Some general properties of U(1) gauged Q-balls

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Simplest nontopological soliton in (3+1) space-time

$$S = \int d^4x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right)$$

$$\phi(t, \vec{x}) = \phi(\vec{x})$$

1) There are no solutions for $V(\phi) \geq 0$

(G.H. Derrick, *“Comments on Nonlinear Wave Equations as Models for Elementary Particles”*, J. Math. Phys. **5** (1964) 1252)

2) Such solutions (if exist) are always classically unstable

$$\phi(t, x) = \phi_0(x) + \varphi(x) e^{\gamma t}$$

Q-balls

How to overcome the Derrick theorem and the classical instability? One should take time-dependent solutions.

The simplest case is a complex scalar field with U(1) global symmetry.

$$S = \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi - V(\phi^* \phi)\right)$$

$$\phi(t, \vec{x}) = e^{i\omega t} f(\vec{x})$$

Now non-topological solitons in such theories are called Q-balls.

Q-ball charge

$$Q = -i \int d^3x \left(\phi^* \dot{\phi} - \dot{\phi}^* \phi \right)$$

Q-ball energy

$$E = \int d^3x \left(\dot{\phi}^* \dot{\phi} + \partial_i \phi^* \partial_i \phi + V(\phi^* \phi) \right)$$

It is easy to show that

$$\frac{dE}{dQ} = \omega$$

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U(1) gauged Q-balls

What if the symmetry is not global, but local?

The simplest case – U(1) symmetry.

$$S = \int d^4x \left((\partial^\mu \phi^* - ieA^\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\begin{aligned}
\phi(t, \vec{x}) &= \mathrm{e}^{i\omega t} f(r), & f(r)|_{r\to\infty} &\to 0, & \left.\frac{df(r)}{dr}\right|_{r=0} &= 0, \\
A_0(t, \vec{x}) &= A_0(r), & A_0(r)|_{r\to\infty} &\to 0, & \left.\frac{dA_0(r)}{dr}\right|_{r=0} &= 0, \\
A_i(t, \vec{x}) &\equiv 0, & & & &
\end{aligned}$$

$$r = \sqrt{\vec{x}^2} \qquad f(0) > 0$$

We define

$$g = eA_0,\ V(f) = V(\phi^*\phi)$$

Equations of motion

$$2e^2(\omega + g)f^2 = \Delta g,$$
$$2(\omega + g)^2 f + 2\Delta f - \frac{dV}{df} = 0,$$

where $\Delta = \sum_{i=1}^{3} \partial_i \partial_i$

K.-M. Lee, J.A. Stein-Schabes, R. Watkins and L.M. Widrow, *"Gauged Q-balls"*, Phys. Rev. D **39** (1989) 1665.

G. Rosen, *"Charged particlelike solutions to nonlinear complex scalar field theories"*, J. Math. Phys. **9** (1968) 999.

The charge

$$Q = 2\int d^3x(\omega + g)f^2$$

1) Symmetry of equations of motion $\omega \to -\omega,\ g \to -g$

2) It is easy to show (K.-M. Lee, J.A. Stein-Schabes, R. Watkins, L.M. Widrow), that the sign of $\omega + g$ coincides with the sign of ω

3) If $g \equiv 0$, then $\omega = 0$ (G. Rosen)

For simplicity, we can consider $\omega \geq 0$, which corresponds to $Q \geq 0$

The energy of gauged Q-ball is

$$E = \int d^3x \left((\omega + g)^2 f^2 + \partial_i f \partial_i f + V(f) + \frac{1}{2e^2}\partial_i g \partial_i g \right)$$

dE/dQ for U(1) gauged Q-balls

$$
\begin{aligned}
\frac{dE}{d\omega} &= \int \left(2\frac{d(\omega+g)}{d\omega}(\omega+g)f^2 + 2(\omega+g)^2 f\frac{df}{d\omega} + 2\partial_i f\partial_i\frac{df}{d\omega} + \frac{dV}{df}\frac{df}{d\omega} \right.\\
&+ \left. \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{d\omega}\right) d^3x = \int \left(2\frac{d(\omega+g)}{d\omega}(\omega+g)f^2 + 2(\omega+g)^2 f\frac{df}{d\omega} \right.\\
&+ \left. \left(-2\Delta f + \frac{dV}{df}\right)\frac{df}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{d\omega}\right) d^3x\\
&= \int \left((\omega+g)\left(2\frac{d(\omega+g)}{d\omega}f^2 + 4(\omega+g)f\frac{df}{d\omega}\right) + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{d\omega}\right) d^3x
\end{aligned}
$$

