Construction of the NSVZ scheme for Abelian supersymmetric theories, regularized by higher derivatives

A. L. Kataev[a*], K. V. Stepanyantz[b†]

[a] *Institute for Nuclear Research of the Russian Academy of Sciences*
60-letiya Oktyabrya prospekt 7a, Moscow 117312, Russia
[b] *Moscow State University, Physical Faculty*
Leninskie Gory, Moscow 119991, Russia

Abstract

We construct in all loops a renormalization prescription giving the NSVZ scheme (in which the β-function coincides with the NSVZ expression relating the β-function with the anomalous dimension of the matter superfields) for Abelian supersymmetric theories regularized by higher derivatives. The NSVZ scheme for the renormalization group functions defined in terms of the renormalized coupling constant is obtained in this case by imposing some simple boundary conditions on the renormalization constants. The renormalization group functions defined in terms of the bare coupling constant satisfy the NSVZ relation for an arbitrary renormalization prescription, if the higher derivatives are used for regularization.

1 Introduction

The NSVZ β-function [1, 2, 3, 4] is an equation which relates the β-function of $\mathcal{N} = 1$ supersymmetric theories to the anomalous dimensions of the matter superfields:

$$\beta(\alpha) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)}{2\pi(1 - C_2\alpha/2\pi)}, \qquad (1)$$

where

$$\begin{aligned} &\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad && (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;\\ &f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad && r \equiv \delta_{AA}. \end{aligned} \qquad (2)$$

The NSVZ β-function was constructed using some general arguments: structure of instanton contributions [1, 3] (see Ref. [5] for a recent review), structure of the anomaly supermultiplet [2, 4, 6], non-renormalization of the topological term [7].

Here we pay especial attention to the Abelian case, namely, to the $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) with N_f flavors. In the massless case this theory is described by the action

* **e-mail**: kataev@ms2.inr.ac.ru
† **e-mail**: stepan@phys.msu.ru

$$S = \frac{1}{4e_0^2}\mbox{Re}\int d^4x\, d^2\theta\, W^a W_a + \sum\limits_{i=\alpha}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\phi_\alpha^* e^{2V}\phi_\alpha + \widetilde\phi_\alpha^* e^{-2V}\widetilde\phi_\alpha\Big), \qquad (3)$$

where V is a real gauge superfield, ϕ_α and $\widetilde\phi_\alpha$ with $\alpha = 1,\ldots,N_f$ are chiral matter superfields, and (in the Abelian case) $W_a = \bar D^2 D_a V/4$. For this theory

$$C_2 = 0;\qquad C(R) = I;\qquad T(R) = 2N_f\qquad r = 1, \qquad (4)$$

where I is the $2N_f\times 2N_f$ unit matrix. This implies that for $\mathcal{N}=1$ SQED with N_f flavors the NSVZ β-function has the form [8, 9]

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big). \qquad (5)$$

The NSVZ β-function (obtained from the formal arguments) can be compared with the results of explicit calculations in the lowest orders, if a certain regularization and a certain renormalization prescription are chosen. Although the dimensional technique is the most popular regularization, one encounters problems applying it to supersymmetric theories [10]. The dimensional regularization [11, 12, 13, 14] breaks the supersymmetry [15] and is not convenient for calculations in supersymmetric theories. Its modification, called the dimensional reduction [16], is either mathematically inconsistent [17], or is not manifestly supersymmetric [18] that can lead to supersymmetry breaking in higher loops [19, 20]. However, using the dimensional reduction and the $\overline{\mbox{DR}}$-scheme the β-function for general $\mathcal{N}=1$ supersymmetric theories was calculated up to the four-loop approximation [21, 22, 23, 24], see [25] for a recent review. The results agree with the NSVZ β-function only in the one- and two-loop approximations. In the higher loops it is necessary to make a special tuning of the coupling constant [22, 26]. Some other regularization techniques are mostly used in the one- and two-loop approximations [27, 28], where the problems related to the scheme dependence are not essential.

It appears that a very convenient regularization for supersymmetric theories is the higher covariant derivative regularization [29, 30]. It is a consistent regularization, which does not break supersymmetry in the supersymmetric case [31, 32]. This regularization can be also constructed for $\mathcal{N}=2$ supersymmetric theories [33, 34]. In order to regularize a theory by higher derivatives it is necessary to add a term with higher degrees of covariant derivatives. Then divergences remain only in the one-loop approximation [35]. These remaining divergences are regularized by inserting the Pauli–Villars determinants [36] into the generating functional.

