BLTP, JINR & University of Oldenburg

Isospinning hopfions and baby Skyrmions

Ya Shnir

(works in collaboration with
A Acus, P Dorey, A Halavanau, D Harland, J Jakka, B Malomed, J M Speight and G Zhilin)

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PRD 88 (2013) 085028

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Outline

- Skyrme family
- Baby Skyrmions
- Faddeev-Skyrme model
- Soliton solutions of the Faddeev-Skyrme model
- Isorotations of the Skyrmions
- Isospinning Hopfions
- Gauged Hopfions
- Summary

Skyrme family

- *(2+1)-dim*: **Baby Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)$$

$$\phi : S^2 \to S^2; \quad \phi_\infty = (0,0,1)$$

Standard choice: $V(\phi) = \mu^2(1-\phi_3)$

$$Q \in \mathbb{Z} = \pi_2(S^2)$$

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\partial_1 \phi \times \partial_2 \phi) d^2 x$$

- *(3+1)-dim*: **Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)$$

$$\phi : S^3 \to S^3; \quad \phi_\infty = (0,0,0,1)$$

$$R_\mu = \partial_\mu U U^\dagger; \quad U = \phi_0 \mathbb{I} + i\sigma^a \cdot \phi^a$$

$$\mathcal{L} = -\ \mathrm{Tr}\left\{\frac{1}{2}(R_\mu R^\mu) + \frac{1}{16}([R_\mu, R_\nu][R^\mu, R^\nu]) + \mu^2(U - \mathbb{I})\right\}$$

$$Q \in \mathbb{Z} = \pi_3(S^3)$$

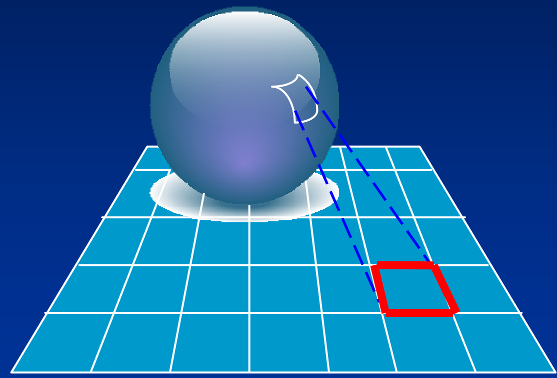
$$Q = \frac{1}{24\pi^2}\ \mathrm{Tr}\int_{\mathbb{R}^3} \varepsilon_{ijk} R_i R_j R_k d^3 x$$

Baby Skyrme model

(Bogolubskaya, Bogolubsky (1989)
R.A. Leese et al (1990)

$$\phi = (\phi^1, \phi^2, \phi^3); \qquad \phi^a \cdot \phi^a = 1; \qquad \phi: \ S^2 \to S^2$$

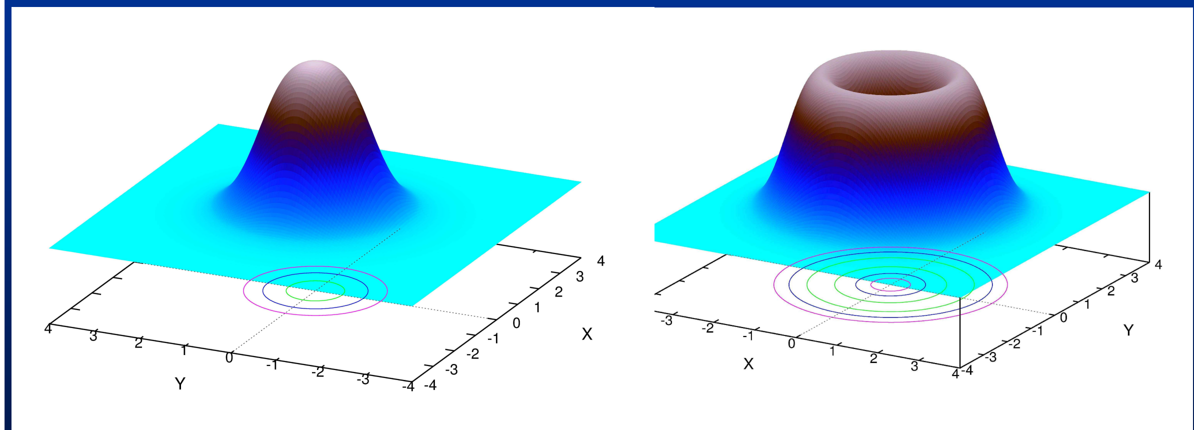
$$Q = \frac{1}{4\pi} \int d^2x \ \varepsilon_{abc} \varepsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c = 1$$



Derrick's scaling theorem: Skyrme term provides a scale but cannot stabilise the soliton: potential term is necessary

$$L = \frac{1}{4}(\partial_\mu \phi^a)^2 - \frac{\kappa}{8}\left[(\partial_\mu \phi^a \partial_\mu \phi^a)^2 - (\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^a \partial^\nu \phi^a)\right] + m^2(1 - \phi^3)$$

$E \geq \pm 4\pi Q$ equality is possible if $\kappa = 0$ and $m = 0$



Q=1 Q=2

Axially symmetric ansatz:

