Linearized solutions for U(1) gauged Q-balls

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Abstract

Explicit examples of $U(1)$ gauged Q-balls in the case, when the back-reaction of the gauge field on the scalar field is small and the linearized theory can be used, are presented.

As it was demonstrated in [1] (see [2] for details), there exists a simple method for obtaining the main characteristics of $U(1)$ gauged Q-balls in the case, when the back-reaction of the gauge field on the scalar field is small. In particular, the charge and the energy of gauged Q-ball can be represented as

$$Q = Q_0 + \triangle Q = Q_0 + \frac{dI}{d\omega}, \tag{1}$$

$$E = E_0 + \triangle E = E_0 + \omega\frac{dI}{d\omega} - I, \tag{2}$$

where

$$I = -16\pi e^2\omega^2 \int\limits_0^\infty f_0^2(r) r \int\limits_0^r f_0^2(y) y^2 dy\, dr. \tag{3}$$

Here f_0 is a solution to the equation

$$\omega^2 f_0 + \Delta f_0 - \frac{1}{2}\frac{dV}{df}\bigg|_{f=f_0} = 0 \tag{4}$$

and corresponds to the nongauged Q-ball, which plays the role of the background.

To briefly show how the proposed method can be used for calculations (the detailed analysis and more plots can be found in [2]), we start with the model proposed in [3]. It has the potential of form

$$V(\phi^*\phi) = -\mu^2\phi^*\phi\ln(\beta^2\phi^*\phi), \tag{5}$$

where μ and β are the model parameters. The spherically symmetric background (nongauged) solution for the Q-ball in this model takes the form

$$f_0(r) = \mu\xi \mathrm{e}^{-\frac{\omega^2}{2\mu^2}} \mathrm{e}^{-\frac{\mu^2 r^2}{2}}, \tag{6}$$

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where $0 \le \omega < \infty$ and $\xi = \frac{\mathrm{e}}{\beta\mu}$. The charge and the energy of the nongauged Q-ball look like

$$Q_0 = 2\pi^{\frac{3}{2}}\xi^2\frac{\omega}{\mu}\,\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \quad (7)$$

$$E_0 = 2\pi^{\frac{3}{2}}\xi^2\mu\left(\frac{\omega^2}{\mu^2}+\frac{1}{2}\right)\mathrm{e}^{-\frac{\omega^2}{\mu^2}}. \quad (8)$$

The integral in (3) can be calculated analytically for the background solution defined by (6). The result has the form [2]

$$I = -\mu e^2\frac{\pi^{\frac{3}{2}}}{\sqrt{2}}\xi^4\left(\frac{\omega}{\mu}\right)^2\mathrm{e}^{-\frac{2\omega^2}{\mu^2}}. \quad (9)$$

The corrections $\triangle Q$ and $\triangle E$ can also be calculated analytically [2], and for the charge and the energy of the gauged Q-ball we get

$$Q = Q_0 + \triangle Q = 2\pi^{\frac{3}{2}}\xi^2\left(\tilde{Q}_0 + e^2\xi^2\triangle\tilde{Q}\right) = 2\pi^{\frac{3}{2}}\xi^2\tilde{Q}, \quad (10)$$

$$E = E_0 + \triangle E = \mu\,2\pi^{\frac{3}{2}}\xi^2\left(\tilde{E}_0 + e^2\xi^2\triangle\tilde{E}\right) = \mu\,2\pi^{\frac{3}{2}}\xi^2\tilde{E} \quad (11)$$

with

$$\tilde{Q}_0 = \tilde{\omega}\mathrm{e}^{-\tilde{\omega}^2}, \qquad \triangle\tilde{Q} = \left(\sqrt{2}\,\tilde{\omega}^3 - \frac{\tilde{\omega}}{\sqrt{2}}\right)\mathrm{e}^{-2\tilde{\omega}^2}, \quad (12)$$

$$\tilde{E}_0 = \left(\tilde{\omega}^2+\frac{1}{2}\right)\mathrm{e}^{-\tilde{\omega}^2}, \qquad \triangle\tilde{E} = \left(\sqrt{2}\,\tilde{\omega}^4 - \frac{\tilde{\omega}^2}{2\sqrt{2}}\right)\mathrm{e}^{-2\tilde{\omega}^2}, \quad (13)$$

where $\tilde{\omega} = \frac{\omega}{\mu}$. In Fig. 1 one can see an example of the $E(Q)$ diagram for the gauged Q-ball in this model. We see from Fig. 1 that the energy of the gauged Q-ball is larger than the energy of the corresponding nongauged Q-ball with the same charge, as it was noted in [1].