Quantum corrections calculated with the higher derivative regularization in $\mathcal{N}=1$ supersymmetric theories have an interesting feature: integrals giving the β-function defined in terms of the bare coupling constant are integrals of (double) total derivatives [37, 38]. One of these integrals can be calculated analytically giving the NSVZ relation. In the Abelian case this was proved exactly in all loops [39, 40].

However, the ordinary renormalization group (RG) functions defined in terms of the renormalized coupling constant [41] are scheme dependent. For these RG functions the NSVZ relation appears only with a certain renormalization prescription, which is called the NSVZ scheme. In the case of using the dimensional reduction the NSVZ scheme can be constructed by a special choice of finite renormalization which should be made in each order. The higher derivative regularization allows to find a simple prescription which gives the NSVZ scheme exactly in all orders [42]. In this paper we describe how this can be made.

The paper is organized as follows: In section 2 we introduce the higher derivative regularization for $\mathcal{N}=1$ SQED with N_f flavors. Then in section 3 we explain why the NSVZ relation arises for the RG functions defined in terms of the bare coupling constant in the case of using this regularization. The NSVZ scheme for the RG functions defined in terms of the renormalized coupling constant is formulated in section 4. The result is verified by the explicit three-loop calculation in section 5. Scheme dependence of the NSVZ relation and its scheme-idenpendent consequences are discussed in section 6.

2 $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives

In order to regularize the theory by higher derivatives it is necessary to add the higher derivative term to the action:

$$S_{\rm reg} = \frac{1}{4e_0^2}\mbox{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a + \sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_\alpha^* e^{2V}\phi_\alpha + \widetilde\phi_\alpha^* e^{-2V}\widetilde\phi_\alpha\Big), \qquad (6)$$

where $(R(\partial^2/\Lambda^2)-1)$ is a regulator. For example, one can choose $R = 1+\partial^{2n}/\Lambda^{2n}$.

Adding the higher derivative term allows to remove all divergences beyond the one-loop approximation. In order to remove one-loop divergencies we insert in the generating functional the Pauli–Villars determinants [36]:

$$Z[J,\mathbf{\Omega}] = \int D\mu \prod\limits_I \Big(\det PV(V,M_I)\Big)^{N_f c_I} \exp\Big\{iS_{\rm reg} + iS_{\rm gf} + S_{\rm Sources}\Big\}, \qquad (7)$$

where the coefficients c_I are restricted by the following constrains:

$$\sum\limits_I c_I = 1;\qquad \sum\limits_I c_I M_I^2 = 0 \qquad (8)$$

needed for canceling the remaining one-loop divergences. It is essential that the Pauli–Villars masses are proportional to the parameter Λ in the higher derivative term, the ratios being independent on the bare coupling constant:

$$M_I = a_I\Lambda,\qquad a_I \ne a_I(e_0). \qquad (9)$$

A part of the effective action corresponding to the two-point Green functions can be written in the form

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta\,\Big(- \frac{1}{16\pi} V(-p,\theta)\,\partial^2\Pi_{1/2} V(p,\theta)\, d^{-1}(\alpha_0,\Lambda/p)$$
$$+\frac{1}{4}\sum_{\alpha=1}^{N_f}\Big(\phi_\alpha^*(-p,\theta)\phi_\alpha(p,\theta) + \widetilde\phi_\alpha^*(-p,\theta)\widetilde\phi_\alpha(p,\theta)\Big)\, G(\alpha_0,\Lambda/p)\Big), \qquad (10)$$

where $\partial^2\Pi_{1/2}$ denotes a supersymmetric transversal projection operator. After calculating these functions in a certain order, we construct the renormalized coupling constant $\alpha(\alpha_0,\Lambda/\mu)$, requiring finiteness of the inverse invariant charge $d^{-1}(\alpha_0(\alpha,\Lambda/\mu),\Lambda/p)$ in the limit $\Lambda\to\infty$. Then the renormalization constant Z_3 is given by

$$\frac{1}{\alpha_0} \equiv \frac{Z_3(\alpha,\Lambda/\mu)}{\alpha}. \qquad (11)$$