The charge density $q = 2(\omega+g)f^2$

$$
\frac{dE}{d\omega} = \int \left((\omega+g)\frac{dq}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{d\omega}\right) d^3x = \omega\frac{dQ}{d\omega} + \int \left(g\frac{dq}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{d\omega}\right) d^3x
$$

$$
Q = \int q d^3x
$$

From the equations of motion it follows that $\frac{dq}{d\omega} = \frac{1}{e^2}\Delta\frac{dg}{d\omega}$.

$$\frac{dE}{d\omega} = \omega\frac{dQ}{d\omega} + \frac{1}{e^2}\int\left(g\Delta\frac{dg}{d\omega} + \partial_i g\partial_i\frac{dg}{d\omega}\right)d^3x$$

$$\frac{dE}{d\omega} = \omega\frac{dQ}{d\omega}$$

$$\boxed{\frac{dE}{dQ} = \omega}$$

Maximal charge of gauged Q-balls

K.-M. Lee, J.A. Stein-Schabes, R. Watkins and L.M. Widrow, *"Gauged Q-balls"*, Phys. Rev. D **39** (1989) 1665.

$$S_{\text{scalar}} \approx \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi - M^2 \phi^* \phi\right)$$

The maximal charge Q_{max} is defined from the condition

$$\frac{dE}{dQ} = M$$

For $\frac{dE}{dQ} > M$ it is energetically favorable to have Q-ball with the charge Q_{max} and $Q - Q_{max}$ free particles

Equation of motion for the scalar field

$$2(\omega + g)^2 f + 2\Delta f - \frac{dV}{df} = 0$$

Far away from the Q-ball center

$$(\omega^2 - M^2) f + \Delta f = 0$$

$$\boxed{\omega < M}$$

$$\frac{dE}{dQ} = \omega < M$$

Coulomb repulsion?

$$\Delta g - 2e^2 f^2 g = 2e^2 \omega f^2$$

Gauged Q-balls in the case of small back-reaction of the gauge field

$$|g(r)| \ll \omega, \; |f(r) - f_0(r)| \ll f_0(r)$$

Linearized equations of motion for $\varphi(r) = f(r) - f_0(r)$

$$\Delta g - 2e^2 \omega f_0^2 = 0,$$

$$\Delta\varphi + \omega^2\varphi + 2\omega g f_0 - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0}\varphi = 0,$$

Equation for the background Q-ball solution

$$\omega^2 f_0 + \Delta f_0 - \frac{1}{2}\frac{dV}{df}\bigg|_{f=f_0} = 0$$

Applicability of the linear approximation – smallness of the coupling constant e?

Let us change the variables as $f_0' = \gamma f_0,\ \varphi' = \gamma\varphi$

and introduce the notation $e = \gamma e'$

$$\Delta g - 2\, e'^2 \omega f_0'^2 = 0,$$

$$\Delta\varphi' + \omega^2\varphi' + 2\omega g f_0' - \frac{1}{2}\frac{d^2V'(f')}{df'^2}\bigg|_{f'=f_0'}\varphi' = 0,$$

$$V'(f') = \gamma^2 V\left(\frac{f'}{\gamma}\right)$$

The charge and the energy in the linear approximation

$$Q = Q_0 + \triangle Q = Q_0 + 4\pi \int\limits_0^\infty drr^2(2gf_0^2 + 4\omega f_0\varphi)$$

$$E = E_0 + \triangle E = E_0 + 4\pi\omega \int\limits_0^\infty drr^2(gf_0^2 + 4\omega f_0\varphi)$$

How to calculate the charge and the energy?

