Holography of Wilson loops in rotating black holes

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Abstract

We study the potential of quark-antiquark pair using the AdS/CFT correspondence in the theory dual to the BTZ black hole with non-zero angular momentum (with zero charge). As a result we find out, that Wick rotation affects on the average of rectangular Wilson (Polyakov) loop operator that provides information about the potential due to change of the sign in angular momentum.

1   Introduction

In this talk we present the result of our paper [1] and make new comments and clarifications about the results obtained. The main purpose of our talk is to study non-local operators arising in the gauge/gravity duality. Polyakov and Wilson loops arising in gauge theories are important and well-studied observables. They can be used as an elegant and simple way to study quark-antiquark potentials as in presence of the temperature, as for the real-time both.

One of the most prominent of the AdS/CFT duality conjectures is that one can evaluate the average of such a loop operator (in the theory dual to asymptotically AdS space) using classical Nambu-Goto string. A lot of papers considered the AdS/CFT duality to elaborate on these non-local operators and use them to investigate the behaviour of the quark-antiquark pairs potentials in the different cases [2, 3, 4, 5, 6]. The temporal Wilson loop at the zero-temperature has been calculated using the AdS/CFT in the pioneering paper [2] and the non-zero temperature Polyakov loop correlators have been calculated in [3, 4], see also refs. in [7, 8, 9]. In the temperature formalism the structure of correlator of two non-local operators (Polyakov lines) is more complicated: there are contributions in the singlet and the adjoint potentials for $SU(N_c)$ theory, see [10]. In the context of gauge/gravity duality on the gravity side it was shown how to relate this potentials with a different string configurations arising in black hole background [11].

So there is a simple wisdom - the Euclidean version of black hole is needed when computing Polyakov lines average, while Wilson loop can be extracted when considering the Lorentzian one [12]. In [12] it was found, that string configurations contributing to the Wilson loop can be continued through the horizon, leading to the complex potential. Hoverer, faraway the horizon quantities extracted in different signatures coincide.

In [1] in was considered the case, when Wick rotation of the gravity background affects average of these non-local observables crucially even faraway from horizon. Moreover this leads to a different behaviour of the space where real string configurations exist.

We discuss the holographic Wilson and Polyakov loops in the BTZ-black hole with non-zero angular momentum and extract interquark interaction potentials from these quantities. In accordance with the holographic prescription we consider the Lorentzian version of the BTZ

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for the Wilson loop and the Euclidean one for the Polyakov lines and obtain corresponding potentials.

First we remind the BTZ black hole geometry in the Lorentzian and Euclidean versions. Then we discuss the holographic prescription for the Wilson and Polyakov loops for black holes in different signatures and discuss the renormalization in different signature cases. Next we derive formulae for the classical Nambu-Goto string action and the relation between maximum string profile and interquark distance. After the discussion of the "living space" for real string configuration we extract quark-antiquark potential from the renormalized action. We end we the conclusion and discussion of the results obtained.

2 The BTZ black hole.

The BTZ black hole [13] is the solution of 2+1 gravity with cosmological term. In this case gravity is topological (i.e. dynamical degrees of freedom are absent). It provides low-dimensional model of the space, that exhibits black hole properties with non-zero angular properties or charge and admits simple form. It also admits non-trivial charge. The properties and the solution's form of this space with both angular momentum and charge are much more pathological and complicated.

We start with the metric of the BTZ black hole (in the Lorentzian version) in the coordinates (r, t, ϕ)

$$ds^2 = -(-M + \frac{r^2}{l^2})dt^2 + \frac{dr^2}{-M + \frac{r^2}{l^2} + \frac{a^2}{r^2}} - 2adtd\phi + r^2d\phi^2, \tag{1}$$

where $-\infty < t < \infty$, $0 < \phi < 2\pi$, $r > 0$, M is the mass of the BTZ black hole, a is its angular momentum and l is the scale parameter. We set $l = 1$ until the end of the paper. One can rewrite this metric using the change of variable $r = \frac{1}{z}$

$$ds^2 = -(-M + \frac{1}{z^2})dt^2 + \frac{1}{z^4}\frac{dz^2}{-M + \frac{1}{z^2} + a^2z^2} - 2adtd\phi + \frac{1}{z^2}d\phi^2. \tag{2}$$

The Euclidean version of metric (2) is

$$ds_E^2 = (-M + \frac{1}{z^2})dt^2 + \frac{1}{z^4}\frac{dz^2}{-M + \frac{1}{z^2} - a^2z^2} + 2adtd\phi + \frac{1}{z^2}d\phi^2. \tag{3}$$