The renormalization constant Z is constructed, requiring that the renormalized two-point Green function ZG is finite in the limit $\Lambda\to\infty$:

$$G_{\rm ren}(\alpha,\mu/p) = \lim\limits_{\Lambda\to\infty} Z(\alpha,\Lambda/\mu) G(\alpha_0,\Lambda/p). \qquad (12)$$

3 Why the NSVZ relation is satisfied by the RG functions defined in terms of the bare coupling constant

In most original papers [1, 3, 8, 9] the NSVZ β-function was constructed for the RG functions defined in terms of the bare coupling constant

$$\beta\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv \frac{d\alpha_0(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}; \qquad \gamma_i{}^j\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv -\frac{d\ln Z_i{}^j(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}. \tag{13}$$

These RG functions

1. are scheme independent for a fixed regularization;
2. depend on the regularization;
3. in all loops satisfy the NSVZ relation in the case of $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives.

The independence of the renormalization prescription follows from the possibility of expressing them via unrenormalized Green functions, e.g.,

$$0 = \lim_{p\to 0}\frac{dd^{-1}(\alpha_0,\Lambda/p)}{d\ln\Lambda}\Big|_{\alpha=\text{const}} = \lim_{p\to 0}\Big(\frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\ln p}\Big), \tag{14}$$

where in the last equality α_0 and p are considered as independent variables. Similarly, differentiating

$$\ln G(\alpha_0,\Lambda/q) = \ln G_{\text{ren}}(\alpha,\mu/q) - \ln Z(\alpha,\Lambda/\mu) \\ +(\text{terms vanishing in the limit } q\to 0) \tag{15}$$

with respect to $\ln\Lambda$ at a fixed value of α, in the limit $q\to 0$ we obtain

$$\gamma(\alpha_0) = \lim_{q\to 0}\Big(\frac{\partial\ln G(\alpha_0,\Lambda/q)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial\ln G(\alpha_0,\Lambda/q)}{\partial\ln q}\Big). \tag{16}$$

Therefore, $\beta(\alpha_0)$ and $\gamma(\alpha_0)$ do not depend on an arbitrariness of choosing the renormalization constants.

In the case of using the higher derivative regularization for Abelian supersymmetric theories the RG functions defined in terms of the bare coupling constant satisfy the NSVZ relation in all loops [39, 40]. This follows from the fact that with the higher covariant derivative regularization loop integrals giving the β-function defined in terms of the bare coupling constant are integrals of total derivatives [37] and even integrals of double total derivatives [38]. As a consequence, one of the loop integrals can be calculated analytically. This gives the NSVZ relation for the RG functions defined in terms of the bare coupling constant. It was also proved for Abelian theories in all loops in Refs. [39, 40]:

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} \\ = \frac{N_f}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big). \tag{17}$$

In the non-Abelian case the similar calculations have been done only in the two-loop approximation and reveal the same features [43, 44, 45, 46, 47].

In order to illustrate the factorization of integrals for the β-function defined in terms of the bare coupling constant into integrals of double total derivatives we present the result of the three-loop calculation for $\mathcal{N}=1$ SQED with N_f flavors:

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = 2\pi N_f \frac{d}{d\ln\Lambda}\Big\{ \sum_I c_I \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu} \frac{\ln(q^2+M^2)}{q^2} + 4\pi \int \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{e^2}{k^2R_k^2}$$

$$\times \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big(\frac{1}{q^2(k+q)^2} - \sum_I c_I \frac{1}{(q^2+M_I^2)((k+q)^2+M_I^2)}\Big)\Big[R_k\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$

$$-2e^2N_f\bigg(\int\frac{d^4t}{(2\pi)^4}\,\frac{1}{t^2(k+t)^2} - \sum_J c_J \int \frac{d^4t}{(2\pi)^4}\frac{1}{(t^2+M_J^2)((k+t)^2+M_J^2)}\bigg)\Big]$$

$$+4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{e^4}{k^2R_kl^2R_l}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big\{\Big(-\frac{2k^2}{q^2(q+k)^2(q+l)^2(q+k+l)^2}$$

$$+\frac{2}{q^2(q+k)^2(q+l)^2}\Big) - \sum_I c_I\Big(-\frac{2(k^2+M_I^2)}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)}$$

$$\times\frac{1}{((q+k+l)^2+M_I^2)} + \frac{2}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)} - \frac{1}{(q^2+M_I^2)^2}$$

$$\times\frac{4M_I^2}{((q+k)^2+M_I^2)((q+l)^2+M_I^2)}\Big) + O(e^6)\Big\}. \qquad (18)$$