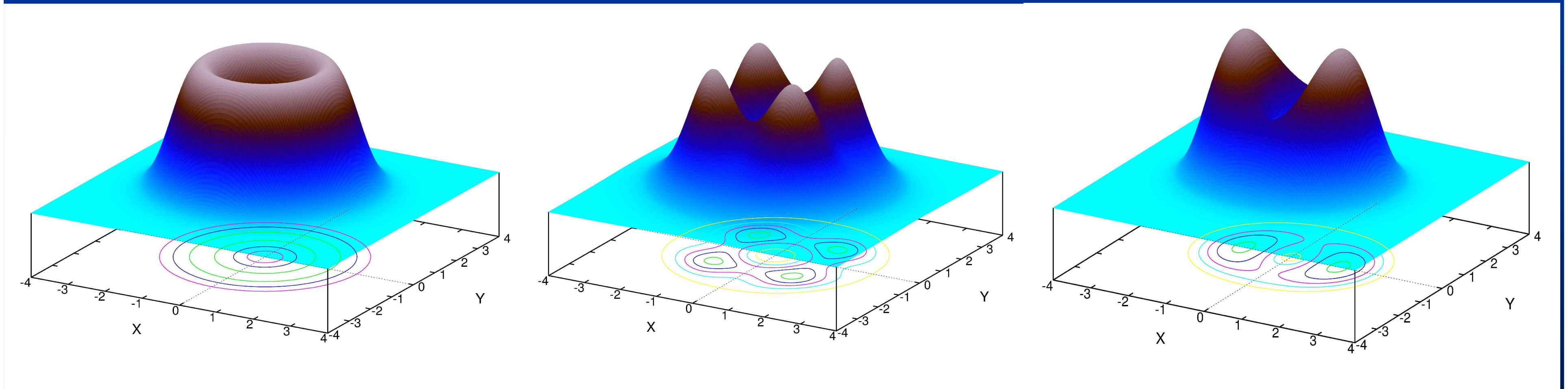
$$\phi^1 = \sin f(r) \cos(Q\varphi - \delta);$$
$$\phi^1 = \sin f(r) \sin(Q\varphi - \delta);$$
$$\phi^3 = \cos f(r)$$

Baby Skyrme model

Potential of the baby Skyrme model: potential term U(ϕ) may be chosen almost arbitrarily, however must vanish at infinity for a given vacuum field value in order to ensure existance of the finite energy solutions: $\phi^a_{(0)} = (0,0,1)$

Several potential terms have been studied in great detail:

- "Old" model, with $U(\phi) = m^2(1-\phi_3)$
- Holomorphic model, with $U(\phi) = m^2(1-\phi_3)^4$
- "Double vacuum" model, with $U(\phi) = m^2(1-\phi_3^2)$



B=2

Baby Skyrme model

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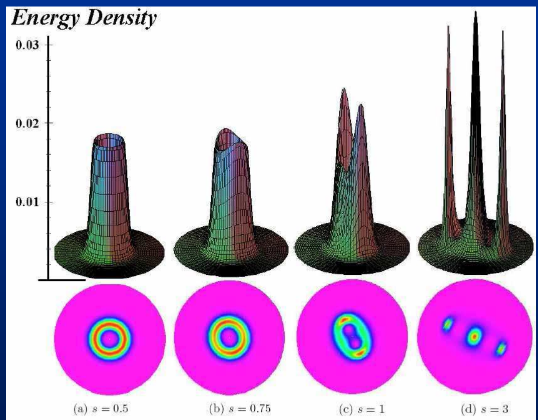
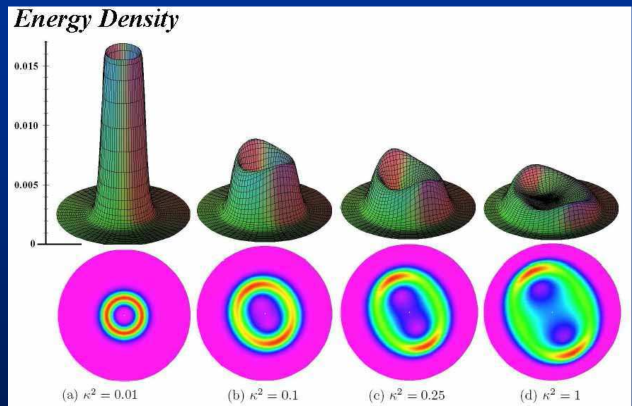
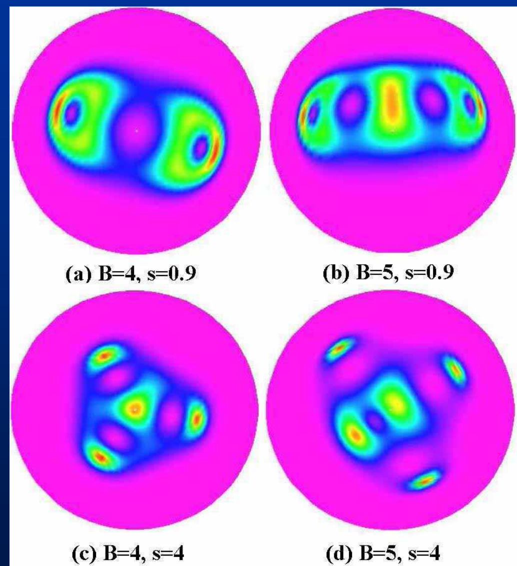
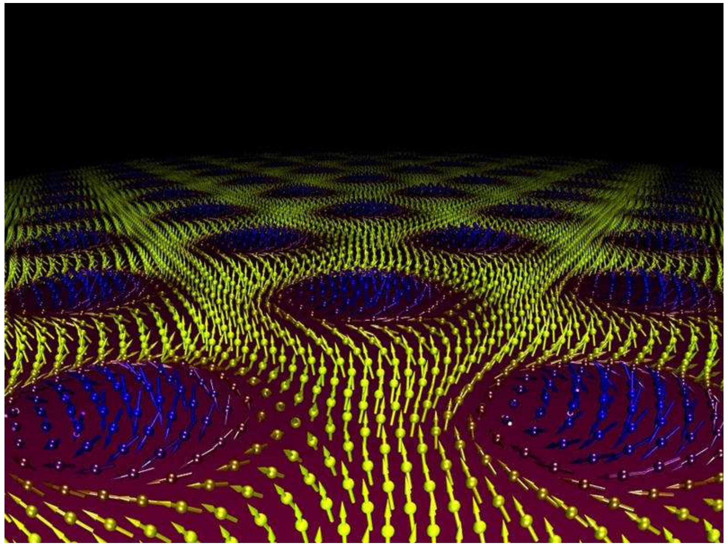


