

Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology

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Exact solutions in Lovelock gravity:

- Power-law vacuum solutions:

$$ds^2 = -dt^2 + \sum_k t^{2p_k} dx_k^2, \quad \sum_{i < j < l < m} p_i p_j p_l p_m = 0, \quad \sum_n p_n = 3$$

- Exponential solutions:

$$ds^2 = -dt^2 + \sum_{k=1}^N e^{2H_k t} dx_k^2, \quad H_k \equiv \text{const}$$

Exponential solutions

$$S = \frac{1}{2\kappa^2} \int d^{N+1}x \sqrt{-g} \{ \mathcal{L}_E + \alpha \mathcal{L}_{GB} + \mathcal{L}_{matter} \}$$

$$\mathcal{L}_E = R, \quad \mathcal{L}_{GB} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2$$

Matter is a perfect fluid with the equation of state $p = \omega\rho$

$$\sum_{i \neq j} H_i^2 + \sum_{\{i>k\} \neq j} H_i H_k + 4\alpha \sum_{i \neq j} H_i^2 \sum_{\{k>l\} \neq \{i,j\}} H_k H_l +$$

$$+ 12\alpha \sum_{\{i>k>l>m\} \neq j} H_i H_k H_l H_m = -\omega \varkappa, \quad j = \overline{1, N}$$

$$\sum_{i > j} H_i H_j + 12\alpha \sum_{i > j > k > l} H_i H_j H_k H_l = \varkappa, \quad \varkappa = \frac{\kappa^2 \rho}{2}$$

Continuity equation:

$$\dot{\rho} + (\rho + p) \sum_i H_i = 0 \iff (\rho + p) \sum_i H_i = 0 \iff \begin{cases} \rho = 0 \\ p = -\rho \\ \sum_k H_k = 0 \end{cases}$$

Exponential solutions are divided into two different types:

① with constant volume:

$$dV = \sqrt{-g} \exp \left(\sum_k H_k \right) d^N x, \quad \sum_k H_k = 0$$

② with volume changing in time:

- vacuum solutions ($\rho = 0$)
- Λ -term solutions ($p = -\rho$)

Solutions with changing volume.

Space splits into isotropic subspaces

spatial dimensions $N = 4$:

- isotropic case: $H_1 = H_2 = H_3 = H_4 \equiv H \in \mathbb{R}$
- (3+1)-splitting: $H_1 = H_2 = H_3 \equiv H = \frac{1}{\sqrt{-4\alpha}}, H_4 \equiv h \in \mathbb{R}$
- (2+2)-splitting: $H_1 = H_2 \equiv H, H_3 = H_4 \equiv h, Hh = -\frac{1}{4\alpha}$

α is coupling constant

Solutions with changing volume.

Space splits into isotropic subspaces

spatial dimensions $N = 5$:

- isotropic case: $H_1 = H_2 = H_3 = H_4 = H_5 \equiv H \in \mathbb{R}$
- (4+1)-splitting:

$$H_1 = H_2 = H_3 = H_4 \equiv H = \frac{1}{\sqrt{-12\alpha}}, H_5 \equiv h \in \mathbb{R}$$

- (2-2+1)-splitting:

$$H_1 = H_2 = -H_3 = -H_4 \equiv H = \frac{1}{\sqrt{4\alpha}}, H_5 \equiv h \in \mathbb{R}$$

- (3+2)-splitting:

$$H_1 = H_2 = H_3 \equiv H, H_4 = H_5 \equiv h, h = -\frac{4H^2\alpha + 1}{8H\alpha}$$

Solutions with changing volume.