In Fig. 2, the plots of corrections $\triangle\tilde{Q}$ and $\triangle\tilde{E}$ are presented. One sees from these plots that $\triangle Q$ and $\triangle E$ can be negative or positive for a given ω (although the energy of gauged Q-ball is always larger than the energy of the corresponding nongauged Q-ball with the same charge).

It is not difficult to show that if $e^2\xi^2 \ll 1$, then the parameter $\alpha(\omega) \ll 1$ (see [2], see also [1] for the definition of $\alpha(\omega)$).

For completeness below we present the exact, in the linear approximation, solution for the fields g and φ in this model, which describe the deviation of the gauged Q-ball from the nongauged Q-ball (see [1] for the definition of g and φ). For the first time this solution was obtained in [4], and in our notations, it has the form

$$g(r) = \mu\alpha_1\,\Phi_g(\omega)F_g(r), \quad (14)$$

$$\varphi(r) = \mu\alpha_1\xi\,\Phi_\varphi(\omega)F_\varphi(r), \quad (15)$$

where

$$\Phi_g(\omega) = \frac{\sqrt{\pi}}{2}\frac{\omega}{\mu}\,\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \quad (16)$$

$$F_g(r) = -\frac{1}{\mu r}\mathrm{erf}(\mu r), \quad (17)$$

$$\Phi_\varphi(\omega) = \sqrt{\pi}\left(\frac{\omega}{\mu}\right)^2\mathrm{e}^{-\frac{3\omega^2}{2\mu^2}}, \quad (18)$$

$$F_\varphi(r) = \mathrm{e}^{-\frac{3\mu^2r^2}{2}}\left(\frac{1}{4\sqrt{\pi}} + \frac{1}{4}\mathrm{e}^{\mu^2r^2}\left(\mu r + \frac{1}{2\mu r}\right)\mathrm{erf}(\mu r)\right) \quad (19)$$