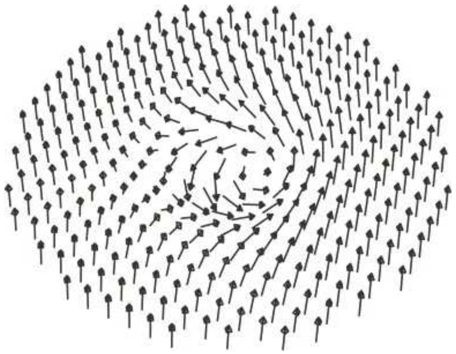




Karliner, Hen (2007) $U(\phi) = m^\alpha(1-\phi_3^\beta)$

Baby Skyrmions in action: Condensed Matter Systems

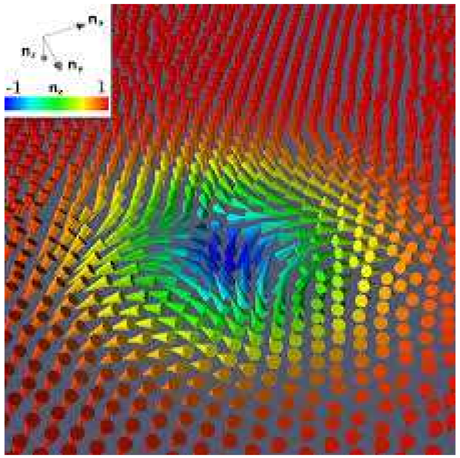




Rößler et al.
Nature 442 (2006) 797

Chiral Skyrmions in noncentrosymmetric magnets

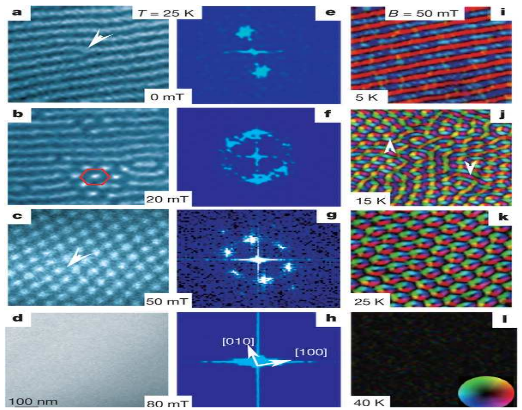
J. Garaud, J. Carlström, and E. Babaev,
Phys. Rev. Lett. 107, 197001 (2011);



Three-component superconductors

Yu, Onose et al. Nature 465, 901 (2010)

Experimental realization in a thin film of $Fe_{0.5}Co_{0.5}Si$



Isospinning Skyrmions

R.Battye and M. Haberichter
Phys.Rev. D88 (2013) 125016

A.Halavanau and Ya Shnir
Phys.Rev. D88 (2013) 085028

$\phi : S^2 \to S^2; \quad \phi_\infty = (0,0,1)$ → SO(2) internal rotational symmetry group

$$\phi_1 + i\phi_2 \to (\phi_1 + i\phi_2)e^{i\omega t}$$

Conserved quantity - angular momentum

$$J = \omega\Lambda = \omega \int\limits_{\mathbb{R}_2} \left\{ (\boldsymbol{\phi}_\infty \times \boldsymbol{\phi})^2 [1 + (\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi})] - [\boldsymbol{\phi}_\infty \cdot (\boldsymbol{\phi} \times \partial_i \boldsymbol{\phi})]^2 \right\}$$

Two equivalent variational problems:

I For fixed ω extremize

$$F_\omega[\psi] = V[\psi] - \frac{1}{2}\omega^2\Lambda[\psi]$$

II For fixed J extremize

$$E[\psi] = V[\psi] + T[\psi] = V[\psi] + \frac{J^2}{2\Lambda[\psi]}$$

Pseudoenergy vs Energy

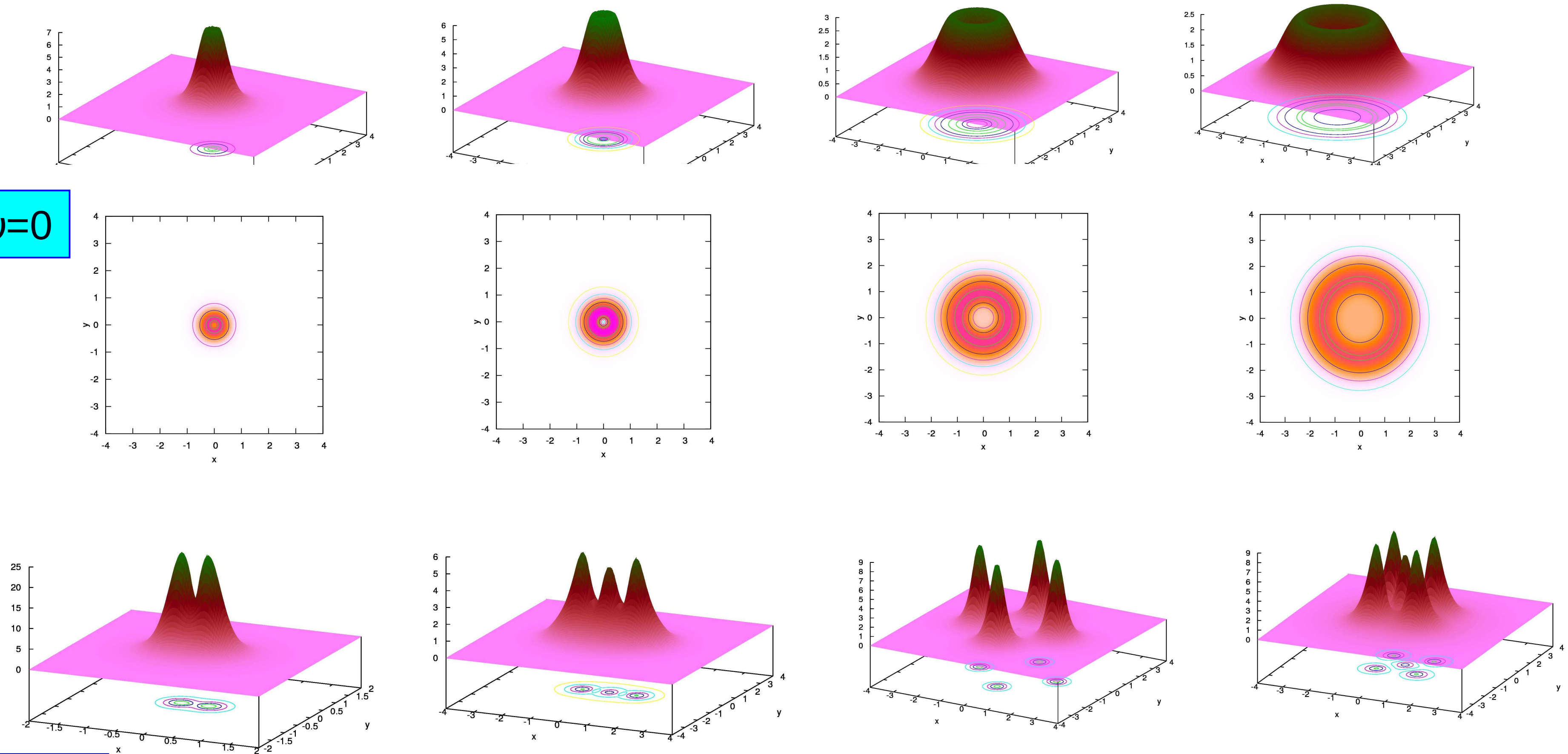
Approach II: Rigid body approximation, for fixed J extremize $E[\psi] = V[\psi] + \frac{J^2}{2\Lambda[\psi]}$