Taking these integrals of the double total derivatives we obtain the $(\alpha_0)^0$, $(\alpha_0)^1$, and $(\alpha_0)^2$ terms of the relation

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big), \qquad (19)$$

where the two-loop Green function of the matter superfields is given by

$$G(\alpha_0,\Lambda/p) = 1 - \int\frac{d^4k}{(2\pi)^4}\frac{2e_0^2}{k^2R_k(k+p)^2} + \int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4}{k^2R_kl^2R_l}\bigg(\frac{1}{(k+p)^2(l+p)^2}$$

$$+\frac{1}{(l+p)^2(k+l+p)^2} - \frac{(k+l+2p)^2}{(k+p)^2(l+p)^2(k+l+p)^2}\bigg) + \int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4N_f}{k^2R_k^2(k+p)^2}$$

$$\times\bigg(\frac{1}{l^2(k+l)^2} - \sum_{I=1}^n c_I\frac{1}{(l^2+M_I^2)\,((k+l)^2+M_I^2)}\bigg) + O(e_0^6), \qquad (20)$$

4 The NSVZ scheme for the RG functions defined in terms of the renormalized coupling constant

RG function defined in terms of the bare coupling constant are scheme-independent for a fixed regularization. However, RG functions are usually defined by a different way, in terms of the renormalized coupling constant:

$$\widetilde\beta\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\alpha(\alpha_0,\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad (21)$$

$$\widetilde\gamma_i{}^j\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\ln Z_i{}^j(\alpha(\alpha_0,\Lambda/\mu),\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}. \qquad (22)$$

These RG functions are scheme-dependent. It is possible to prove [42, 48] that they coincide with the RG functions defined in terms of the bare coupling constant, if the boundary conditions

$$Z_3(\alpha, x_0) = 1; \qquad Z_i{}^j(\alpha, x_0) = 1 \tag{23}$$

are imposed on the renormalization constants, where x_0 is an arbitrary fixed value of $\ln\Lambda/\mu$. Really, for example, the anomalous dimension (22) can be presented as

$$\widetilde{\gamma}\left(\alpha(\alpha_0, x)\right) = -\frac{d\ln Z\left(\alpha(\alpha_0, x), x\right)}{dx} = -\frac{\partial \ln Z(\alpha, x)}{\partial\alpha}\cdot\frac{\partial\alpha(\alpha_0, x)}{\partial x} - \frac{\partial \ln Z\left(\alpha(\alpha_0, x), x\right)}{\partial x}, \tag{24}$$

where the total derivative with respect to $x = \ln\Lambda/\mu$ also acts on x inside α. Calculating these expressions at the point $x = x_0$ and taking into account that $\partial\ln Z(\alpha, x_0)/\partial\alpha = 0$ we obtain $\widetilde{\gamma}(\alpha_0) = \gamma(\alpha_0)$. The similar equality for the β-functions can be proved in the same way.

The RG functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ (defined in terms of the renormalized coupling constant) are scheme-dependent. They satisfy the NSVZ relation only in a certain subtraction scheme, called the NSVZ scheme, which is evidently fixed in all loops by the boundary conditions (23) if the theory is regularized by higher derivatives. (This is so, because the functions β and γ satisfy the NSVZ relation in the case of using this regularization.)

5 NSVZ and other schemes in the three-loop approximation

For $R_k = 1 + k^{2n}/\Lambda^{2n}$ it is possible to find the expressions for the divergent parts of the function d^{-1} in the three-loop approximation and of the function $\ln G$ in the two-loop approximation, and construct the corresponding renormalization constants. Instead of the expression for Z_3, it is more convenient to write the result for the renormalized coupling constant. In the three-loop approximation it is given by

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big) - \frac{\alpha^2 N_f}{\pi^3}\Big(\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu}$$
$$-\ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} - N_f b_1\Big) + b_3\Big) + O(\alpha^3), \tag{25}$$

where b_i are arbitrary finite constants. Similarly, the renormalization constant Z (in the two-loop approximation) for the matter superfields is given by

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{\alpha^2(N_f+1)}{2\pi^2}\ln^2\frac{\Lambda}{\mu}$$
$$-\frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I\ln a_I - N_f b_1 + N_f + \frac{1}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3), \tag{26}$$

where g_i are other arbitrary finite constants. The subtraction scheme is fixed by fixing values of these constants b_i and g_i.