Vacuum solutions:

- spatial dimensions $N = 4$:

$$H_1 = H_2 = H_3 = H_4 = \frac{1}{\sqrt{-2\alpha}}, \quad \alpha < 0$$

- spatial dimensions $N = 5$:

- ① isotropic:

$$H_1 = H_2 = H_3 = H_4 = H_5 = \frac{1}{\sqrt{-6\alpha}}, \quad \alpha < 0$$

- ② (3+2)-splitting:

$$H_1 = H_2 = H_3 = H, \quad H_4 = H_5 = \xi H$$

$$H^2 = \frac{f(\xi)}{4\alpha} \Bigg|_{\xi = \frac{\sqrt[3]{10}}{3} - \frac{\sqrt[3]{100}}{6} - \frac{2}{3} \approx -0.722}, \quad f(\xi) = -\frac{\xi^2 + 6\xi + 3}{3\xi(3\xi + 2)}, \quad \alpha > 0$$

Solutions with changing volume, $N = 4$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2$$

| | |
|------------------------|---|
| (H_1, H_2, H_3, H_4) | $\Lambda < -\frac{3}{2\kappa^2 \alpha}, \quad \alpha < 0$ |
| (H, H, h, h) | No |
| (H, H, H, h) | No |
| (H, H, H, H) | $H^2 = \frac{1}{4\alpha} \left(-1 \pm \sqrt{1 + \frac{2\kappa^2 \alpha \Lambda}{3}} \right)$ |

| | |
|------------------------|---|
| (H_1, H_2, H_3, H_4) | $\Lambda = -\frac{3}{2\kappa^2 \alpha}, \quad \alpha < 0$ |
| (H, H, h, h) | $H^2 = h^2 = -\frac{1}{4\alpha}$ |
| (H, H, H, h) | $H^2 = -\frac{1}{4\alpha},$ $h \in \mathbb{R}$ |
| (H, H, H, H) | $H^2 = -\frac{1}{4\alpha}$ |

Solutions with changing volume, $N = 4$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2$$

| | |
|------------------------|---|
| (H_1, H_2, H_3, H_4) | $\Lambda > -\frac{3}{2\kappa^2 \alpha}, \quad \alpha < 0$ |
| (H, H, h, h) | $H^2 = \frac{1}{4} \left(\kappa^2 \Lambda + \frac{1}{2\alpha} \pm \sqrt{\left(\kappa^2 \Lambda + \frac{1}{2\alpha} \right)^2 - \frac{1}{\alpha^2}} \right),$ $h = -\frac{1}{4\alpha H}$ |
| (H, H, H, h) | No |
| (H, H, H, H) | No |

Solutions with changing volume, $N = 4$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2$$

| | |
|------------------------|---|
| (H_1, H_2, H_3, H_4) | $\Lambda > \frac{1}{2\kappa^2\alpha}, \alpha > 0$ |
| (H, H, h, h) | $H^2 = \frac{1}{4} \left(\kappa^2 \Lambda + \frac{1}{2\alpha} \pm \sqrt{\left(\kappa^2 \Lambda + \frac{1}{2\alpha} \right)^2 - \frac{1}{\alpha^2}} \right)$ $h = -\frac{1}{4\alpha H}$ |
| (H, H, H, h) | No |
| (H, H, H, H) | $H^2 = \frac{1}{4\alpha} \left(-1 + \sqrt{1 + \frac{2\kappa^2\alpha\Lambda}{3}} \right)$ |

Solutions with changing volume, $N = 4$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2$$

| | |
|------------------------|---|
| (H_1, H_2, H_3, H_4) | $0 < \Lambda \leq \frac{1}{2\kappa^2 \alpha}, \alpha > 0$ |
| (H, H, h, h) | No |
| (H, H, H, h) | No |
| (H, H, H, H) | $H^2 = \frac{1}{4\alpha} \left(-1 + \sqrt{1 + \frac{2\kappa^2 \alpha \Lambda}{3}} \right)$ |

Solutions with changing volume, $N = 5$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2 + e^{2H_5 t} dx_5^2$$

| | |
|-----------------------------|---|
| $(H_1, H_2, H_3, H_4, H_5)$ | $\alpha > 0$ |
| (H, H, H, H, H) | $H^2 = \frac{1}{12\alpha} \left(-1 + \sqrt{1 + \frac{6\kappa^2\alpha\Lambda}{5}} \right), \Lambda > 0$ |
| (H, H, H, H, h) | No |
| $(H, H, -H, -H, h)$ | $H^2 = \frac{1}{4\alpha}, h \in \mathbb{R}, \Lambda = \frac{1}{2\kappa^2\alpha}$ |