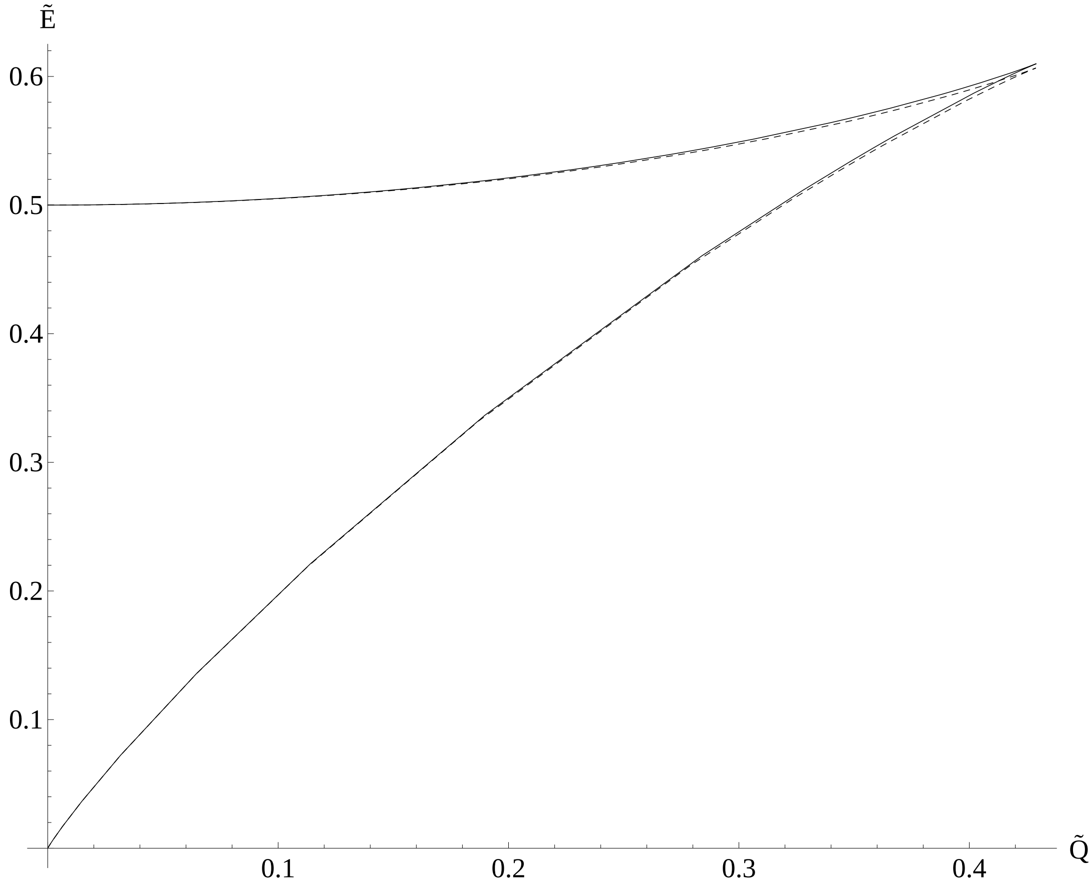

Figure 1: $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. Here, $e^2\xi^2 = 0.05$ and $0 \leq \tilde{\omega} \leq 10$.

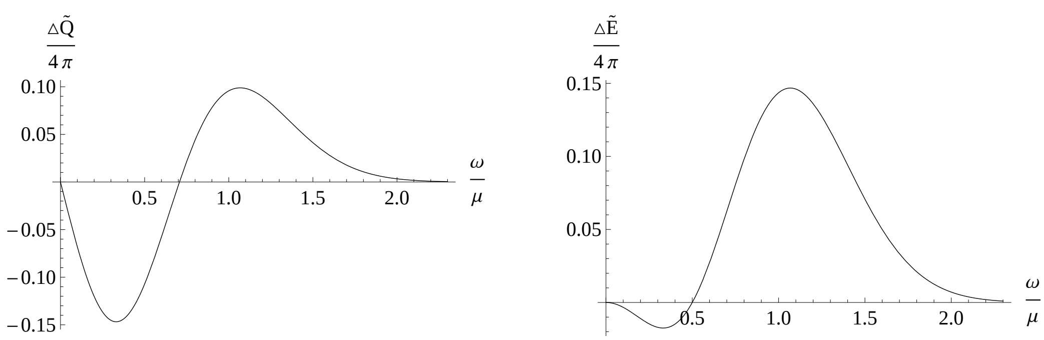

Figure 2: $\triangle\tilde{Q}$ and $\triangle\tilde{E}$ for $0 \leq \tilde{\omega} \leq 2.3$.

with $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^z \mathrm{e}^{-t^2} dt$.

Now we turn to the second model with a piecewise scalar field potential, which was proposed in [5] and examined in [6]:

$$V(\phi^*\phi) = M^2\phi^*\phi\,\theta\left(1 - \frac{\phi^*\phi}{v^2}\right) + \left(m^2\phi^*\phi + v^2(M^2 - m^2)\right)\theta\left(\frac{\phi^*\phi}{v^2} - 1\right), \qquad (20)$$

where $M^2 > 0$, $M^2 > m^2$, and θ is the Heaviside step function with the convention $\theta(0) = \frac{1}{2}$.

