2i}\ln\frac{\Gamma(1-i\nu+\frac{n}{2})\Gamma(1-i\nu-\frac{n}{2})\Gamma(i\nu+\frac{1+n}{2}+i\frac{q}{2\pi T})\Gamma(i\nu+\frac{1-n}{2}-i\frac{q^*}{2\pi T})}{\Gamma(1+i\nu+\frac{n}{2})\Gamma(1+i\nu-\frac{n}{2})\Gamma(-i\nu+\frac{1-n}{2}+i\frac{q}{2\pi T})\Gamma(-i\nu+\frac{1+n}{2}-i\frac{q^*}{2\pi T})}\,. \tag{76}$$

Assuming that $q_y=0$ and matching these results at $t=\ln|q|^2/\Lambda_s^2$ with eigenfunction (57) multiplied by corresponding factors depending on $\beta_n^T(\vec{\rho},\vec{q})$ we obtain the following equations describing the spectrum of the Regge trajectories $\omega=\omega_k^{\pm}(n,|q|^2)$

$$\frac{\pi}{2}-\gamma^T_{\widetilde{\nu}_\omega,n}(\vec{\rho},\vec{q})-\delta^p_{\omega_k^+(n,|q|^2)}(t,n)=k\pi\,,\;\gamma^T_{\widetilde{\nu}_\omega,n}(\vec{\rho},\vec{q})-\delta^p_{\omega_k^-(n,|q|^2)}(t,n)=k\pi. \tag{77}$$

Here $\widetilde{\nu}_\omega$ is calculated from eq.(60). One can verify, that the leading Regge trajectory $\omega(q^2,T)$ at $n=0$ increases with temperature at some interval of T [46]. In QCD the parameter T^{-1} should be of the order of Λ_s^{-1}. It plays role of the radius of a bag containing two reggeized gluons. The bag is surrounded by the physical vacuum in which the gluons and their color electric field can not penetrate due to the dual Meissner effect. It is assumed usually, that the physical vacuum arises as a result of the monopols condensation. For comparatively large $|q|$ the cylinder-type configuration for the bag of the bare vacuum in the physical vacuum looks natural and leads to the confining potential between gluons. But with a decrease of $|q|$ one can expect a similar periodicity of the Pomeron wave functions also in the direction along $\vec{q}$ leading to a torus-type configuration of the impact parameter space.

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