- **Advantages of the approach I:** (for fixed ω extremize $F_\omega[\psi] = V[\psi] - \frac{1}{2}\omega^2\Lambda[\psi]$)

- It works beyond rigid body approximation;
- The corresponding Euler-Lagrange equation is a PDE; for the problem **II** it is a differential-integral equation;
- It reveals ***two*** critical frequencies $\omega_1 = 1; \quad \omega_2 = \mu$, F_ω is unbounded from below if $\omega \in [1; \mu]$; the isospinning configurations are destabilized by nonlinear velocity terms generated by the Skyrme term;
- The second critical frequency ω_2 corresponds to the instability of the isospinning soliton w.r.t. radiation of mesons;
- This observation should be generic for ***all models*** from the Skyrme family
- The conservation of the total energy and isospin means that for an isospinning soliton the quantity $T + V - \omega J$ conserves - orbital stability of a local minimum of F_ω

Isospinning Baby Skyrmions: Numerical Results

$\omega=0$



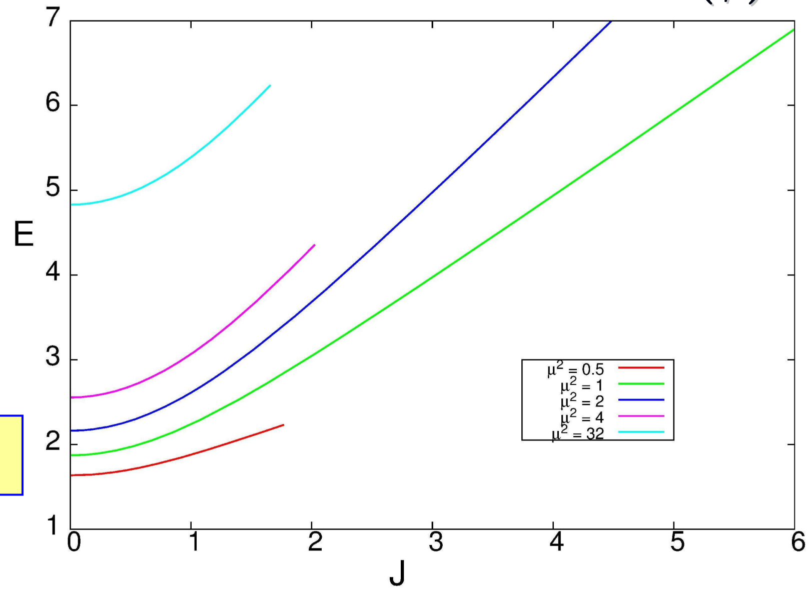
$\omega=0.8$

B=2 B=3 B=4 B=5

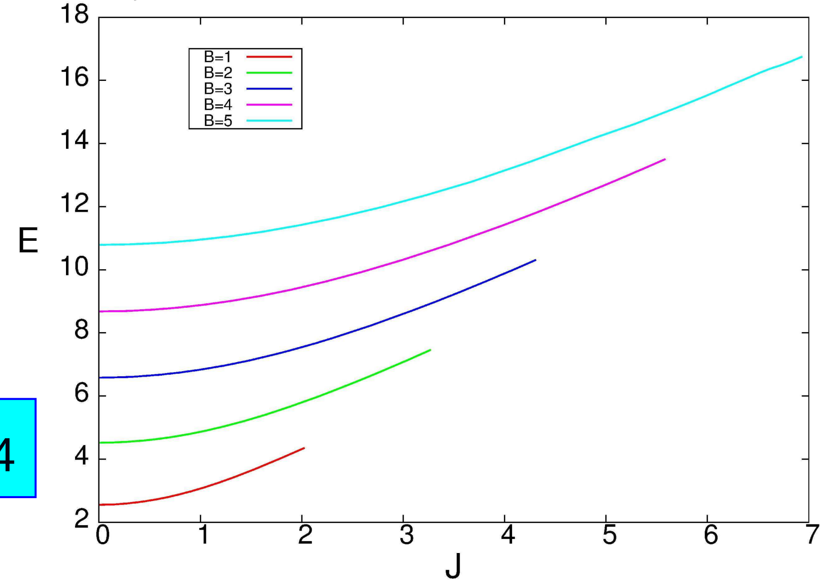