Solutions with changing volume, $N = 5$

$$ds^2 = -dt^2 + e^{2H_1 t} dx_1^2 + e^{2H_2 t} dx_2^2 + e^{2H_3 t} dx_3^2 + e^{2H_4 t} dx_4^2 + e^{2H_5 t} dx_5^2$$

| | |
|---------------------|---|
| (H_1, \dots, H_5) | $\alpha < 0$ |
| (H, H, H, H, H) | $H^2 = \frac{1}{12\alpha} \left(-1 - \sqrt{1 + \frac{6\kappa^2\alpha\Lambda}{5}} \right), \Lambda < -\frac{5}{6\kappa^2\alpha}$ $H^2 = \frac{1}{12\alpha} \left(-1 + \sqrt{1 + \frac{6\kappa^2\alpha\Lambda}{5}} \right), \Lambda < 0$ |
| (H, H, H, H, h) | $H^2 = -\frac{1}{12\alpha}, h \in \mathbb{R}, \Lambda = -\frac{5}{6\kappa^2\alpha}$ |
| $(H, H, -H, -H, h)$ | No |

Solutions with changing volume, $N \geq 4$

Vacuum isotropic solution:

$$H_1 = \dots = H_N \equiv H$$

$$H^2 = -\frac{1}{\alpha(N-2)(N-3)}, \quad \alpha < 0$$

Solutions with changing volume, $N \geq 4$

Isotropic solutions with Λ -term:

$$H_1 = \dots = H_N \equiv H$$

- $\alpha > 0$

$$H^2 = \frac{\sqrt{1 + 4\kappa^2\alpha\Lambda \frac{(N-2)(N-3)}{N(N-1)}} - 1}{2\alpha(N-2)(N-3)}, \quad \Lambda > 0$$

- $\alpha < 0$

$$H^2 = \frac{\pm\sqrt{1 + 4\kappa^2\alpha\Lambda \frac{(N-2)(N-3)}{N(N-1)}} - 1}{2\alpha(N-2)(N-3)}, \quad \Lambda < -\frac{N(N-1)}{4\kappa^2\alpha(N-2)(N-3)}$$

Solutions with changing volume, $N \geq 4$

Solutions with $([N - 1] + 1)$ -splitting:

$$H_1 = \dots = H_{N-1} \equiv H, \quad H_N \equiv h$$

$$H^2 = -\frac{1}{2\alpha(N-2)(N-3)}, \quad \Lambda = -\frac{N(N-1)}{4\kappa^2\alpha(N-2)(N-3)}, \quad \alpha < 0$$
$$h \in \mathbb{R}$$

Solutions with constant volume, $\sum_k H_k = 0$

$$\sum_{i \neq j} H_i^2 + \sum_{\{i>k\} \neq j} H_i H_k + 4\alpha \sum_{i \neq j} H_i^2 \sum_{\{k>l\} \neq \{i,j\}} H_k H_l +$$

$$+ 12\alpha \sum_{\{i>k>l>m\} \neq j} H_i H_k H_l H_m = -\omega \varkappa, \quad j = \overline{1, N}$$

$$\sum_{i>j} H_i H_j + 12\alpha \sum_{i>j>k>l} H_i H_j H_k H_l = \varkappa, \quad \varkappa = \frac{\kappa^2 \rho}{2}$$

$$\sum_{k=1}^N H_k = 0$$

Solutions with constant volume, $\sum_k H_k = 0$

Necessary conditions for solutions to exist:

$$\omega < \frac{1}{3}$$

$$\rho \geq \rho_{\lim}(\omega), \quad \rho_{\lim}(\omega) = \begin{cases} \frac{2}{9\kappa^2\alpha} \frac{1}{2\sigma_+ - 1} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha > 0 \\ \frac{2}{9\kappa^2\alpha} \frac{1}{2\sigma_- - 1} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2}, & \alpha < 0 \end{cases}$$

