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Gravitational waves from an early matter era: numerical approach.

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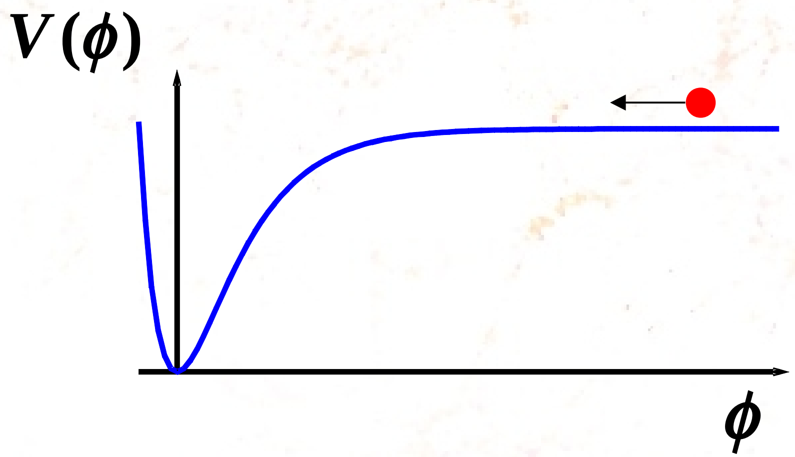
The effect

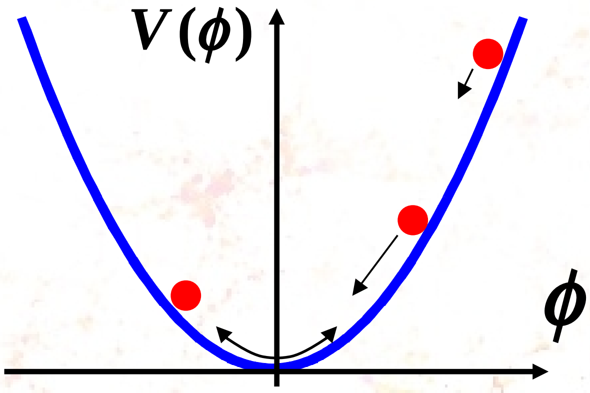
1. Inflation

- $$\frac{M_{Pl}V'(\phi)}{V(\phi)} \ll 1, \quad \phi \gg M_{Pl}$$
- Scale factor $a \propto e^{Ht}$
- Perturbations $\delta\rho_k/\rho \propto H_k/M_{Pl} \approx const$



2. Postinflationary phase

- Consider $V(\phi) \approx m^2\phi^2/2, \quad \phi \ll M_{Pl}$
- Scale factor $a \propto t^{\frac{2}{3}}$ ⇨ dust-like stage!
- Perturbations $\delta\rho_k/\rho \propto a(t), \quad \mathrm{k/a(t)} \gtrsim H$



If the reheating temperature is small enough

$$\mathrm{T_{reh}} \lesssim 10^9\ \mathrm{GeV}$$

- **Perturbations growth and become nonlinear at small scales:**

$$\delta\rho_k/\rho \sim 1$$

- **Gravitationally-bounded scalar field objects (boson stars) are formed**
- **They are evolve during some time (collapse, merge, etc.)**
- **They are destroyed during reheating**

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Gravitational waves sources!

Linear analyses

H. Assadullahi, D. Wands, arXiv:0901.0989

In the approximation of a pressureless matter-like stage: $\Psi = \Phi = \Phi(x)$

$$h''_{ij} + \frac{4}{\eta} h'_{ij} - \Delta h_{ij} = S^{TT}_{ij} \ .$$

2nd-order!

For the source term we have: $S_{ij}(x) = 4\Phi \partial_i \partial_j \Phi - \frac{2}{3} \partial_i \Phi \partial_j \Phi$

After Fourier transformation we have for one polarization mode

$$h''_k + \frac{4}{\eta} h'_k + k^2 h_k = S_k \ , \qquad S_k = \frac{40}{3(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{p}\, p^2 \sin^2\theta_{\mathbf{pk}} \Phi_{\mathbf{k}-\mathbf{p}} \Phi_{\mathbf{p}}$$

Solution with $h = h' = 0$ initial conditions is

$$h_k = \frac{S_k}{k^2}\left[1 + 3\frac{k\eta\cos(k\eta) - \sin(k\eta)}{k^3\eta^3}\right] \to \frac{S_k}{k^2}\frac{(k\eta)^2}{10} \quad \text{for } k\eta \ll 1$$

Power spectrum of gravitational waves

$$\mathcal{P}_h(k) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{40}{3}\right)^2 \int_0^{v_{\max}} dv \int_{|v-1|}^{|v+1|} dy \frac{v^2}{y^2}(1-\mu^2)^2 \mathcal{P}_\zeta(ky)\mathcal{P}_\zeta(kv) g^2(x)$$

where $x = k\eta, \quad v = p/k, \quad y = \sqrt{1 + v^2 - 2v\cos\theta_{\mathbf{pk}}}, \quad \text{and}$

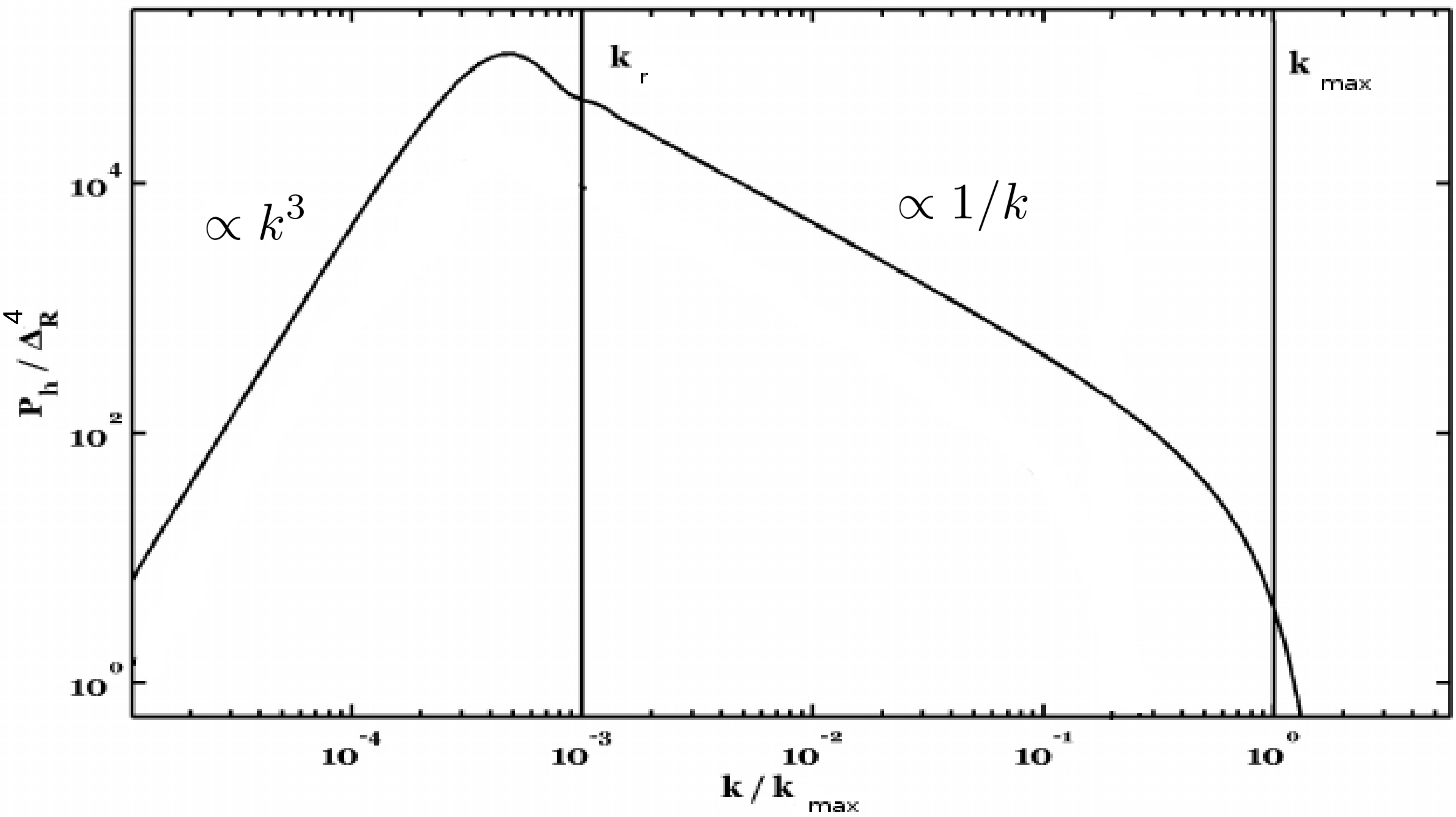
$$g(x) = \left[1 + 3\frac{x\cos(x) - \sin(x)}{x^3}\right]$$