The horizons for the Lorentzian version of metric (2) in the z-coordinate are of the form:

$$z_+ = \frac{1}{2}\frac{\sqrt{2}\sqrt{M + \sqrt{M^2 - 4a^2}}}{a}, \tag{4}$$

$$z_- = \frac{1}{2}\frac{\sqrt{2}\sqrt{M - \sqrt{M^2 - 4a^2}}}{a}, \tag{5}$$

and for the Euclidean version of metric (3) in the z-coordinate they are:

$$z_C^{(E)} = \frac{1}{2}\frac{\sqrt{2M + 2\sqrt{M^2 + 4a^2}}}{ia}, \tag{6}$$

$$z^{(E)} = \frac{1}{2}\frac{\sqrt{2\sqrt{M^2 + 4a^2} - 2M}}{a}. \tag{7}$$

These horizons exhibit different behaviour under the change of a.

3 Holographic description of non-local operators.

3.1 Non-local operators in gauge/gravity duality

According to Maldacena's proposal one calculates the average of the Wilson loop operator $< W(C) >$ using the following formula

$$< W(C) >= e^{-S_{ren}}, \tag{8}$$

where C is a contour on the boundary of BTZ, $W(C)$ is the Wilson loop operator

$$W(C) = \frac{1}{N_c} \text{Tr}(\text{P} \exp(ig \int A_\mu dx^\mu)), \tag{9}$$

and S_{ren} is the renormalized action of the string ending on the contour S and hanging in the BTZ space-time.

The Euclidean version of the BTZ black hole provides us the connected correlator of two Polyakov loops

$$< L^\dagger(-\phi_0/2)L(\phi_0/2) >= e^{-S_{ren}}, \tag{10}$$

where $L(r)$ is given by

$$L(r) = \frac{1}{N_c} \text{Tr}(\text{P} \exp(ig \int\limits_0^\beta A_4(r,\tau)d\tau)). \tag{11}$$

Extraction of the quark-antiquark potential is related to the square infinite contour, where "long" sides (in time) of the contours correspond to heavy quarks path's in the presence of external gluon field. Now let's briefly remind few points in calculation of quark-antiquark potential in $SU(N_c)$ theory.

In quenched approximation (i.e. neglecting virtual quark loops) and in the large N_c limit we can set fermionic determinants arising in calculation to one. In addition our quarks are static - so we neglect time derivatives when calculating propagators. In this way we obtain, that four-point singlet Green function is proportional to the product of static propagators and the Wilson loop. For infinitely long (in time T) rectangular contour from four-point static Green function we get the following relation:

$$V(r) = -\frac{1}{iT} \ln W(\mathcal{C}), \ \ T \to \infty \tag{12}$$

In the case of temperature field theory the correlator of two Polyakov loops [10] gives us the adjoint and singlet potentials of interquark interaction (in the contrast of real-time Wilson loops, that provide us only the singlet one):

$$< L^\dagger(-\phi_0/2)L(\phi_0/2) >= \frac{e^{(-\beta V_{singl.})} + (N_c^2 - 1)e^{(-\beta V_{adj.})}}{N_c^2}, \tag{13}$$

where $V_{singl.}$ is the singlet potential and $V_{adj.}$ is the adjoint potential for $SU(N_c)$ gauge theory.

In the context of Polyakov loops correlator S_{ren} on the gravity dual side should split on different configuration actions [11] that contribute in different potentials:

$$< L^\dagger(-\phi_0/2)L(\phi_0/2) >= \frac{e^{(-S_{hanging})} + (N_c^2 - 1)e^{(-S_{stretched})}}{N_c^2}. \tag{14}$$

Here $S_{hanging}$ is the action the Nambu-Goto string that hanging in the bulk of the BTZ and $S_{stretched}$ is the action of two strings stretched between the conformal boundary of our space and Euclidean horizon. When renormalizing $S_{stretched}$ we get zero due to counterterms (we discuss the renormalization of string configurations in the next section) equal to $S_{stretched}$. So we get only the singlet potential, the adjoint potential is zero at leading order in N_c^2 here.

3.2 Renormalization and static quarks in BTZ background.

In the case of the Lorentzian signature there is no well established procedure of renormalization that can be relevant to the complex string configurations as well. In [12] substraction was performed by real part of the action infinitely hanged complex string configuration. We discuss the renormalization scheme [1, 12] for the Lorentzian and the Euclidean signatures. Counterterms for renormalization in different signatures differs only in their finite part, see [1].