The RG functions defined in terms of the bare coupling constant calculated on the base of Eqs. (25) and (26) have the form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^{n} c_I\ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3); \tag{27}$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(N_f\sum_{I=1}^{n} c_I\ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3). \tag{28}$$

We see that these RG functions in the considered approximation do not depend on the finite constants b_i and g_i (i.e. they are scheme-independent) and satisfy the NSVZ relation.

The RG functions defined in terms of the renormalized coupling constant are given by

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} + N_f(b_2 - b_1)\Big) + O(\alpha^3); \qquad (29)$$

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I - N_f b_1 + N_f g_1\Big) + O(\alpha^3) \qquad (30)$$

and depend on a subtraction scheme.

The NSVZ scheme (with the higher derivative regularization) is determined by the conditions (23). For simplicity we set $g_1 = 0$ (this constant can be excluded by a redefinition of μ). In this case $x_0 = 0$ and the conditions (23) give $g_2 = b_1 = b_2 = b_3 = 0$. Then in the considered approximations

$$\widetilde{\gamma}_{\text{NSVZ}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3) = \gamma(\alpha); \qquad (31)$$

$$\widetilde{\beta}_{\text{NSVZ}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3)\Big) = \beta(\alpha). \qquad (32)$$

As a consequence, in this scheme the NSVZ relation is satisfied.

Let us also present the results of similar calculations in other subtraction schemes. In the MOM scheme the results obtained with the dimensional reduction and with the higher derivative regularization coincide and have the form

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3); \qquad (33)$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\left(1 - \zeta(3)\right)\Big) + O(\alpha^3)\Big). \qquad (34)$$

In the $\overline{\text{DR}}$-scheme the result of Ref. [22] (in the notation used in this paper) can be written as

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3); \qquad (35)$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big). \qquad (36)$$

From the above expressions for the RG functions we see that in the considered approximations only terms proportional to $(N_f)^2\alpha^4$ in the β-function and to $N_f\alpha^3$ in the anomalous dimension are scheme dependent. The other terms are same in all schemes.

6 Finite renormalizations and the NSVZ relation

Different subtraction schemes can be related by finite renormalizations

$$\alpha \to \alpha'(\alpha); \qquad Z'(\alpha', \Lambda/\mu) = z(\alpha) Z(\alpha, \Lambda/\mu). \qquad (37)$$

Under such a finite renormalization the β-function and the anomalous dimension defined in terms of the renormalized coupling constant are changed:

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\widetilde{\beta}(\alpha); \qquad \widetilde{\gamma}'(\alpha') = \frac{d\ln z}{d\alpha}\cdot\widetilde{\beta}(\alpha) + \widetilde{\gamma}(\alpha). \tag{38}$$

Using these equations one can see [48] that if $\widetilde{\beta}(\alpha)$ and $\widetilde{\gamma}(\alpha)$ satisfy the NSVZ relation, then

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2 N_f}{\pi}\frac{1-\widetilde{\gamma}'(\alpha')}{1-\alpha^2 N_f (d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}. \tag{39}$$

Quantum corrections to the coupling constant are produced by diagrams which contain at least one loop of the matter superfields, which is proportional to N_f. Thus, it is reasonable to make finite renormalizations of the coupling constant of the first order in N_f:

$$\alpha'(\alpha) - \alpha = O(N_f); \qquad z(\alpha) = O\left((N_f)^0\right). \tag{40}$$

This implies that all scheme dependent terms in the β-function are proportional at least to $(N_f)^2$ in all loops. Moreover, it is evident that the terms proportional to $(N_f)^0$ in the anomalous dimension do not depend on a subtraction scheme. Therefore, due to existence of the NSVZ scheme, the NSVZ relation is satisfied for terms proportional to $(N_f)^1$ in all orders, while terms proportional to $(N_f)^\alpha$ with $\alpha \geq 2$ are scheme-dependent.