Critical behavior

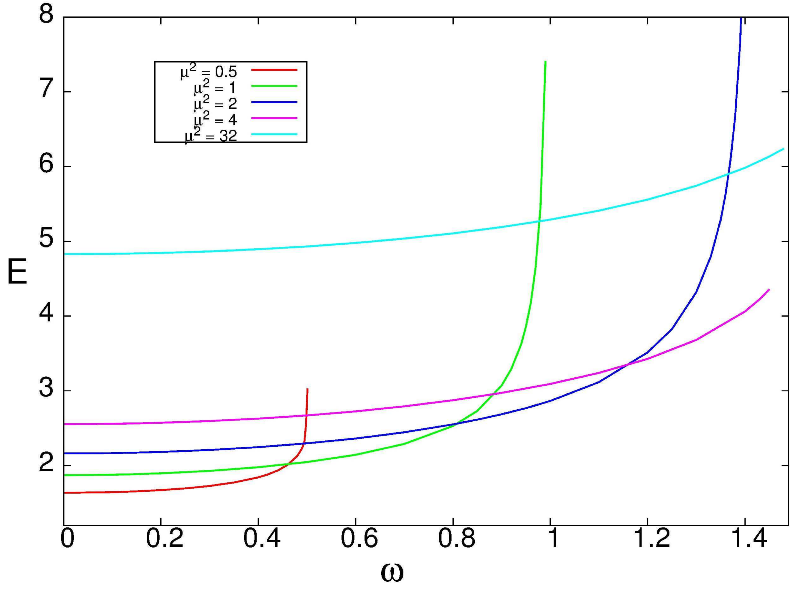
$$U(\phi) = m^2(1 - \phi_3)$$

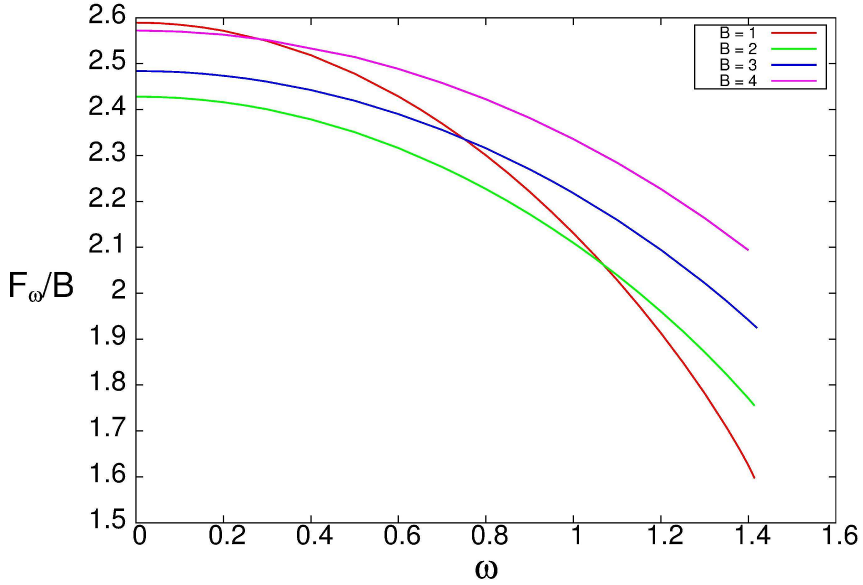
B=1



m=4







Faddeev-Skyrme model

- **(3+1) dim:** $\phi : S^3 \to S^2; \quad \phi_\infty = (0,0,1)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\left(\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\right)^2 - V(\phi)$$

$$V(\phi) = \mu^2(1-\phi_3^2)^2$$

First Hopf map

Pull-back: $\phi^* : H^2(S^2) \to H^2(S^3)$

$$Q \in \mathbb{Z} = \pi_3(S^2)$$

$$F = dA = \phi^*\Omega$$

Area form on S^2

$$Q = \frac{1}{16\pi^2}\int_{\mathbb{R}^3} A \wedge dA d^3x$$

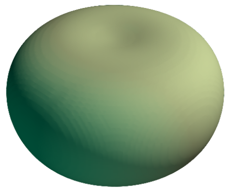
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{2}\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c$$

Chern-Simons 3-form over S^3

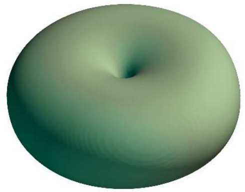
Note: unlike the other models from the Skyrme family the topological charge is not equal to the degree of the map ϕ – this is the linking number of the loops in $\mathbb{R}_3$, preimages of two distinct points on the target space **S^2**