The background solution for the Q-ball in this model takes the form

$$f_0(r<R)=f_0^{<}(r) \;=\; v\frac{R\sin\left(\sqrt{\omega^2-m^2}\,r\right)}{r\sin\left(\sqrt{\omega^2-m^2}\,R\right)}, \tag{21}$$

$$f_0(r>R)=f_0^{>}(r) \;=\; v\frac{R\mathrm{e}^{-\sqrt{M^2-\omega^2}r}}{r\mathrm{e}^{-\sqrt{M^2-\omega^2}R}}, \tag{22}$$

where R is defined as

$$R=R(\omega)=\frac{1}{\sqrt{\omega^2-m^2}}\left(\pi-\arctan\left(\frac{\sqrt{\omega^2-m^2}}{\sqrt{M^2-\omega^2}}\right)\right). \tag{23}$$

The charge and the energy of the Q-ball look like

$$Q_0 \;=\; 4\pi R^2\omega v^2\left(\frac{(M^2-m^2)(R\sqrt{M^2-\omega^2}+1)}{(\omega^2-m^2)\sqrt{M^2-\omega^2}}\right), \tag{24}$$

$$E_0 \;=\; \omega Q_0+4\pi\frac{R^3v^2(M^2-m^2)}{3}. \tag{25}$$

As in the previous case, the integral in (3) can be calculated analytically for the background solution defined by (21) and (22). The result looks like [2]

$$\begin{aligned}\frac{I}{4\pi}=e^2\omega^2\Bigg[&a^4\left(\frac{\sin(2\sqrt{\omega^2-m^2}\,R)}{2\sqrt{\omega^2-m^2}}-R+\frac{\mathrm{Si}(2\sqrt{\omega^2-m^2}\,R)}{2\sqrt{\omega^2-m^2}}-\frac{\mathrm{Si}(4\sqrt{\omega^2-m^2}\,R)}{4\sqrt{\omega^2-m^2}}\right)\\ &-4b^2\left(a^2\left(\frac{R}{2}-\frac{\sin(2\sqrt{\omega^2-m^2}\,R)}{4\sqrt{\omega^2-m^2}}\right)+\frac{b^2\mathrm{e}^{-2\sqrt{M^2-\omega^2}\,R}}{2\sqrt{M^2-\omega^2}}\right)\mathrm{E}_1(2\sqrt{M^2-\omega^2}\,R)\\ &+\frac{2b^4}{\sqrt{M^2-\omega^2}}\mathrm{E}_1(4\sqrt{M^2-\omega^2}\,R)\Bigg],\end{aligned} \tag{26}$$

where

$$\mathrm{Si}(y)=\int\limits_0^y\frac{\sin(t)}{t}dt,\qquad \mathrm{E}_1(y)=\int\limits_y^\infty\frac{\mathrm{e}^{-t}}{t}dt \tag{27}$$

and

$$a=a(\omega)=\frac{vR}{\sin\left(\sqrt{\omega^2-m^2}\,R\right)},\qquad b=b(\omega)=\frac{vR}{\mathrm{e}^{-\sqrt{M^2-\omega^2}R}}. \tag{28}$$

The charge and the energy can be represented as

$$Q=Q_0+\triangle Q \;=\; \frac{v^2}{M^2}\left(\tilde{Q}_0+\frac{e^2v^2}{M^2}\triangle\tilde{Q}\right), \tag{29}$$

$$E=E_0+\triangle E \;=\; \frac{v^2}{M}\left(\tilde{E}_0+\frac{e^2v^2}{M^2}\triangle\tilde{E}\right) \tag{30}$$

with $\tilde{Q}_0$, $\tilde{E}_0$, $\triangle\tilde{Q}$ and $\triangle\tilde{E}$ being dimensionless functions depending on $\frac{\omega}{M}$ and $\frac{m^2}{M^2}$ only.

For a numerical analysis, we choose the case $m^2<0$. In Fig. 3, one can see an example of the $E(Q)$ diagram for the gauged Q-ball in this model (we calculated the values of $\triangle Q$ and $\triangle E$ numerically using (26)). Again we see from Fig. 3 that the energy of the gauged Q-ball is larger than the energy of the corresponding nongauged Q-ball for the same values of charge. As in the previous case, we present the plots of $\triangle\tilde{Q}$ and $\triangle\tilde{E}$; see Fig. 4. And again, one sees from these plots that the corrections $\triangle Q$ and $\triangle E$ can be negative or positive for a given ω.