In the non-Abelian case one should take into account existence of the Yukawa couplings λ^{ijk}. Under the finite renormalizations

$$\alpha \to \alpha'(\alpha,\lambda); \quad \lambda \to \lambda'(\alpha,\lambda); \quad {Z'}_i{}^j(\alpha',\lambda',\Lambda/\mu) = z_i{}^k(\alpha,\lambda) Z_k{}^j(\alpha,\lambda,\Lambda/\mu), \tag{41}$$

where we assume that z and Z commute, the NSVZ relation is changed according to the following rule:

$$\begin{aligned}
\widetilde{\beta}'(\alpha',\lambda') = &-\frac{\alpha^2}{2\pi(1-C_2\alpha/2\pi)\partial\alpha/\partial\alpha' - \alpha^2 C(R)_l{}^k \partial\ln z_k{}^l/\partial\ln\alpha'}\Big\{3C_2 \\
&-T(R) + \frac{1}{r}C(R)_m{}^n\Big[\widetilde{\gamma}'_n{}^m(\alpha',\lambda') - \frac{3}{2}\Big((\lambda')^{ljk}\,\widetilde{\gamma}'_l{}^i(\alpha',\lambda')\,\frac{\partial\ln z_n{}^m}{\partial(\lambda')^{ijk}} + \text{c.c.}\Big)\Big] \\
&+\frac{3}{2}\cdot\frac{2\pi}{\alpha^2}\Big(1 - C_2\frac{\alpha}{2\pi}\Big)\Big((\lambda')^{ljk}\,\widetilde{\gamma}'_l{}^i(\alpha',\lambda')\,\frac{\partial\alpha}{\partial(\lambda')^{ijk}} + \text{c.c}\Big)\Big\}_{\alpha=\alpha(\alpha',\lambda')}.
\end{aligned} \tag{42}$$

We observe that in L loops the terms proportional to $\text{tr}\left(C(R)^L\right)$ are the same in both sides of this equation for an arbitrary renormalization prescription [49].

7 Conclusion

For $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives, the NSVZ β-function is naturally obtained for the RG functions defined in terms of the bare coupling constant. These functions do not depend on the renormalization prescription. The NSVZ β-function appears because integrals which determine the β-function defined in terms of the bare coupling constant are factorized into integrals of double total derivatives.

If the RG functions are defined in terms of the renormalized coupling constant, the NSVZ β-function is obtained in a special subtraction scheme, called the NSVZ scheme. In case of using the higher derivative regularization for Abelian supersymmetric theories this scheme is obtained by imposing the boundary conditions (23) in all orders.

Although the NSVZ relation is not valid for an arbitrary renormalization prescription, it is possible to prove that terms proportional to $(N_f)^1$ (or proportional to $\text{tr}\, C(R)^L$ in L loops in the non-Abelian case) are scheme independent and satisfy the NSVZ relation in all schemes.

Acknowledgements.

The work of A.K. was supported by the RFBR grant No 14-01-00647. The work of K.S. was supported by the RFBR grant No 14-01-00695.

References

[1] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983) 381.

[2] D. R. T. Jones, Phys. Lett. B **123** (1983) 45.

[3] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Lett. B **166** (1986) 329; Sov. J. Nucl. Phys. **43** (1986) 294; [Yad. Fiz. **43** (1986) 459.]

[4] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986) 456; Sov. Phys. JETP **64** (1986) 428; [Zh. Eksp. Teor. Fiz. **91** (1986) 723.]

[5] M. A. Shifman and A. I. Vainshtein, In *Shifman, M.A.: ITEP lectures on particle physics and field theory, vol. 2* 485-647 [hep-th/9902018].

[6] N. Arkani-Hamed and H. Murayama, JHEP **0006** (2000) 030.

[7] E. Kraus, C. Rupp and K. Sibold, Nucl. Phys. B **661** (2003) 83.

[8] A. I. Vainshtein, V. I. Zakharov and M. A. Shifman, JETP Lett. **42** (1985) 224 [Pisma Zh. Eksp. Teor. Fiz. **42** (1985) 182].

[9] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Lett. B **166** (1986) 334.