Renormalization of the Nambu-Goto string is related with quark self-energy and therefore with two static strings hanging from the boundary[2, 3, 4]. In the zero temperature case counterterms obtained from the static strings contain only divergent part (i.e. no finite contribution due to counterterms), namely $2/\epsilon$ (compare with the regularization used for local correlators [15]) In the black hole background with the Euclidean signature counterterms usually contain finite part (related to thermal temperature-dependent self-energy of quarks) due to finite integration interval, namely $2 \int_\epsilon^{z_H} \frac{dz}{z^2}$. This is an action of two strings, stretched between the boundary and the horizon of the black hole (here z_H is the coordinate of the horizon).

Certain values of counterterms, their analysis and general discussion can be found in [1]

4 The Nambu-Goto action of the string in the BTZ background.

The Nambu-Goto action is

$$S = - \int d\sigma d\tau \sqrt{- \det(h_{\alpha\beta})}, \tag{15}$$

where $h_{\alpha\beta}$ is the induced metric

$$h_{\alpha\beta} = g_{MN} \partial_\alpha X^M \partial_\beta X^N, \tag{16}$$

g_{MN} is the background metric and $X^M(\sigma, \tau)$ is a worldsheet.

The Euclidean version of the action is

$$S_E = \int d\sigma d\tau \sqrt{\det(h_{\alpha\beta}^{(E)})}, \tag{17}$$

where $h_{\alpha\beta}^{(E)}$ is the Euclidean version of the induced metric

$$h_{\alpha\beta}^{(E)} = g_{MN}^{(E)} \partial_\alpha X^M \partial_\beta X^N, \tag{18}$$

and $g_{MN}^{(E)}$ is the Euclidean version of metric g_{MN}.

The ansatz appropriate for description of Nambu-Goto string ending on two infinitely straight lines (we set $\sigma = \phi$, $\tau = t$ and $X^M(\sigma, \tau)$) is the following :

$$X^M(t, \phi) = \begin{pmatrix} t \\ z(\phi) \\ \phi \end{pmatrix}. \tag{19}$$

Using (19) we get the action (we take the metric (2)):

$$S = - \int_0^T dt \int_{-\phi_0/2}^{\phi_0/2} d\phi \sqrt{- \det(h_{\alpha\beta})} = -T \int_{-\phi_0/2}^{\phi_0/2} d\phi \sqrt{- \det(h_{\alpha\beta})}, \tag{20}$$

where $\sqrt{- \det(h_{\alpha\beta})}$ is

$$\sqrt{- \det(h_{\alpha\beta})} = \sqrt{\frac{-z^8 a^4 + 2 M z^6 a^2 + (-M^2 - 2 a^2) z^4 + 2 M z^2 - 1 + (M z^2 - 1) z'^2}{z^4 (-a^2 z^4 + M z^2 - 1)}}. \tag{21}$$

We can consider $\sqrt{-\det(h_{\alpha\beta})}$ as a Lagrangian of an one-dimensional system

Since Lagrangian (21) doesn't contain an explicit dependence on ϕ, it admits the integral of motion and the solution for the action. Therefore, we get the expression [1] for the interquark distance ϕ_0 as the function of z_m, a and M:

$$\frac{\phi_0}{2} = z_+ z_- \int\limits_0^{z_m} \sqrt{\frac{(Mz^2-1)\,\mathcal{F}(a,M,z_m)}{(z^2-z_m^2)\mathcal{G}(M,z,z_m)}} \,\frac{z^2\,dz}{\mathcal{F}(a,M,z)}, \tag{22}$$

where

$$\mathcal{F}(a,M,z) = -1 + z^2\left(-a^2z^2 + M\right) = -a^2(z^2-z_+^2)(z^2-z_-^2), \tag{23}$$
$$\mathcal{G}(M,z,z_m) = Mz_m^2 z^2 - z^2 - z_m^2. \tag{24}$$

Using the expression for derivative $\frac{dz}{d\phi}$ one can write down the expression for the action

$$S = -T \int\limits_{-\phi_m}^{\phi_m} d\phi\,\frac{z_m^2(-a^2z^4 + Mz^2 - 1)}{z^4\sqrt{(a^2z_m^4 - Mz_m^2 + 1)}}, \tag{25}$$

here $z = z(\phi)$.

Formulae (25) and (22) contains all information that is needed to describe quark-antiquark potential (here they are in the Lorentzian case). To get an analog of formula (22) and (25) in the Euclidean case we make the change in (22) as $a \to ia$.

5 Horizons and the living space for string configurations.

As it was pointed out horizons for different signatures of the BTZ black hole exhibits different behaviour. This point is crucial, because horizons define the "living space", where only real string configurations exist (complex string configuration traditionally are interpreted as string breaking and lead to complex-valued potentials, associated with unstable thermal charmonium).