- **Energy bound:** $E \geq cQ^{3/4}$

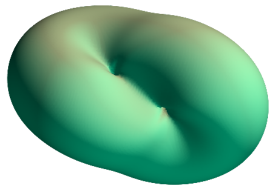
Solitons of the Faddeev-Skyrme model



Q=1 $1\mathcal{A}_{1,1}$

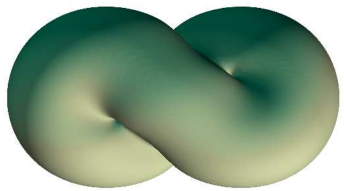


Q=2 $2\mathcal{A}_{2,1}$

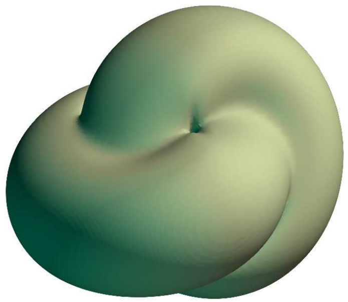


Q=3 $3\widetilde{\mathcal{A}}_{3,1}$

Q=4 $4\mathcal{A}_{2,2}$

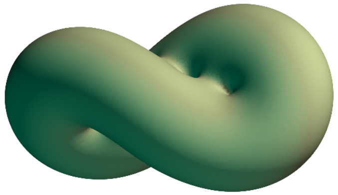


Q=4 $4\widetilde{\mathcal{A}}_{4,1}$

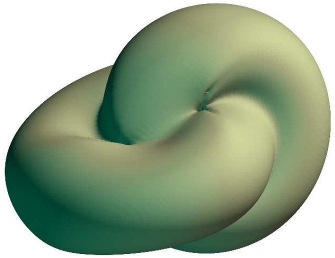


Q=4 $4\mathcal{L}_{1,1}^{1,1}$

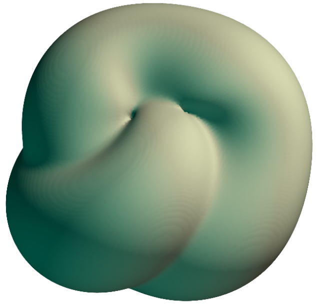
Solitons of the Faddeev-Skyrme model



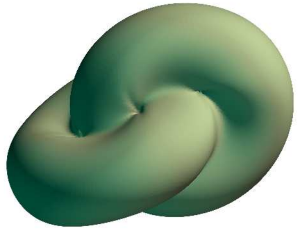
Q=5 $5\widetilde{\mathcal{A}}_{5,1}$



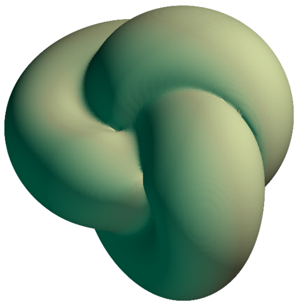
Q=5 $5\mathcal{L}^{1,2}_{1,1}$