σ_{\pm} can be calculated numerically

Solutions with constant volume, $\sum_k H_k = 0$

✓ $([N - 1] + 1)$ -splitting: $H_1 = \dots = H_{N-1} \equiv H$, $H_N \equiv h$:

$$H^2 = -\frac{1}{3\alpha(N-2)(N-3)} \frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = \frac{2}{9\kappa^2\alpha} \frac{N(N-1)}{(N-2)(N-3)} \frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2},$$

$$h = -(N-1)H, \quad \omega < \frac{1}{3}, \quad \alpha < 0$$

✓ $(\frac{N}{2} + \frac{N}{2})$ -splitting, N is even: $H_1 = \dots = H_{\frac{N}{2}} \equiv H \in \mathbb{R}$,
 $H_{\frac{N}{2}+1} = \dots = H_N \equiv h \in \mathbb{R}$

$$H^2 = \frac{1}{3\alpha(N-2)} \frac{\omega-1}{\omega-\frac{1}{3}}, \quad \rho = -\frac{2}{9\kappa^2\alpha} \frac{N}{N-2} \frac{\omega-1}{\left(\omega-\frac{1}{3}\right)^2}, \quad \omega < \frac{1}{3}, \quad \alpha > 0$$

$$h = -H$$

Solutions with constant volume, $\sum_k H_k = 0$

✓ ([n + 1] + n)-splitting, $n \equiv \lfloor \frac{N}{2} \rfloor$, N is odd:

$$H_1 = \dots = H_{n+1} \equiv H, \quad H_{n+2} = \dots = H_N \equiv h$$

$$H^2 = \frac{1}{3\alpha} \frac{(N-1)^2}{(N-3)(N^2+N+2)} \frac{\omega - 1}{\omega - \frac{1}{3}},$$

$$\rho = -\frac{2}{9\kappa^2\alpha} \frac{N(N-1)(N+1)}{(N-3)(N^2+N+2)} \frac{\omega - 1}{\left(\omega - \frac{1}{3}\right)^2},$$

$$h = -(1 + n^{-1})H, \quad \omega < \frac{1}{3}, \quad \alpha > 0$$

Results.

- ① Solutions with volume element changing in time belong to one of the two types:
 - ① vacuum solutions
 - ② solutions with Λ -term

These solutions exist only when space is split into isotropic subspaces.

- ② Solutions with constant volume element exist only when matter density exceeds (or equal to) some critical value which depends on the equation of state of the matter. The parameter ω of the matter should be smaller than $\frac{1}{3}$.

Results.

"Realistic" solutions:

★ (4+1)-dimensional spacetime:

- Λ -term solution

$$H_1 = H_2 = H_3 = \frac{1}{\sqrt{-4\alpha}}, \quad H_4 \in \mathbb{R}, \quad \Lambda = -\frac{3}{2\kappa^2\alpha}, \quad \alpha < 0$$

★ (5+1)-dimensional spacetime:

- vacuum solution: $H_1 = H_2 = H_3 \equiv H, H_4 = H_5 = \xi H$

$$H^2 = \frac{f(\xi)}{4\alpha} \Bigg|_{\xi = \frac{\sqrt[3]{10}}{3} - \frac{\sqrt[3]{100}}{6} - \frac{2}{3} \approx -0.722}, \quad f(\xi) = -\frac{\xi^2 + 6\xi + 3}{3\xi(3\xi + 2)}, \quad \alpha > 0$$

- Λ -term solution: $H_1 = H_2 = H_3 \equiv H, H_4 = H_5 \equiv h$

$$192H^6\alpha^3 - 112H^4\alpha^2 + (256\Lambda\pi\alpha + 4)H^2\alpha - 1 = 0, \quad h = -\frac{4H^2\alpha + 1}{8H\alpha}$$

Papers.

- ① Chirkov D., Pavluchenko S., Toporensky A. Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology // Mod. Phys. Lett. A, Vol. 29, No. 18 (2014) 1450093
- ② Chirkov D., Pavluchenko S., Toporensky A. Constant volume exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology with a perfect fluid / arXiv:1403.4625

Thank you!

Subtracting i -th dynamical equation from j -th one we obtain:

$$(H_j - H_i) \left(\frac{1}{4\alpha} + \sum_{\{k>l\} \neq \{i,j\}} H_k H_l \right) \sum_k H_k = 0$$

\Updownarrow

$$H_i = H_j \quad \vee \quad \sum_{\{k>l\} \neq \{i,j\}} H_k H_l = -\frac{1}{4\alpha} \quad \vee \quad \sum_k H_k = 0$$