[10] I. Jack and D. R. T. Jones, Adv. Ser. Direct. High Energy Phys. **21** (2010) 494.

[11] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972) 189.

[12] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12** (1972) 20.

[13] J. F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289.

[14] G. M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

[15] R. Delbourgo and V. B. Prasad, J. Phys. G **1** (1975) 377.

[16] W. Siegel, Phys. Lett. B **84** (1979) 193.

[17] W. Siegel, Phys. Lett. B **94** (1980) 37.

[18] L. V. Avdeev, G. A. Chochia and A. A. Vladimirov, Phys. Lett. B **105** (1981) 272.

[19] L. V. Avdeev, Phys. Lett. B **117** (1982) 317.

[20] L. V. Avdeev and A. A. Vladimirov, Nucl. Phys. B **219** (1983) 262.

[21] L. V. Avdeev and O. V. Tarasov, Phys. Lett. B **112** (1982) 356.

[22] I. Jack, D. R. T. Jones and C. G. North, Phys. Lett. B **386** (1996) 138.

[23] I. Jack, D. R. T. Jones and C. G. North, Nucl. Phys. B **486** (1997) 479.

[24] R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP **0612** (2006) 024.

[25] L. Mihaila, Adv. High Energy Phys. **2013** (2013) 607807.

[26] I. Jack, D. R. T. Jones and A. Pickering, Phys. Lett. B **435** (1998) 61.

[27] M. A. Shifman and A. I. Vainshtein, Sov. J. Nucl. Phys. **44** (1986) 321 [Yad. Fiz. **44** (1986) 498].

[28] J. Mas, M. Perez-Victoria and C. Seijas, JHEP **0203** (2002) 049 [hep-th/0202082].

[29] A. A. Slavnov, Nucl. Phys. B **31** (1971) 301.

[30] A. A. Slavnov, Theor.Math.Phys. **13** (1972) 1064 [Teor. Mat. Fiz. **13** (1972) 174].

[31] V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978) 745 [Teor. Mat. Fiz. **36** (1978) 291].

[32] P. C. West, Nucl. Phys. B **268** (1986) 113.

[33] V. K. Krivoshchekov, Phys. Lett. B **149** (1984) 128.

[34] I. L. Buchbinder and K. V. Stepanyantz, Nucl. Phys. B **883** (2014) 20

[35] L. D. Faddeev and A. A. Slavnov, "Gauge Fields. Introduction To Quantum Theory," Nauka, Moscow, 1978 and Front. Phys. **50** (1980) 1 [Front. Phys. **83** (1990) 1].

[36] A. A. Slavnov, Theor. Math. Phys. **33** (1977) 977 [Teor. Mat. Fiz. **33** (1977) 210].

[37] A. A. Soloshenko and K. V. Stepanyantz, Theor. Math. Phys. **140** (2004) 1264 [Teor. Mat. Fiz. **140** (2004) 430].

[38] A. V. Smilga and A. Vainshtein, Nucl. Phys. B **704** (2005) 445.

[39] K. V. Stepanyantz, Nucl. Phys. B **852** (2011) 71.

[40] K. V. Stepanyantz, JHEP **1408** (2014) 096.

[41] N. N. Bogolyubov and D. V. Shirkov, "Introduction To The Theory Of Quantized Fields," Nauka, Moscow, 1984 [Intersci. Monogr. Phys. Astron. **3** (1959) 1].

[42] A. L. Kataev and K. V. Stepanyantz, Nucl. Phys. B **875** (2013) 459.

[43] A. B. Pimenov, E. S. Shevtsova and K. V. Stepanyantz, Phys. Lett. B **686** (2010) 293.

[44] K. V. Stepanyantz, Phys. Part. Nucl. Lett. **8** (2011) 321.

[45] K. V. Stepanyantz, arXiv:1108.1491 [hep-th].

[46] K. V. Stepanyantz, J. Phys. Conf. Ser. **343** (2012) 012115.

[47] K. V. Stepanyantz, J. Phys. Conf. Ser. **368** (2012) 012052.

[48] A. L. Kataev and K. V. Stepanyantz, Phys. Lett. B **730** (2014) 184

[49] A. L. Kataev and K. V. Stepanyantz, arXiv:1405.7598 [hep-th], Theor.Math.Phys., to appear.