The function $\alpha(\omega)$ (the details of its calculation can be found in [2]) is presented in Fig. 5 for the set of the model parameters used above. One can see that, according to Fig. 5, the

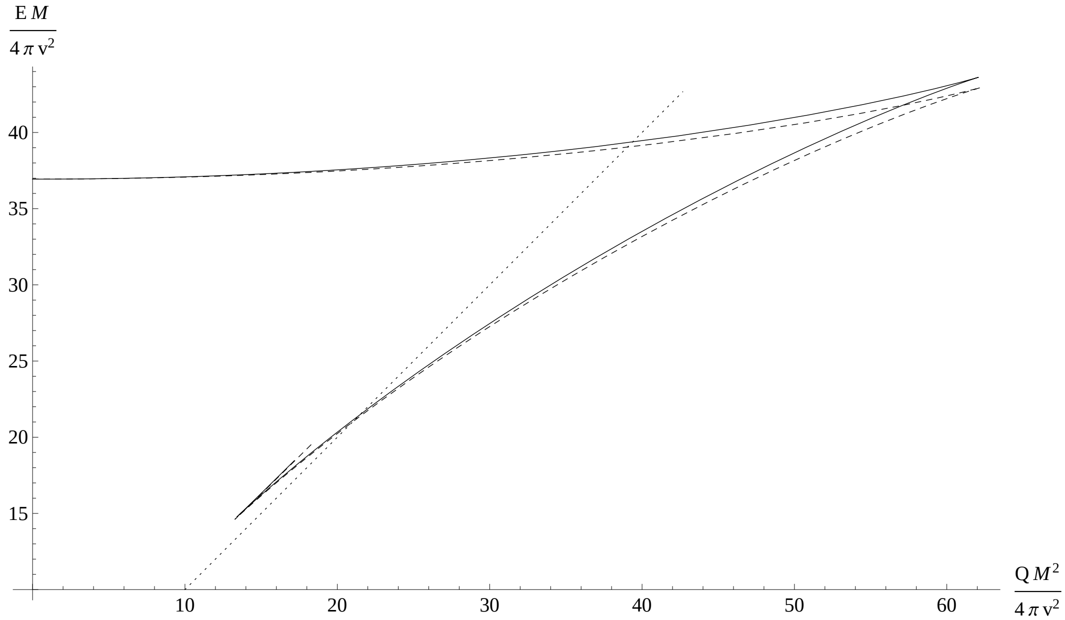

Figure 3: $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. The dotted line stands for free scalar particles of mass M at rest. Here, $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\frac{e^2 v^2}{M^2} = 0.001$, and $0 \leq \frac{\omega}{M} \leq 0.99$.

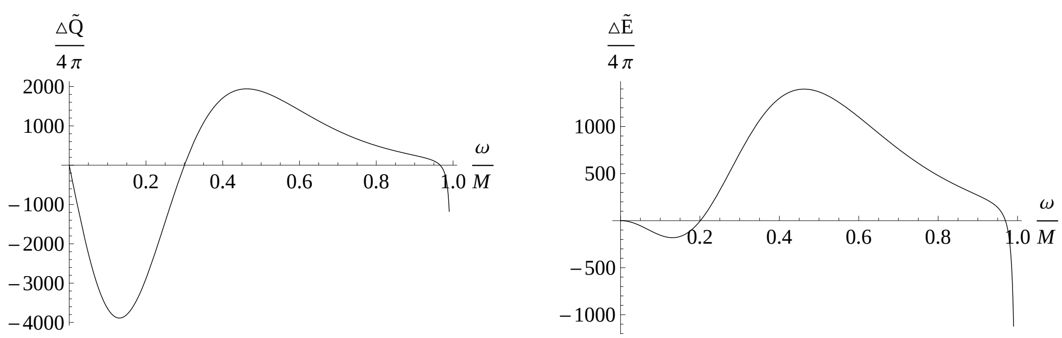

Figure 4: $\triangle\tilde{Q}$ (left plot) and $\triangle\tilde{E}$ (right plot) for $m^2 < 0$, $\frac{|m|}{M} = 0.6$ and $0 \leq \frac{\omega}{M} \leq 0.99$.

linear approximation works well enough only in the vicinity of $\frac{\omega}{M} \approx 0.85$ for the given set of the model parameters.