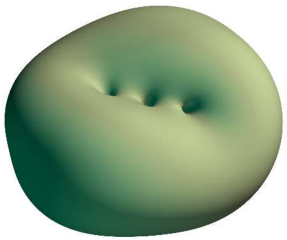
Q=6 $6\mathcal{L}^{2,2}_{1,1}$



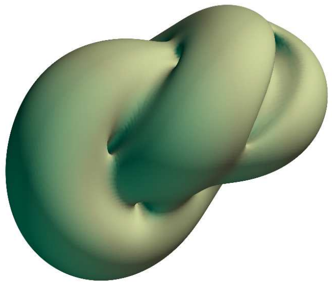
Q=6 $6\mathcal{L}^{3,1}_{1,1}$



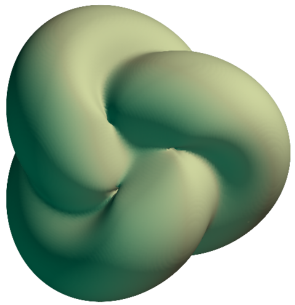
Q=7 $7\mathcal{K}_{3,2}$



Q=8 $8\widetilde{\mathcal{A}}_{4,2}$

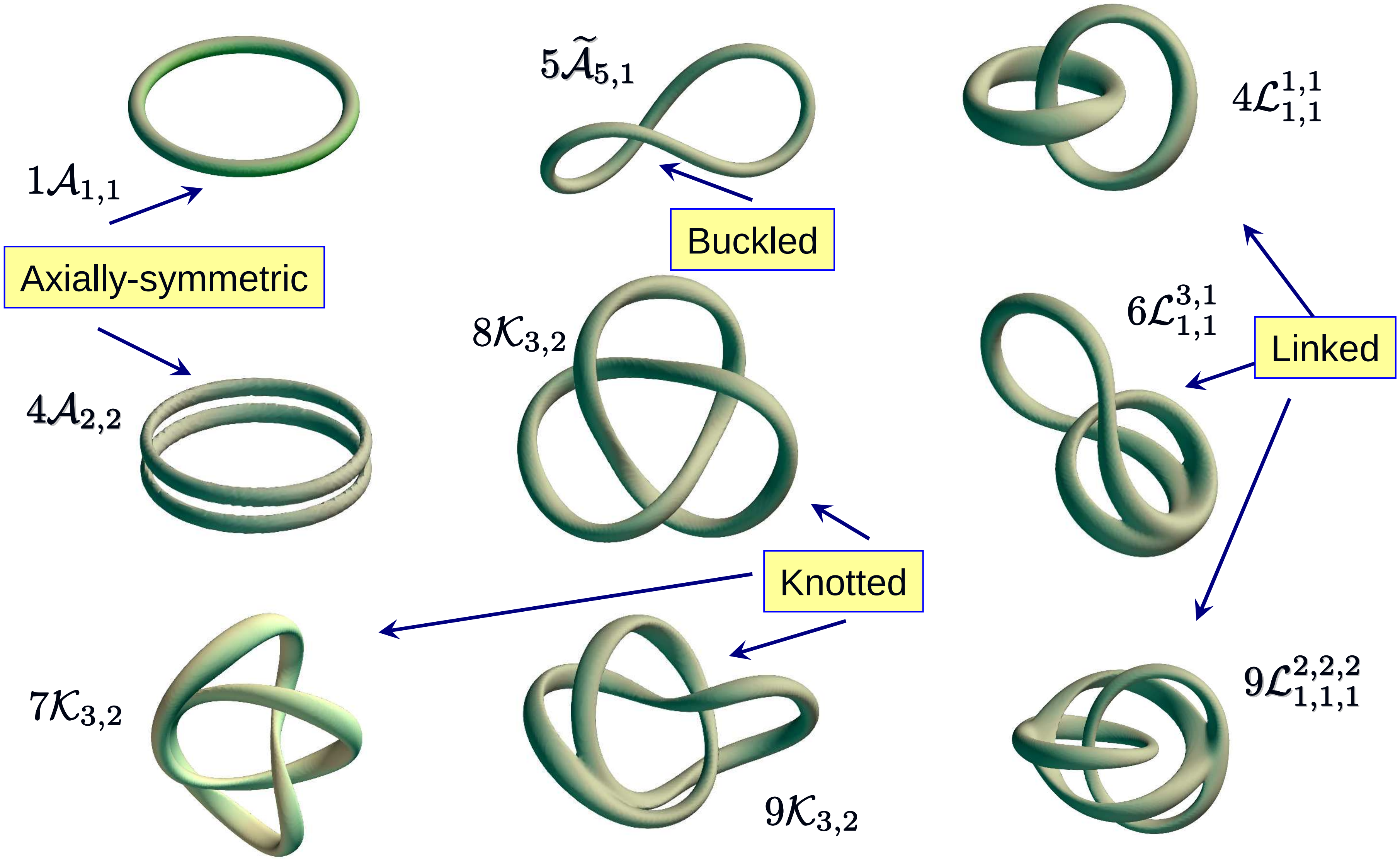


Q=8 $8\mathcal{L}^{3,3}_{1,1}$

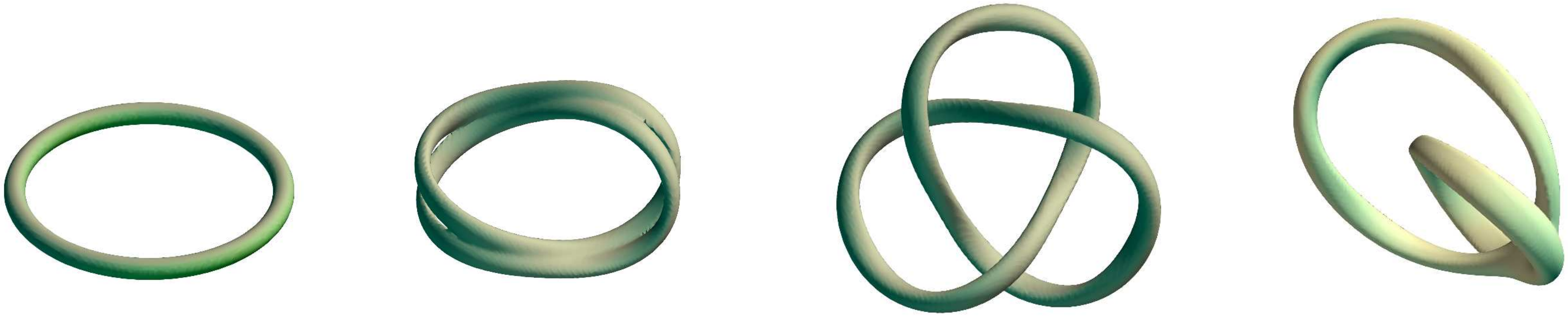


Q=8 $8\mathcal{K}_{3,2}$

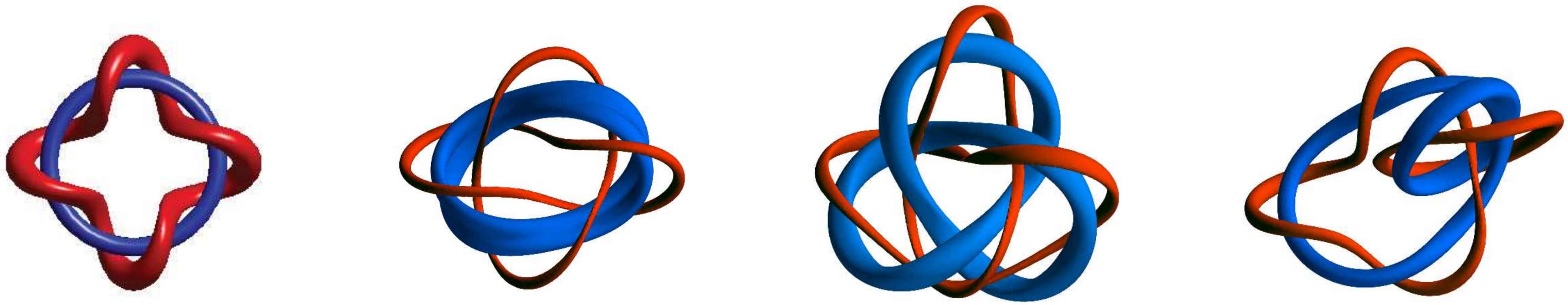
Buckled, linked and knotted hopfions



Position curves and linking numbers



$4\ \mathcal{A}_{4,1}$ $8\widetilde{\mathcal{A}}_{4,2}$ $8\mathcal{K}_{3,2}$ $6\mathcal{L}^{1,1}_{3,1}$



Gauged Hopfions

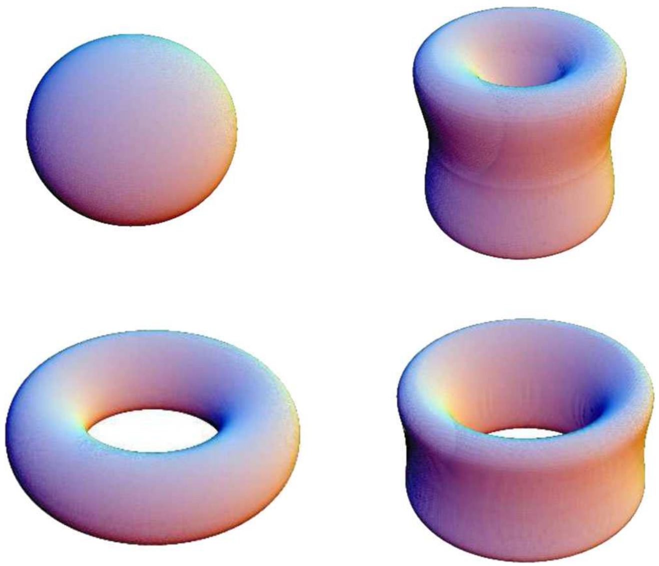
Ya Shnir and G. Zhilin. Phys.Rev. D89 (2014) 105010