$v_{\max}$ — cut-off corresponds to the modes which become nonlinear

For the flat scalar spectrum $\mathcal{P}_\zeta(k) = \Delta_R^2 \simeq 10^{-9}$ and for the scales $k_r < k < k_{max}$

at reheating time we have

$$\frac{\mathcal{P}_h}{\Delta_R^4(k)} \approx 2\left(\frac{40}{3}\right)^2 \left(\frac{16\,k}{35\,\mathrm{k_{max}}} + \frac{16\,\mathrm{k_{max}}}{15\,k} - \frac{4\,\mathrm{k_r}^4}{15\,k^4} + \frac{8\,\mathrm{k_r}^6}{105\,k^6}\right) \sim 380\frac{k_{max}}{k}$$



$k_{\max}(T_{\rm reh})$ – cut-off scale corresponded to the scalar modes which become nonlinear

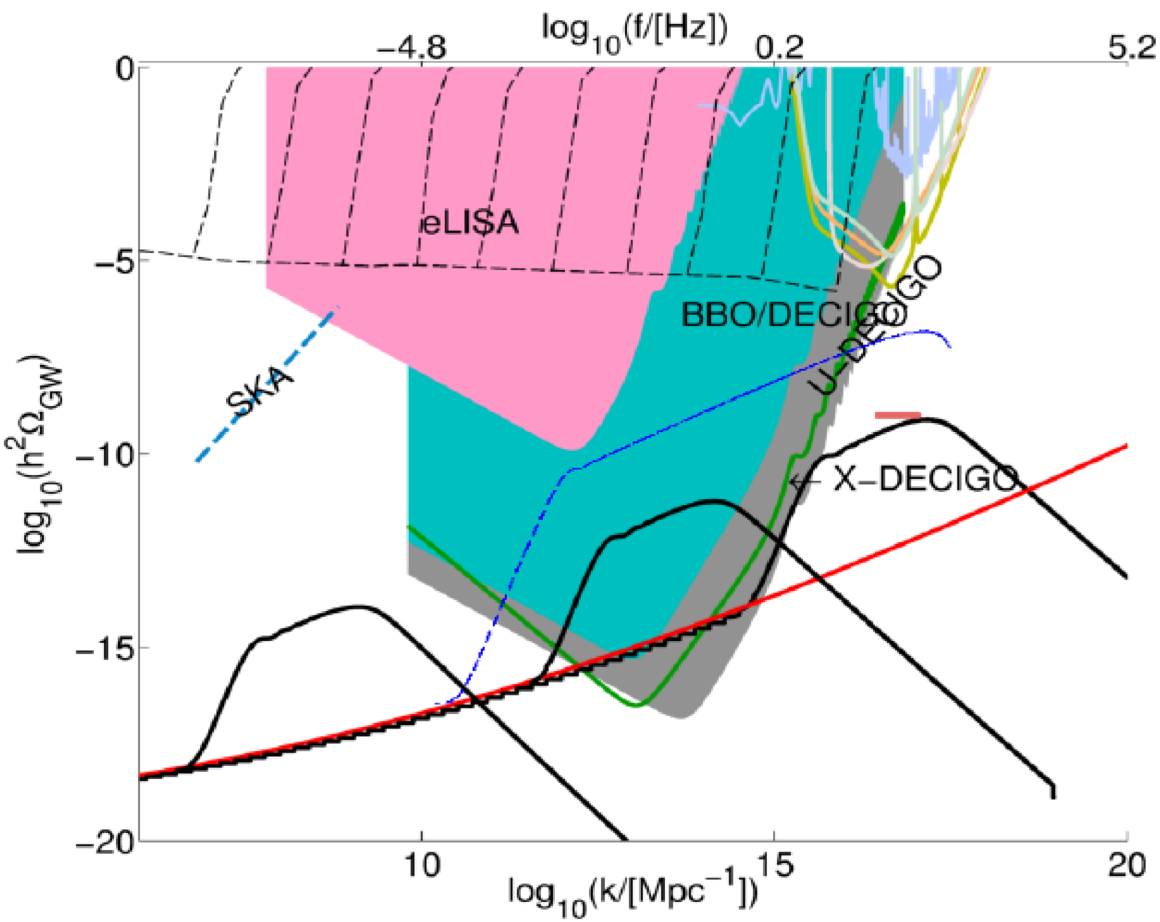
$k_r = H_r$ – reheating scale

Energy density of gravitational waves

$$\rho_{GW} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \frac{1}{32\pi G a^2}\int d(\ln k)\, k^2 \mathcal{P}_h(k,\eta_r)\, t^2(\eta_r,\eta_0) \qquad \Omega_{GW} = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln k}$$

L. Alabidi, K. Kohri, M. Sasaki, Y. Sendouda, arXiv:1303.4519

The Hilltop type model $V = V_0\left(1 + \eta_p \varphi^p - \eta_q \varphi^q\right)$