To study the "living space" we can use simple graphical representation [1] of the positive polynomial roots, that enter in the expression under the square root in (22), see [1] for details and pictures.

It is easy to fix the sign of the expression under the square root for small z and z_m, say at the point p (see pictures in [1]), where the sign is $+$. To get the sign at any given point we connect it with the point p, assuming that the curve crosses the lines in normal direction. The sign is changed when the curve has crossed one of the curves drawn.

As the result we see that z_m can take values from 0 to z_0 in the Lorentzian version and in the Euclidean case z_m can take values only from 0 to $z^{(E)}$. The difference is that when changing a in the Lorentzian version "living space" doesn't change while in the Euclidean case it shrinks.

6 Formulae for the interquark distance.

We can approximate integral in (22) taking into account only the main contribution from the poles at points z_m and z_-, and zero at point $\frac{1}{\sqrt{M}}$ i.e. we put in(22) $\mathcal{G}(M,z_m,z_m)$ instead of $\mathcal{G}(M,z,z_m)$ and $(-z_+^2 + z_m^2)$ instead $(-z_+^2 + z^2)$.

In this approximation the integral in formula (22) takes the form:

$$\frac{\phi_0}{2} \approx z_- z_+ \int\limits_0^{z_m} \frac{z^2\sqrt{(z_-^2 - z_m^2)(-z_+^2 + z_m^2)\,(Mz^2-1)}}{\sqrt{(z^2 - z_m^2)(z_m^2\,(Mz_m^2-1) - z_m^2)}(z_-^2 - z^2)(-z_+^2 + z_m^2)}\,dz. \tag{26}$$

After performing integration in the case of non-zero rotation (26) and some algebra we get [1]:

$$\frac{\phi_0}{2} \approx z_- z_+ \sqrt{\frac{(z_-^2 - z_m^2)}{(z_+^2 - z_m^2)}} \frac{\left(\mathcal{B}(z_-, z_m, M) + M z_-^2 \; \mathcal{E}(z_m \sqrt{M})\right)}{z_m \left(\sqrt{2 - M z_m^2}\right)}, \tag{27}$$

where $\mathcal{B}(z_-, z_m, M)$ is

$$\mathcal{B}(z_-, z_m, M) = \left(1 - M z_-^2\right) \Pi \left(\frac{z_m^2}{z_-^2}, z_m \sqrt{M}\right) \tag{28}$$

and $\Pi(\nu, k)$ is the incomplete elliptic integral of the third kind [16]

$$\Pi(\nu, k) = \int\limits_0^1 \frac{dt}{(1 - \nu t^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}}. \tag{29}$$

For the Euclidean version of the BTZ background the formula for the interquark distance is the same up to the change of horizons to the Euclidean one as following $z_- \to z^{(E)}$, $z_+ \to z_C^{(E)}$. Note, that in this approximation for $a > \frac{M}{2}$ one should take the real part of formula (27).

For different signatures, as can be seen from Fig.1, the behaviour of string profile differs crucially [1].
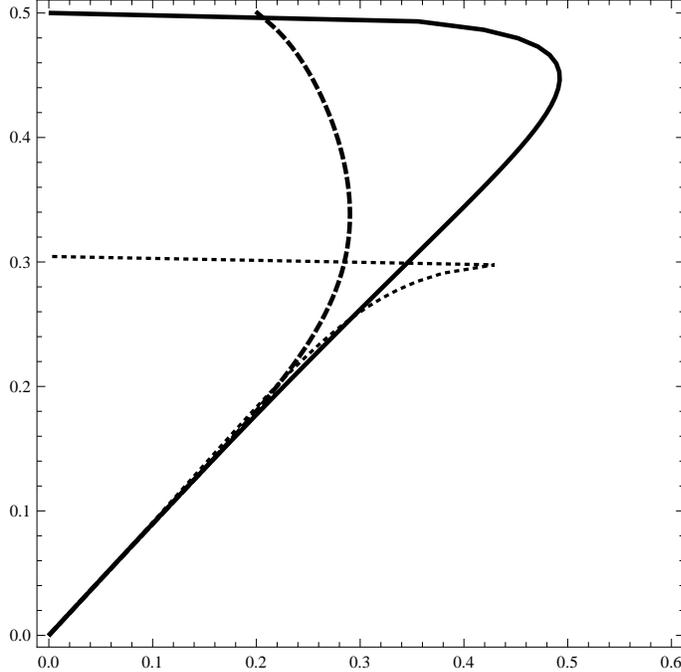

Figure 1: The dependence of the string profile z_m maximum on interquark distance ϕ_0 in the Lorentzian and the Euclidean versions of the BTZ black hole. $M = 4$ is equal to 4. $a = 9$ for the dashed and dotted lines, which correspond to the Lorentzian and the Euclidean case respectively. The solid line corresponds to $a = 0$.