As for the previous model, for completeness below we present an explicit solution for the fields g and φ in this model (see [2] for details):

$$g(r<R) = g_<(r) = C_1\left(\ln(\omega r) - \mathrm{Ci}(2\omega r) + \frac{\sin(2\omega r)}{2\omega r}\right) + C_2, \tag{31}$$

$$g(r>R) = g_>(r) = \frac{C_3}{r} + C_4\left(\frac{\mathrm{e}^{-2\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r} - \mathrm{E}_1(2\sqrt{M^2-\omega^2}\,r)\right), \tag{32}$$

where

$$\mathrm{Ci}(y) = -\int\limits_y^\infty \frac{\cos(t)}{t}dt, \tag{33}$$

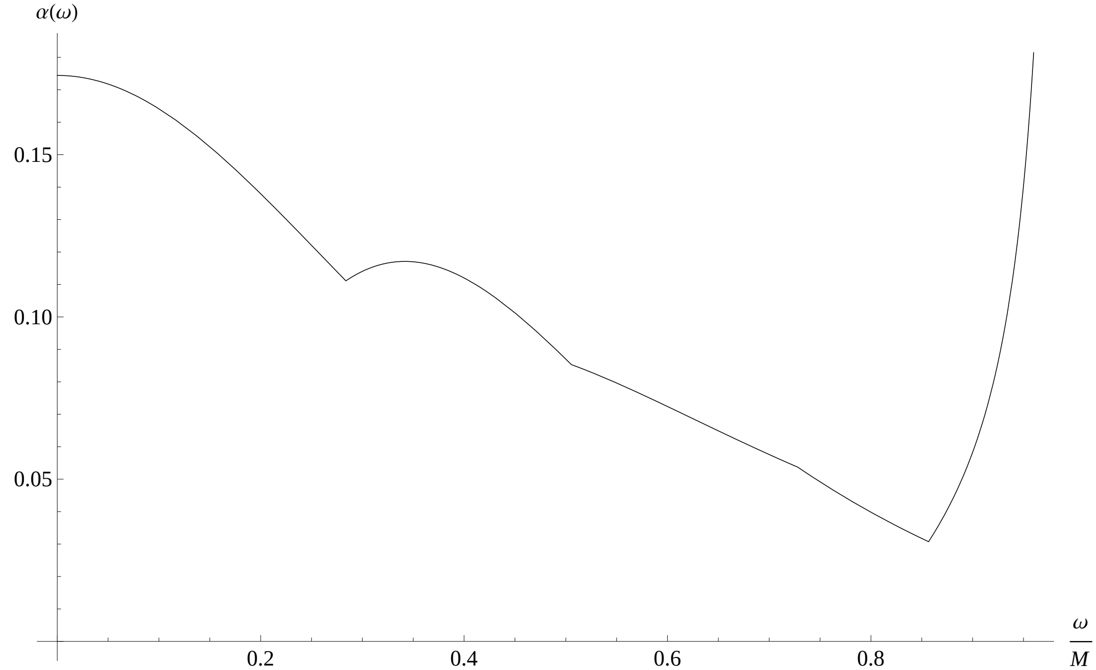

Figure 5: $\alpha(\omega)$ for $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\frac{e^2 v^2}{M^2} = 0.001$, and $0 \leq \frac{\omega}{M} \leq 0.96$.

$$
\begin{aligned}
C_1 = C_1(\omega) &= e^2 v^2 \omega R^2 \frac{1}{\sin^2(\omega R)}, && (34) \\
C_2 = C_2(\omega) &= -e^2 v^2 \omega R^2 \left(2\,\mathrm{e}^{2\sqrt{M^2-\omega^2}R} \mathrm{E}_1(2\sqrt{M^2-\omega^2}R) \right. \\
&+ \left. \frac{-\mathrm{Ci}(2\omega R) + \ln(\omega R) + 1}{\sin^2(\omega R)} \right), && (35) \\
C_3 = C_3(\omega) &= -e^2 v^2 \omega R^2 \left(\frac{M^2}{\omega^2 \sqrt{M^2-\omega^2}} + \frac{R}{\sin^2(\omega R)} \right), && (36) \\
C_4 = C_4(\omega) &= e^2 v^2 \omega R^2 \left(2\,\mathrm{e}^{2\sqrt{M^2-\omega^2}R} \right), && (37)
\end{aligned}
$$