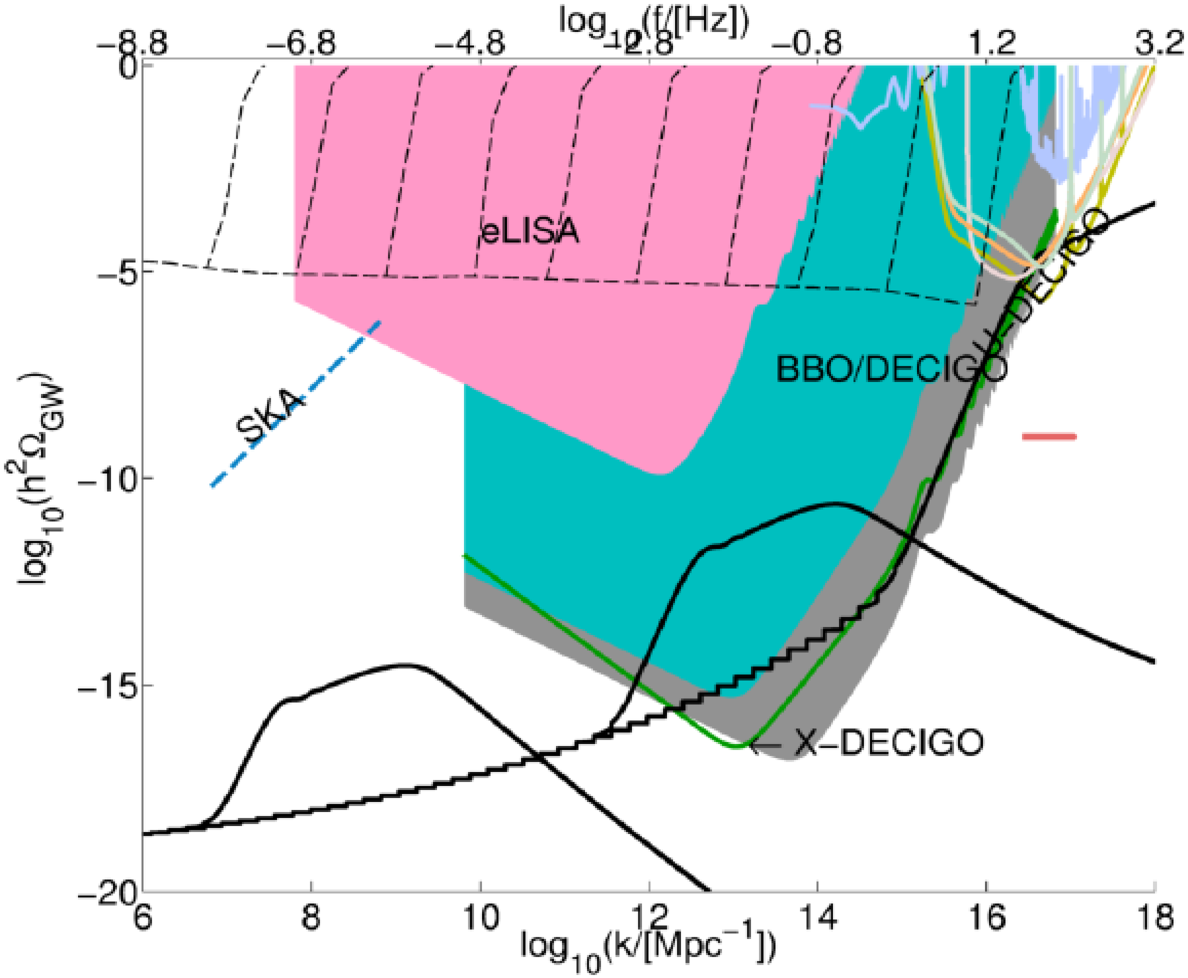


The gravitational waves energy density for p = 2 and q = 2.3. From right to left, the the black solid lines correspond to GW energy density for $T_r = 10^8$ GeV, $T_r = 10^5$ GeV , and 1 GeV.

For illustration, the dashed blue line correspond to GW energy density for $T_r = 10^5$ GeV integrated up to maximum scale instead of no nlinear cut-off scale.

The running mass model $V = V_0 \left(1 - \frac{B_0}{2}\phi^2 + \frac{A\phi^2}{2(1+\alpha \ln \phi)^2} \right)$

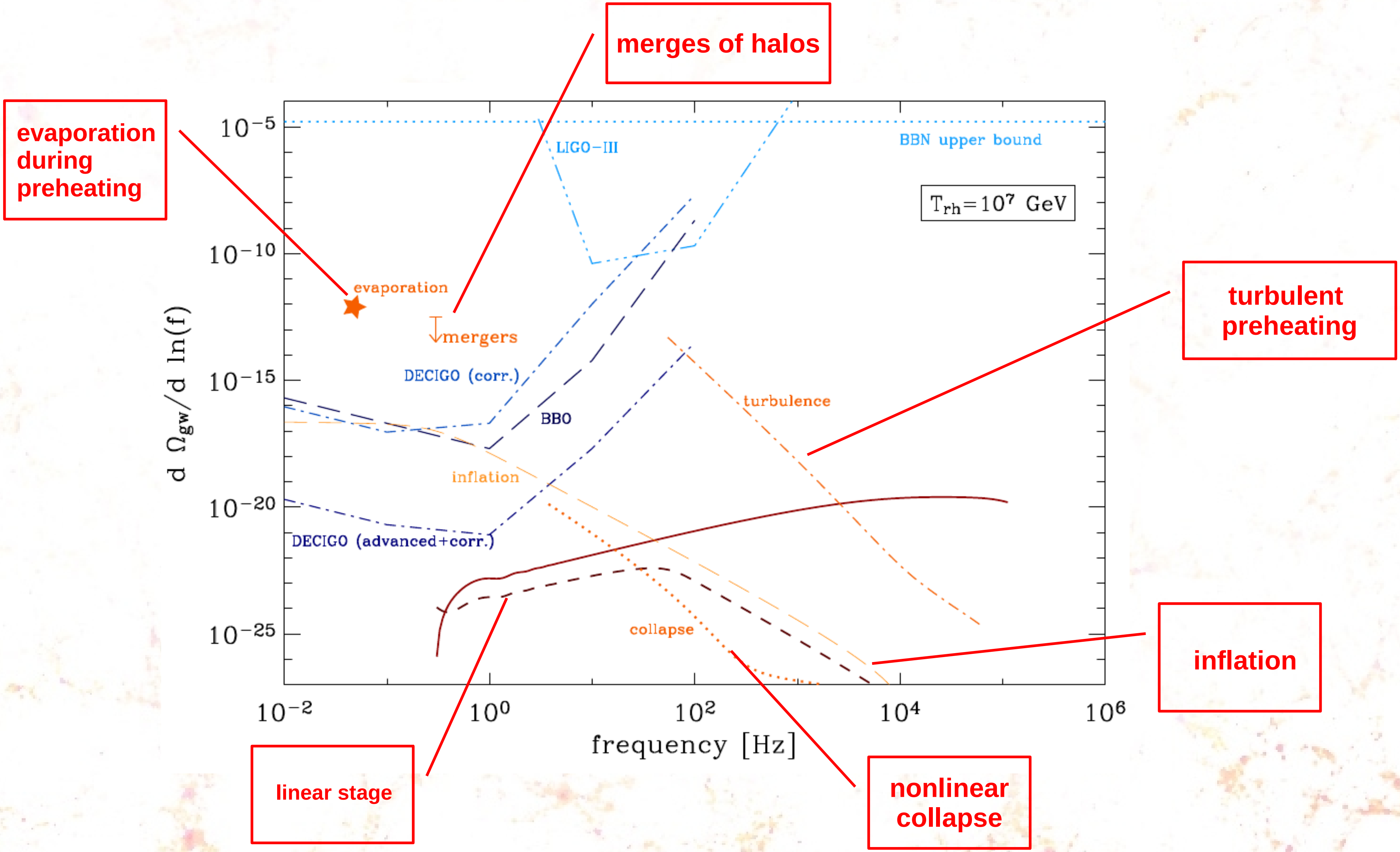
The gravitational waves spectra for $n_s' = 0.0039$. From left to right, the the black solid lines correspond to GW spectra for $T_r = 10^9$ GeV, $T_r = 10^5$ GeV, and $T_r = 1$ GeV



Gravitational waves from nonlinear stage

Order of magnitude estimation have been made by
K. Jedamzik, M. Lemoine, J. Martin, arXiv:1002.3039

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$



Numerical approach

non-relativistic approach

Let us note that:

- we are dealing with the modes $H < k/a(t) < H_{end} \simeq m$

The upper limit corresponds to the modes, which live the horizon at the end of inflation. Modes with higher momentum remain at vacuum state.

- gravity is Newtonian !

The scalar perturbations is a Gaussian random field which spacial form is far from spherical symmetry. This prevents creation of the black holes during evolution.

Schroedinger-Newton system of equations:

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + aU\psi$$

$$\Delta U = |\psi|^2 - |\bar{\psi}|^2$$

Where we introduce $\phi = \frac{1}{4\pi G\, a^{\frac{3}{2}}}\left(\psi\, e^{-imt} + \mathrm{h.c.}\right)$

$$U = a\Phi$$

and rescale the coordinates $\mathbf{x} \to m\mathbf{x}, \quad d\tau = mdt/a^2$

1D evolution

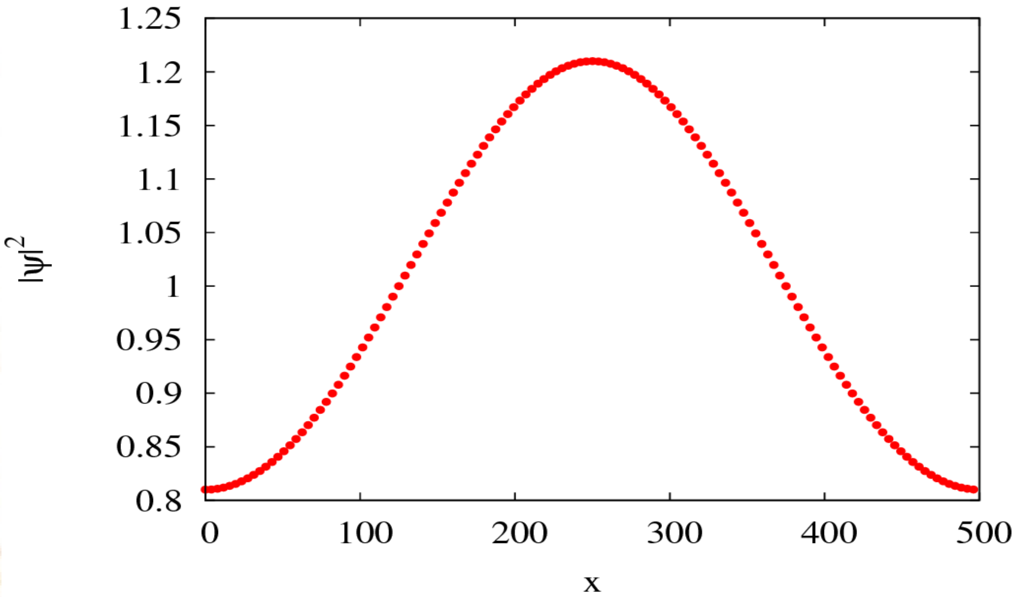
We use the following discretization

$$i\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t} = -\frac{\psi_{j+1}^{n+1}+\psi_{j-1}^{n+1}-2\psi_j^{n+1}}{4(\Delta x)^2} - \{n+1\to n\} + \frac{1}{2}U_j^{n+\frac{1}{2}}\left(\psi_j^{n+1}+\psi_j^n\right)$$