7   The action and the potential.

Changing the variable of integration to z one can write down the regularized action in the following form

6

$$S = -2T \int\limits_{\epsilon}^{z_m} \frac{z_m^2 \sqrt{1 - Mz^2}}{\sqrt{(Mz^2 z_m^2 - z^2 - z_m^2)(z^2 - z_m^2)}} \frac{dz}{z^2}, \tag{30}$$

where ϵ is the finite cut-off. It is interesting, that the action in this form doesn't contain a explicitly. The main dependence on a now is contained in z_m. If we take $\epsilon = 0$ we get a linear divergence of the integral (30). This divergent term should be subtracted in accordance with the section 3.2.

The potential of the interquark interaction can be obtained [1] as

$$V(\phi) = \frac{S_{ren}(\phi)}{T}. \tag{31}$$

We are interested in the behaviour of the potential for small z_m.

To obtain the approximation we expand the integrand (30) in series in M. Then we get

$$V(z_m) \approx -\frac{2A}{z_m} - 2CMz_m, \tag{32}$$

where A is constant (33) and $C = \frac{1}{4}(E(i) - 2K(i)) \approx 0.1779$.

$$A = \frac{\sqrt{2}\pi^{3/2}}{\Gamma^2(\frac{1}{4})} \approx 0.599. \tag{33}$$

Let's consider the potential of interquark interaction in the Euclidean and the Lorentzian case with $a \gg \frac{M}{2}$. From Fig.2 one can see that the potential exhibits strong dependence on angular moment and depends on the signature of the BTZ we use.
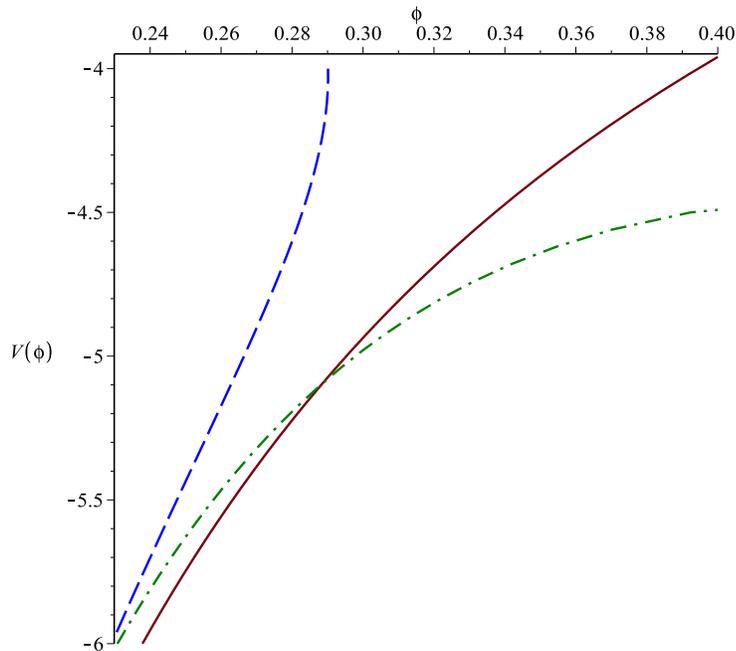

Figure 2: The potential for the different rotation parameter values and signatures (without finite counterterms). The solid red, green dot-dashed and the blue dashed curves correspond to $a = 1$, $a = 9$ for the Euclidean case and $a = 9$ for the Lorentzian case respectively. Here M=4.

8   Conclusions

Using the dual prescription the singlet potential of the quark-antiquark pair in the BTZ black hole with non-zero angular momentum has been calculated [1]. To get this we have considered non-local operators on boundary of the BTZ black hole with non-zero rotation parameter. In this case we find, that Wick rotation of the gravitational background affects the potential. Moreover, we find, that from the different behaviour for BTZ horizons "the living space" for real string configurations relevant for calculation of the potential exhibits different behaviour. First, for the Euclidean signature it shrinks, while for the Lorentzian one it is stable. This affects complex string configurations too. In the Loretzian case two horizons complicates the behaviour of the string dependence and action. In the thermal case there is only one (real) horizon. We relate the results obtained in the Lorentzian signature with real-time thermal correlators. For the Euclidean signature the results should be related with the Euclidean thermal correlators. It would be interesting to analyse the behaviour of the complex interquark potential and it's dependence on the horizon's structure.

Acknowledgments

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