$$
\begin{aligned}
\varphi(r < R) &= B \frac{\sin(\omega r)}{r} + \frac{\sin(\omega r)}{\omega r} \int_0^r G_<(t) \cos(\omega t) dt \\
&- \frac{\cos(\omega r)}{\omega r} \int_0^r G_<(t) \sin(\omega t) dt, && (38) \\
\varphi(r > R) &= A \frac{\mathrm{e}^{-\sqrt{M^2-\omega^2}\,r}}{r} - \frac{\mathrm{e}^{\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r} \int_r^\infty G_>(t) \mathrm{e}^{-\sqrt{M^2-\omega^2}\,t} dt \\
&- \frac{\mathrm{e}^{-\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r} \int_R^r G_>(t) \mathrm{e}^{\sqrt{M^2-\omega^2}\,t} dt, && (39)
\end{aligned}
$$

where

$$
\begin{aligned}
G_<(r) &= -2\omega r g_<(r) f_0^<(r), && (40) \\
G_>(r) &= -2\omega r g_>(r) f_0^>(r), && (41)
\end{aligned}
$$

$$B = B(\omega) = \frac{1}{D} F_1 \frac{e^{\sqrt{M^2-\omega^2}R}}{\sin(\omega R)} - \frac{F_2}{\omega} + \frac{F_3}{\omega^2 R}, \quad (42)$$

$$A = A(\omega) = \frac{e^{\sqrt{M^2-\omega^2}R}}{D} \left(F_1 e^{\sqrt{M^2-\omega^2}R} \left(1 + \frac{D}{2\sqrt{M^2-\omega^2}} \right) + F_3 \frac{M^2 \sin(\omega R)}{\omega^2} \right), \quad (43)$$

$$D = D(\omega) = \frac{M^2 R}{1 + R\sqrt{M^2-\omega^2}}, \quad (44)$$

$$F_1 = F_1(\omega) = \int_R^\infty G_>(t) e^{-\sqrt{M^2-\omega^2}t} dt, \quad (45)$$

$$F_2 = F_2(\omega) = \int_0^R G_<(t) \cos(\omega t) dt, \quad (46)$$

$$F_3 = F_3(\omega) = \int_0^R G_<(t) \sin(\omega t) dt. \quad (47)$$

The plots of this solution for the fields g and φ can be found in [2]. We see, that even for the very simple background solution (21), (22), the solution for g and φ appears to be complicated, its derivation (at least for the field φ) is more bulky than the analytical evaluation of the corresponding integral defined by (3). This clearly demonstrates that without (1), (2), evaluation of the charge and the energy of gauged Q-ball in the general case seems to be a very complicated task, taking into account the necessity to solve the linearized differential equation for the field φ (see eq. (10) in [1]). The use of our method, which is based on the evaluation of the double integral in (3), seems to be a much more simple way for examining the main properties of $U(1)$ gauged Q-balls, even for the background nongauged solution $f_0(r, \omega)$ obtained numerically.

References

[1] I.E. Gulamov, E.Y. Nugaev and M.N. Smolyakov, "Some general properties of U(1) gauged Q-balls", these Proceedings.

[2] I.E. Gulamov, E.Y. Nugaev and M.N. Smolyakov, Phys. Rev. D **89** (2014) 085006.

[3] G. Rosen, Phys. Rev. **183** (1969) 1186.

[4] V. Dzhunushaliev and K.G. Zloshchastiev, Central Eur. J. Phys. **11** (2013) 325.

[5] G. Rosen, J. Math. Phys. **9** (1968) 996.

[6] I.E. Gulamov, E.Y. Nugaev and M.N. Smolyakov, Phys. Rev. D **87** (2013) 085043.