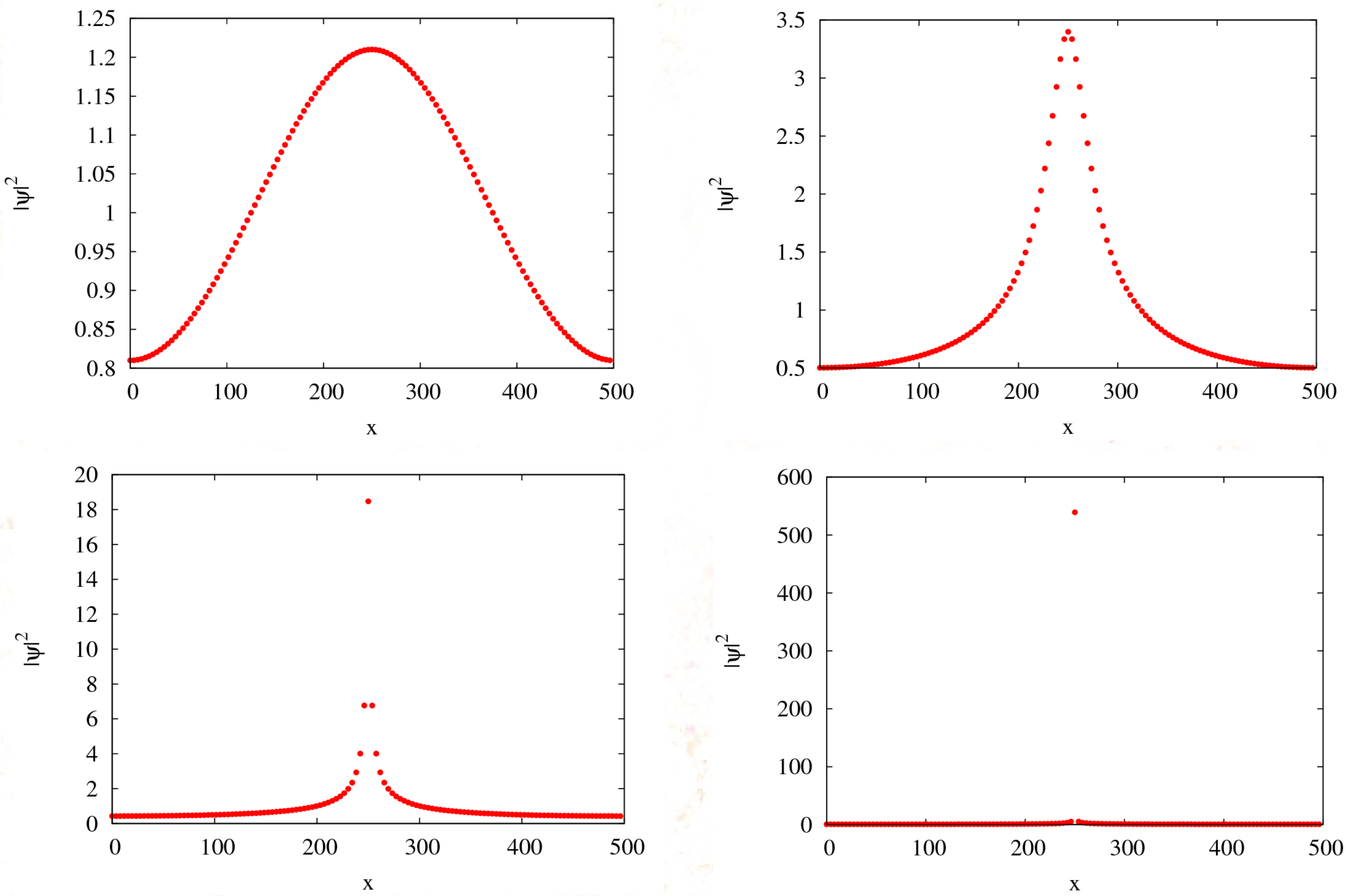
$$\frac{U_{j+1}^{n+\frac{1}{2}}+U_{j-1}^{n+\frac{1}{2}}-2U_j^{n+\frac{1}{2}}}{(\Delta x)^2} = \frac{1}{2}\left(|\psi_j^{n+1}|^2+|\psi_j^n|^2\right) - \langle|\psi|^2\rangle$$

One can solve this system of equations using Crank-Nicolson iterative time stepping algorithm together with some solver for Poisson's equation.

For illustration let's evolve the following configuration



Uniform grid results (N = 128)



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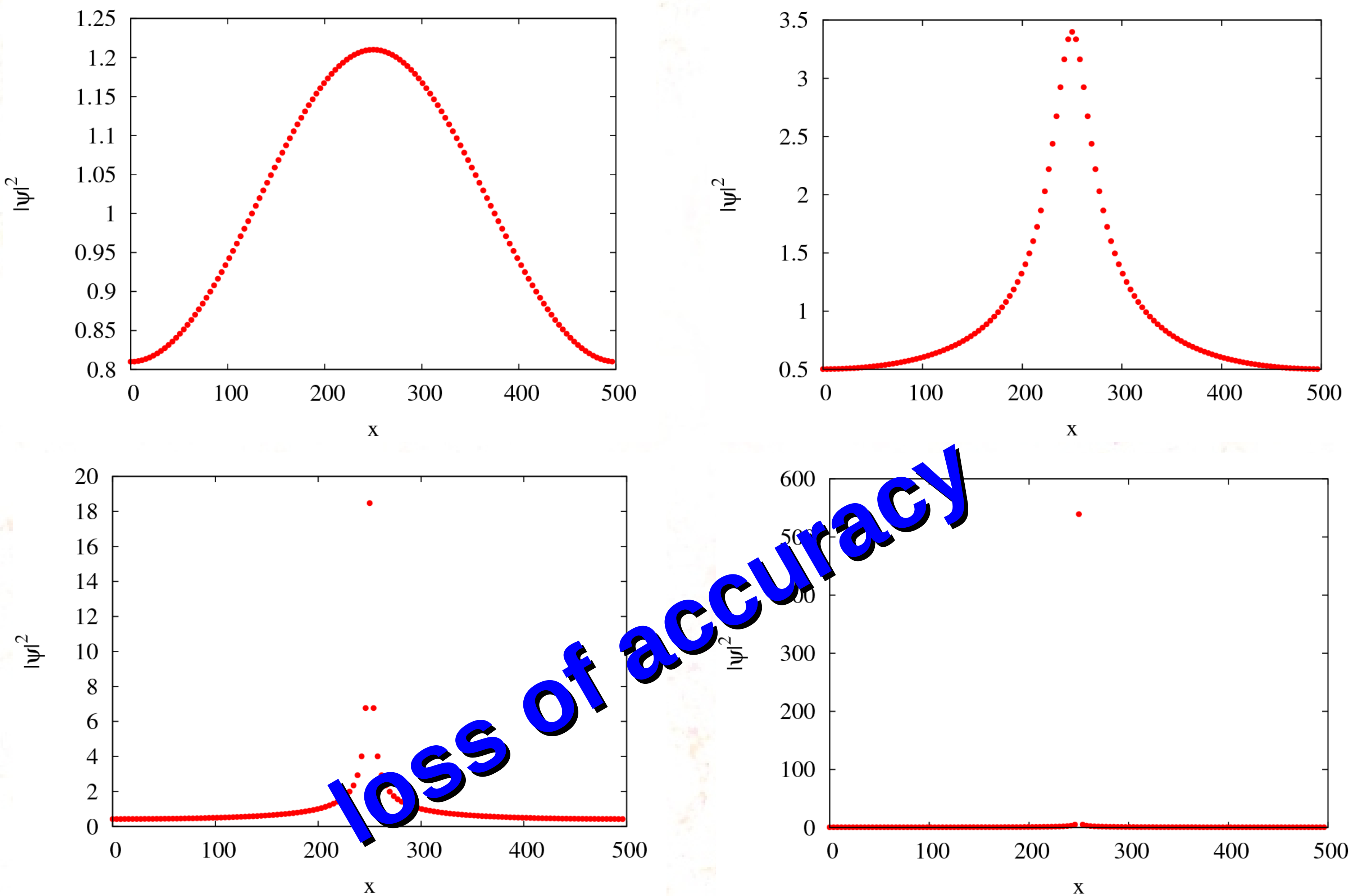