1) Differentiate equation for $f_0(r)$ with respect to ω

$$2\omega f_0 + \omega^2 \frac{df_0}{d\omega} + \Delta \frac{df_0}{d\omega} - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0} \frac{df_0}{d\omega} = 0$$

2) Multiply equation for φ by $\frac{df_0}{d\omega}$, integrate it over the space coordinates and integrate by parts

$$\int \left(\varphi \left(\Delta \frac{df_0}{d\omega} + \omega^2 \frac{df_0}{d\omega} - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0} \frac{df_0}{d\omega} \right) + 2\omega g f_0 \frac{df_0}{d\omega} \right) d^3x = 0$$

We get

$$\omega \int \left(g f_0 \frac{df_0}{d\omega} - \varphi f_0 \right) d^3x = 0.$$

$$\Delta Q = 4\pi \int_0^\infty drr^2(2gf_0^2 + 4\omega f_0\varphi) = 4\pi \int_0^\infty drr^2 \left(2gf_0^2 + 4\omega g f_0 \frac{df_0}{d\omega}\right)$$

$$= 4\pi \int_0^\infty drr^2 g \frac{dq}{d\omega} = \frac{1}{2}\frac{d}{d\omega}\int d^3x g q.$$

Let us introduce the function $I = \frac{1}{2}\int d^3x g q$

Then

$$\boxed{\begin{aligned} \Delta Q &= \frac{dI}{d\omega} \\ \Delta E &= \omega \Delta Q - I = \omega \frac{dI}{d\omega} - I. \end{aligned}}$$

It is not difficult to show that

$$I = -\frac{1}{2e^2} \int d^3x \partial_i g \partial_i g$$

Spherically-symmetric solution for g has the form

$$g = g(r) = -e^2 \int_r^{\infty} q(y) y dy - e^2 \frac{1}{r} \int_0^r q(y) y^2 dy$$

(G. Rosen, *"Charged particlelike solutions to nonlinear complex scalar field theories"*, J. Math. Phys. **9** (1968) 999)

$$\boxed{\frac{I}{4\pi} = -4e^2\omega^2 \int_0^{\infty} f_0^2(r) r \int_0^r f_0^2(y) y^2 dy \, dr}$$

$$\boxed{\begin{aligned}
Q(\omega) &= Q_0(\omega) + \frac{dI(\omega)}{d\omega},\\
E(\omega) &= E_0(\omega) + \omega\frac{dI(\omega)}{d\omega} - I(\omega),\\
I(\omega) &= -16\,\pi e^2\omega^2 \int\limits_0^\infty f_0^2(r,\omega)r\int\limits_0^r f_0^2(y,\omega)y^2 dy\, dr
\end{aligned}}$$

Applicability of the linear approximation

$$\boxed{\begin{aligned} |g(r)| &\ll \omega, \\ |\varphi(r)| &\ll f_0(r) \end{aligned}}$$

One can use the parameter

$$\alpha(\omega) = \max_i \left\{ \frac{|g(0)|}{\omega}, \frac{|2\Delta E - \omega \Delta Q|}{2\omega Q_0}, \left| \frac{g(0)}{f_0(r_i)} \frac{df_0(r_i)}{d\omega} \right| \right\}$$

$$\alpha(\omega) \sim e^2$$

The following inequality should hold

$$\boxed{\alpha(\omega) \ll 1}$$

Comparison of gauged and nongauged Q-balls with the same charge

Nongauged Q-ball $$Q = Q_0(\omega_2)$$

Gauged Q-ball $$Q = Q_0(\omega_1) + \Delta Q(\omega_1)$$

In the linear approximation in $\alpha(\omega)$

$$\Delta Q(\omega_1) = (\omega_2 - \omega_1)\frac{dQ_0}{d\omega}\Big|_{\omega=\omega_1}$$

$$E(\omega_1) - E_0(\omega_2) = E_0(\omega_1) + \Delta E(\omega_1) - E_0(\omega_2) \approx \Delta E(\omega_1) - (\omega_2 - \omega_1)\frac{dE_0}{d\omega}\Big|_{\omega=\omega_1}$$

$$= \Delta E(\omega_1) - \frac{\frac{dE_0}{d\omega}\Big|_{\omega=\omega_1}}{\frac{dQ_0}{d\omega}\Big|_{\omega=\omega_1}}\Delta Q(\omega_1) = \Delta E(\omega_1) - \omega_1 \Delta Q(\omega_1) = -I(\omega_1) = \frac{1}{2e^2}\int d^3x \partial_i g \partial_i g$$

$$\boxed{E(Q_x) = E_0(Q_x) - I(\omega)\Big|_{\omega=Q_0^{-1}(Q_x)}}$$

Thank you
for your attention!