Adaptive mesh refinement

The main idea is to adaptively change the grid spacing during evolution in order to achieve appropriate accuracy for numerical solution.

One can use several grids with different resolution which are automatically created during evolution.

Implementation:

- **Refinement criterion**

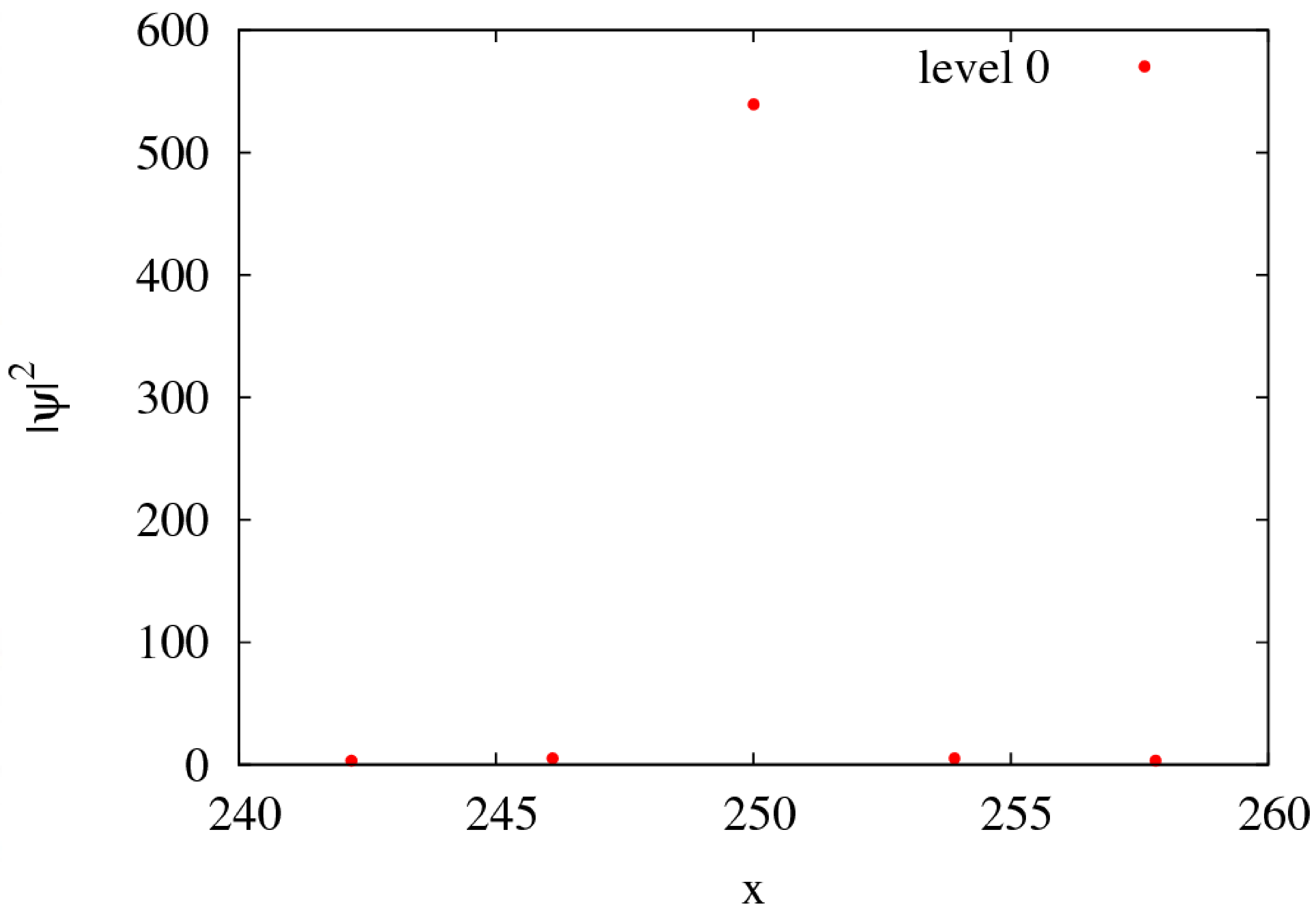
  As the solution proceeds we identify the regions requiring more resolution. One can use some criterion based on truncation error estimation or on some solution properties (local overdensity, smoothness, etc.)

- **Fine grid construction procedure**

  Superimpose finer subgrids on the regions needed to be refined by interpolation solution from coarser grid.

- **Finer and finer subgrids are added recursively during evolution to achieve desired level of accuracy**

Adaptive mesh refinement (AMR)



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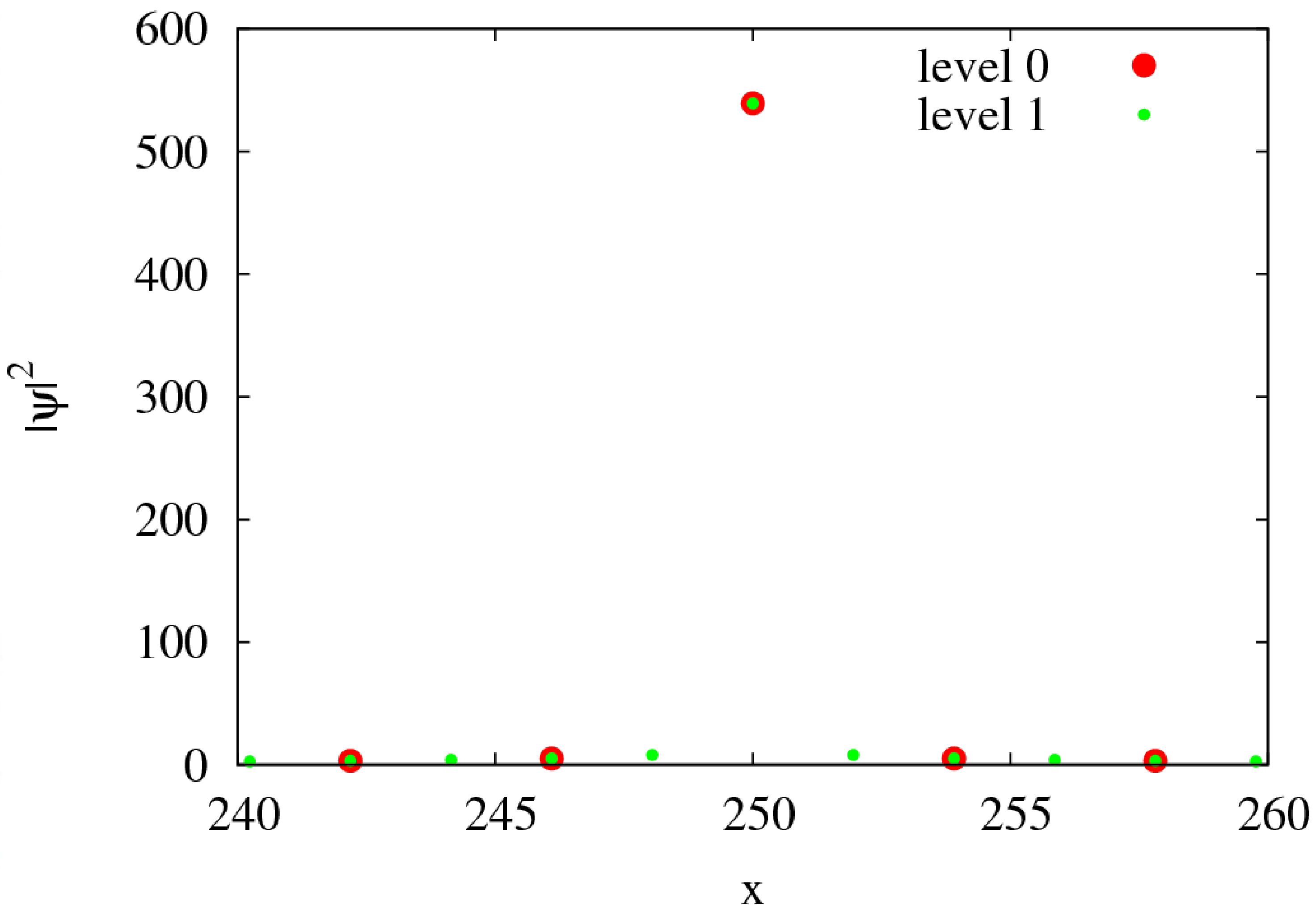

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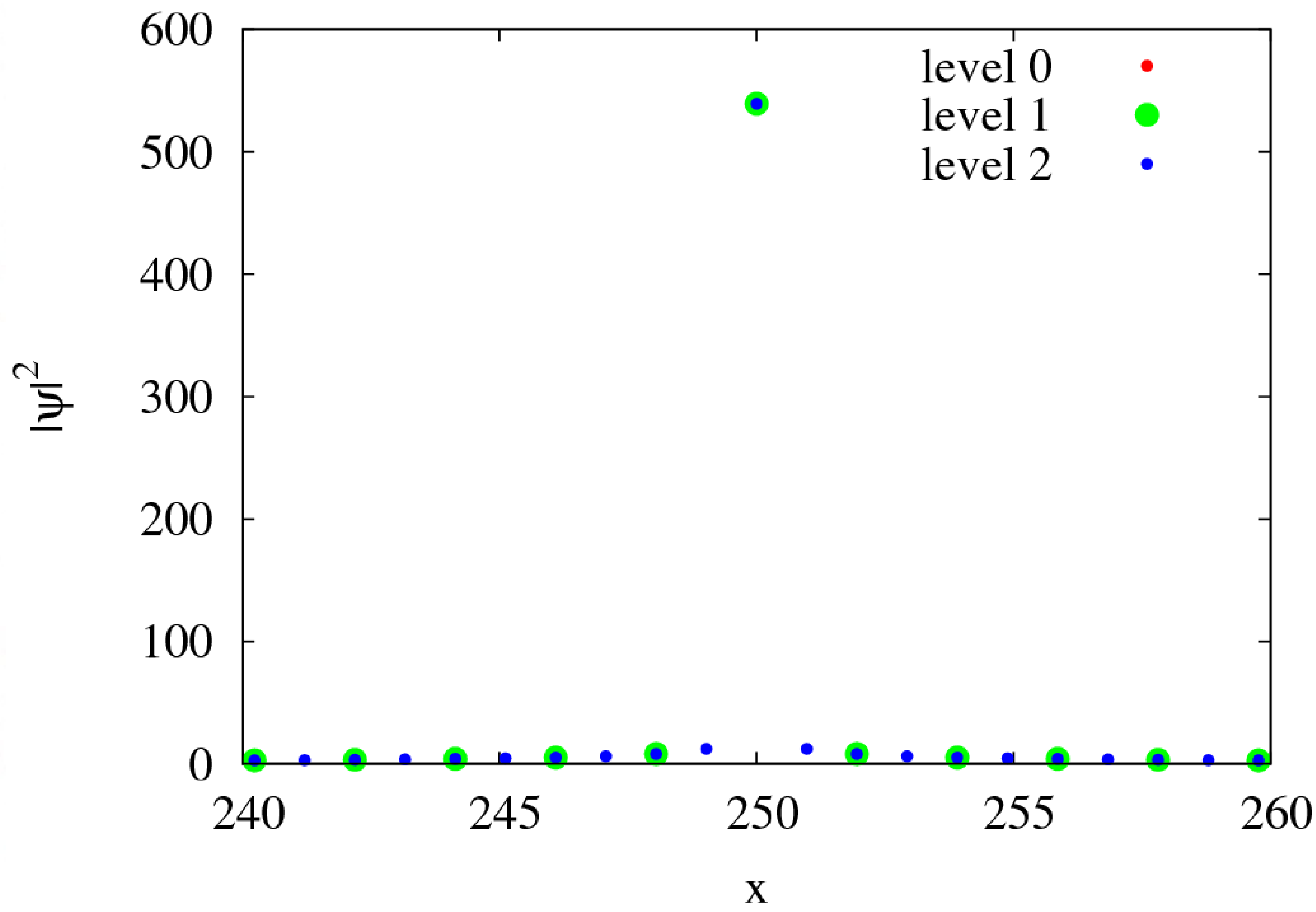

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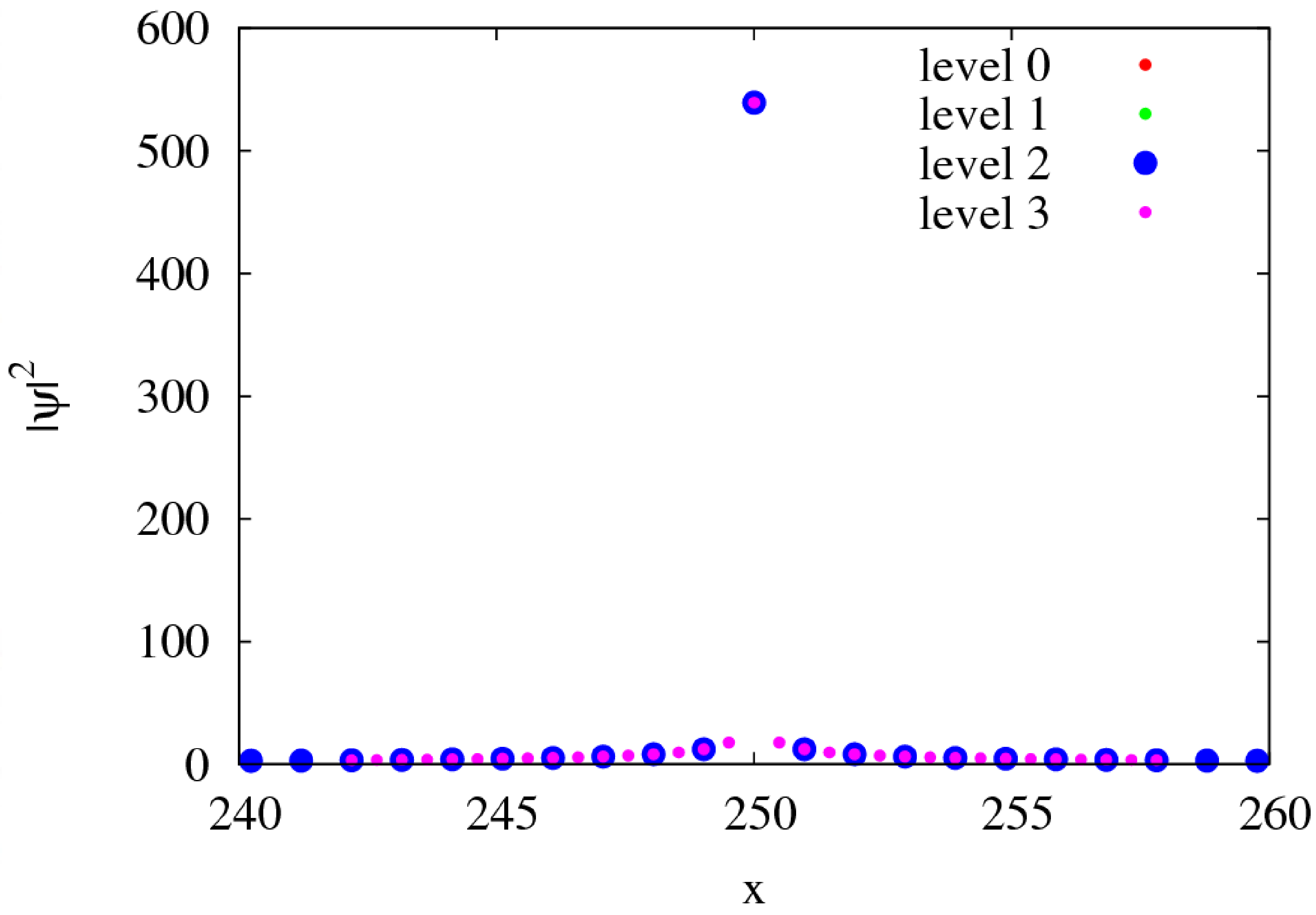

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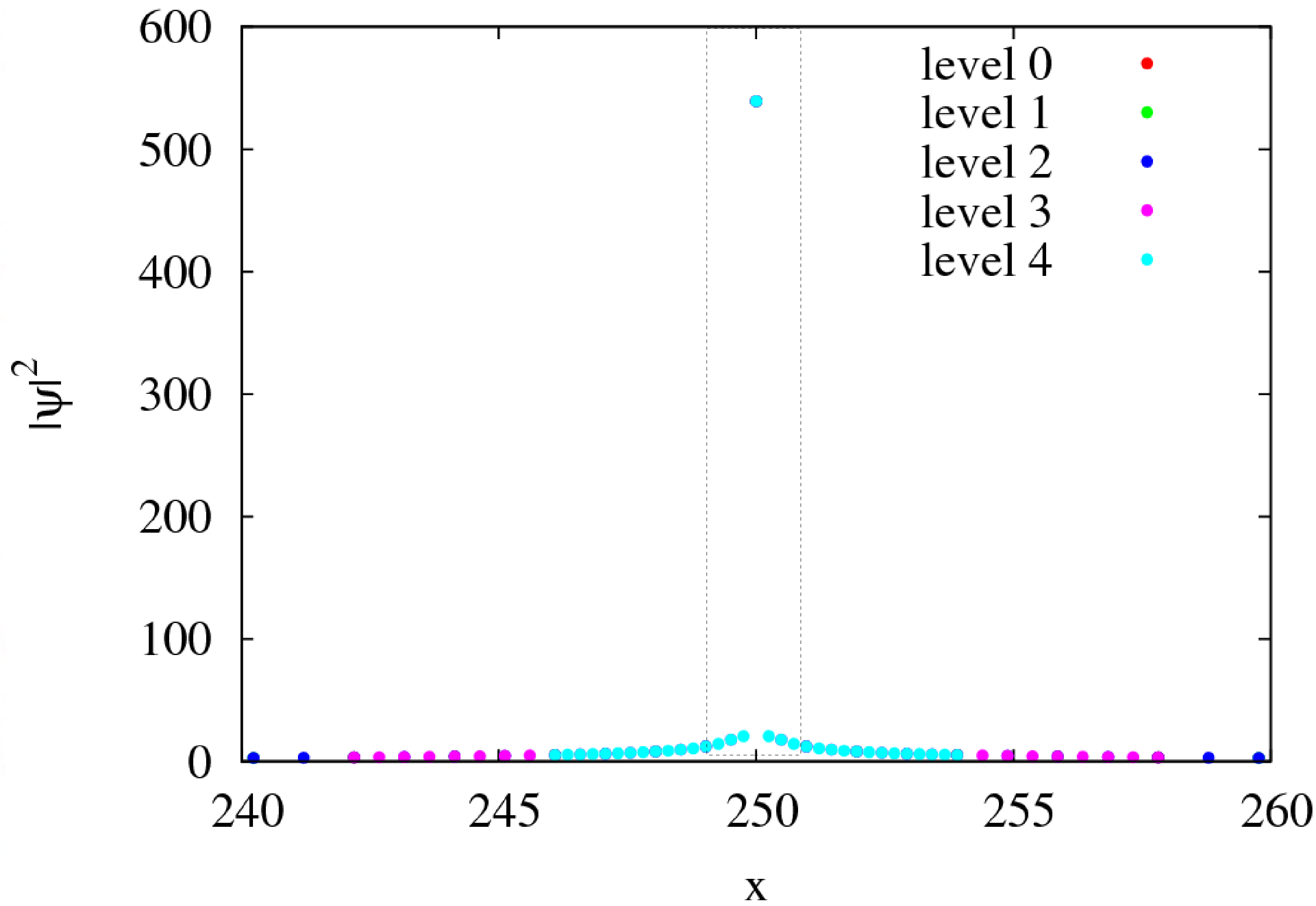
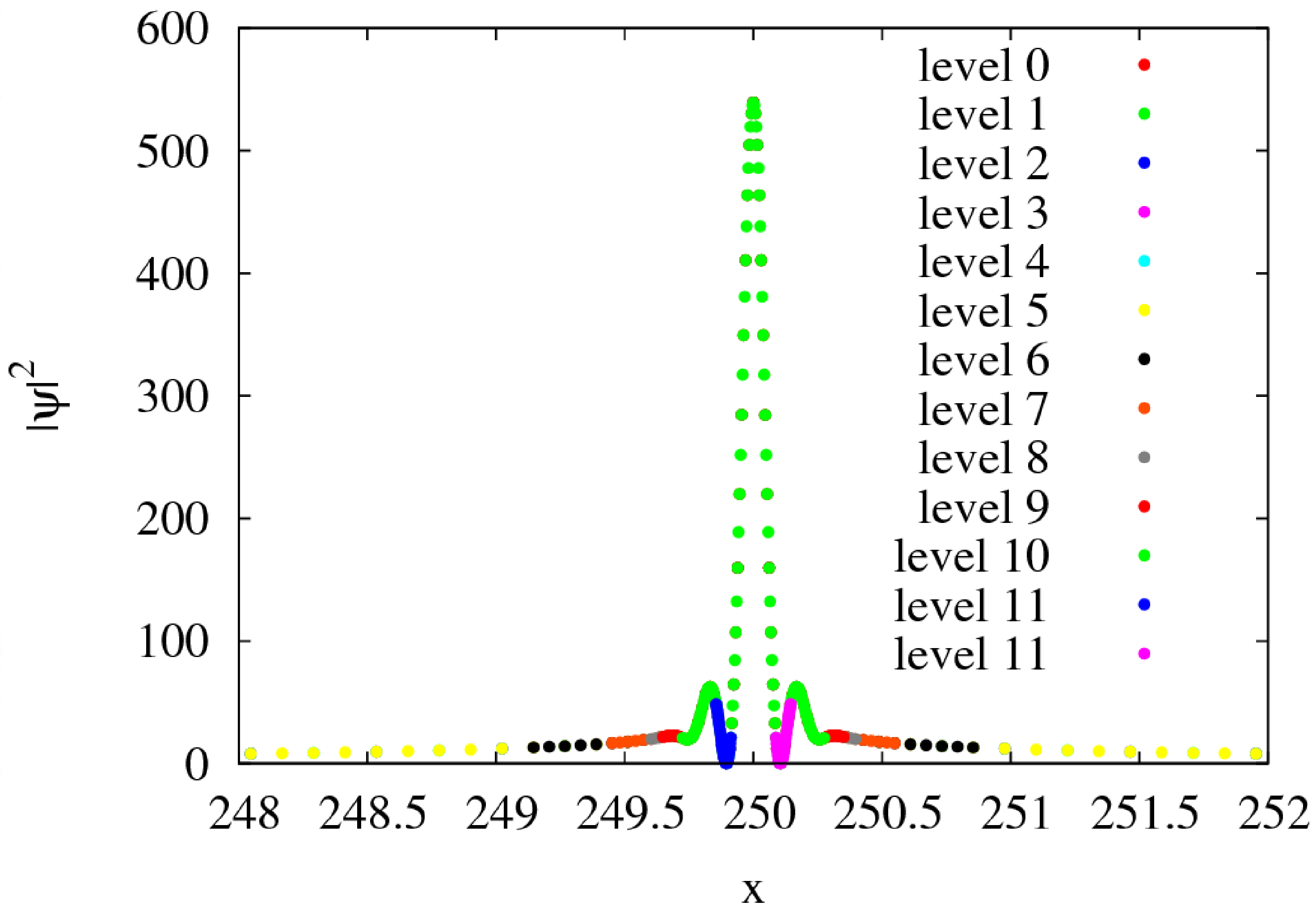
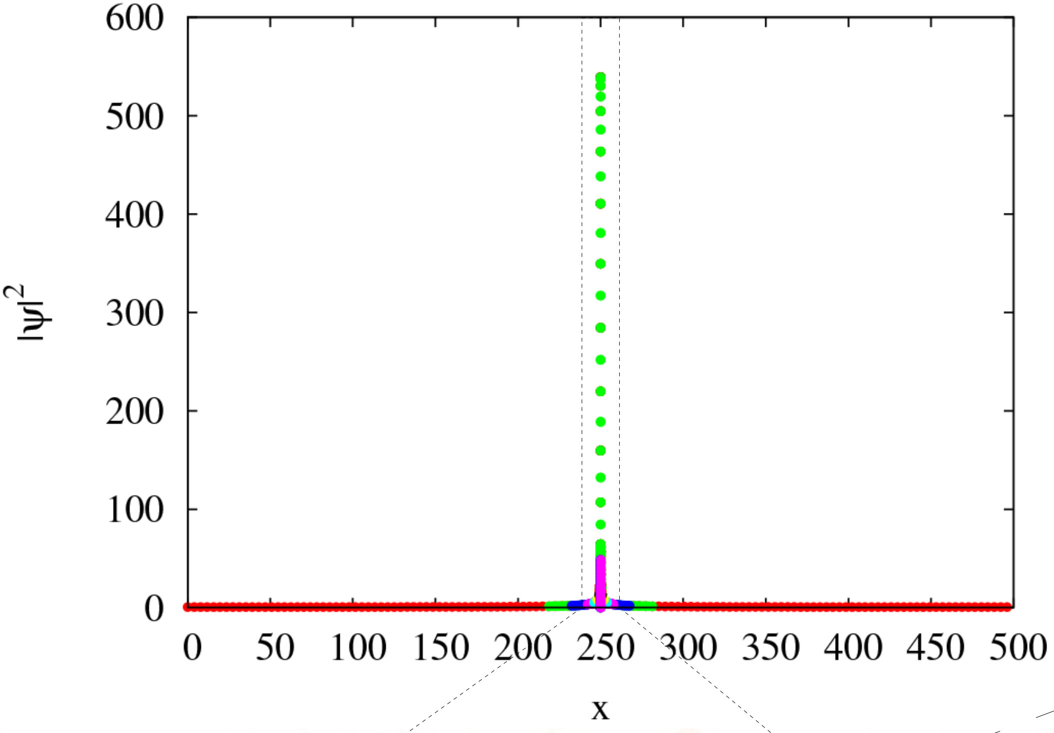


Adaptive mesh refinement (AMR)

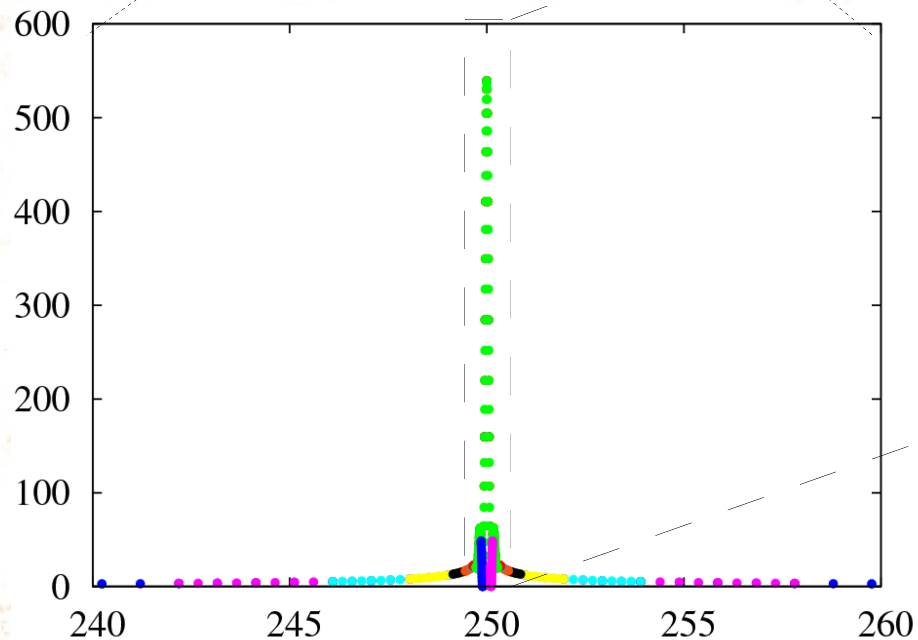


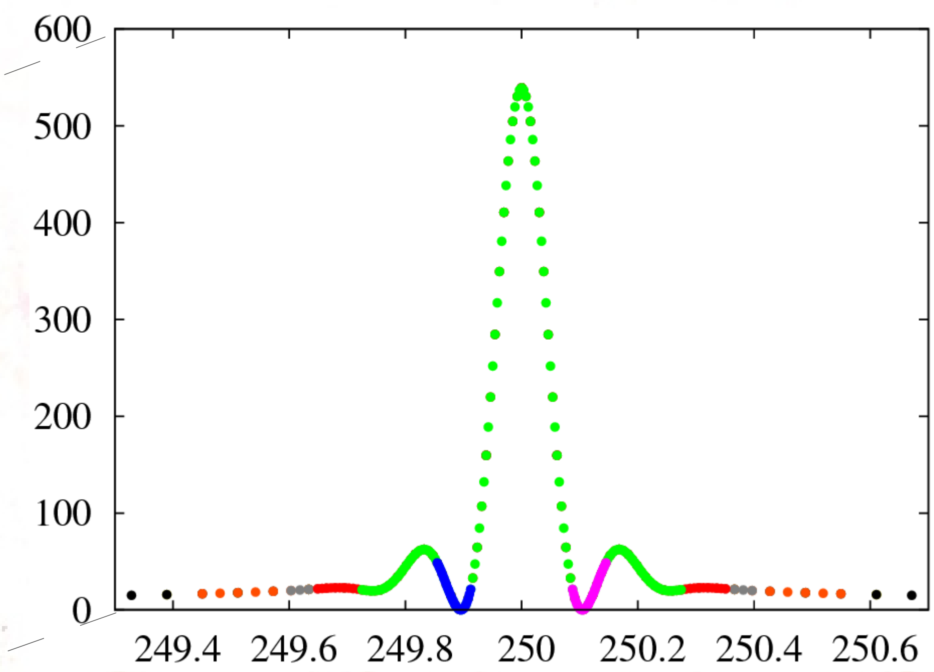
AMR solution



We achieve refinement factor $2^{11} = 2048$ using 1345 grid points only!

In order to obtain the same resolution on a uniform grid, we would need of order of 2×10^5 grid points.





AMR implementation to the Schroedinger-Newtonian system

The main features of our algorithm

Different field variables at different regions

ψ is highly oscillating function. We use $\psi = \rho e^{i\theta}$ everywhere except the regions with $\rho \approx 0$.

The norm of the wave function is conserved

We had to use a first-order discretization at the coarse-fine boundary points. The resulting error (the difference between exact and numerical solutions) remains second-order.

Multigrid solver

Multigrid solves nonlinear system of equations with O(N) operations.

Summary

Gravitational waves from nonlinear stage of the scalar field evolution during early matter-like stage of the Universe history may be sensitive for the future space-based experiments.

This nonlinear stage can be simulated numerically with the help of adaptive mesh refinement technique and multigrid solver.

We plan to implement the developed algorithm for 3D problem and compute the spectra of the gravitational waves emitted during this nonlinear stage of the scalar field evolution.