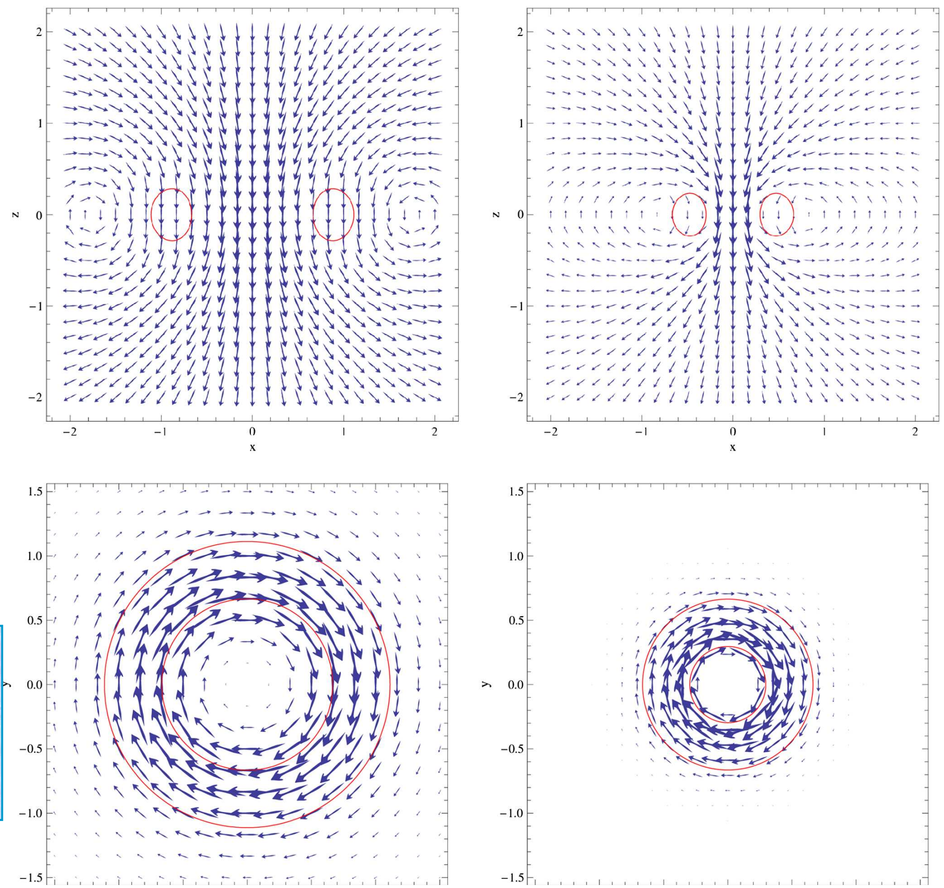
$\phi : S^3 \to S^2; \quad \phi_\infty = (0,0,1)$ → SO(2) symmetry $\phi_1 + i\phi_2 \to (\phi_1 + i\phi_2)e^{i\omega t}$

Covariant derivative: $D_\mu \phi^a = \partial_\mu \phi^a + g A_\mu \varepsilon_{abc} \phi^b \phi^c_\infty$

Faddeev-Skyrme-Maxwell model

$$\mathcal{L} = \frac{1}{2} D_\mu \phi D^\mu \phi - \frac{1}{4} (D_\mu \phi \times D_\nu \phi)^2 - V(\phi)$$





The gauged Hopfion carries two magnetic fluxes, which are quantized in units of 2π, carrying n and m quanta, respectively

Gauged Hopfions

Ya Shnir and G. Zhilin.
Phys.Rev. D89 (2014) 105010

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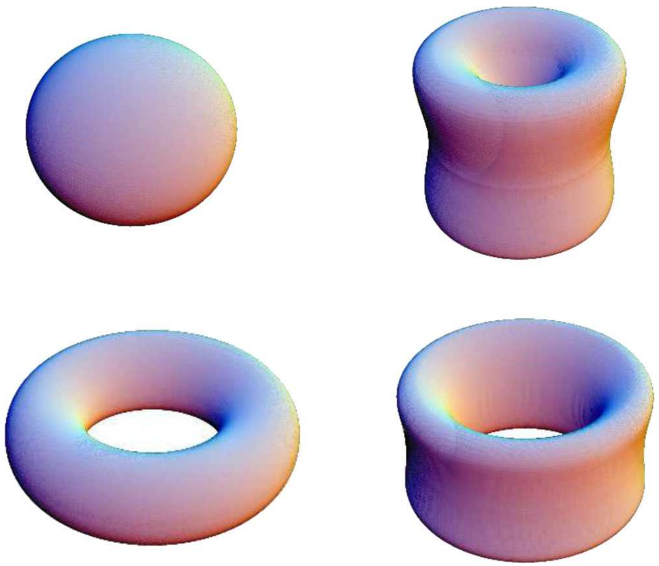

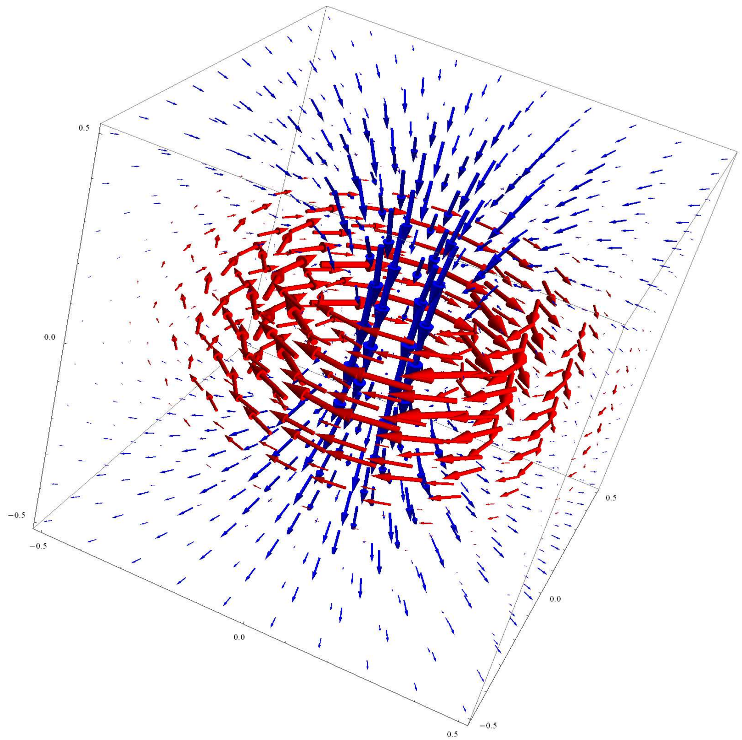

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Elastic rod approximation

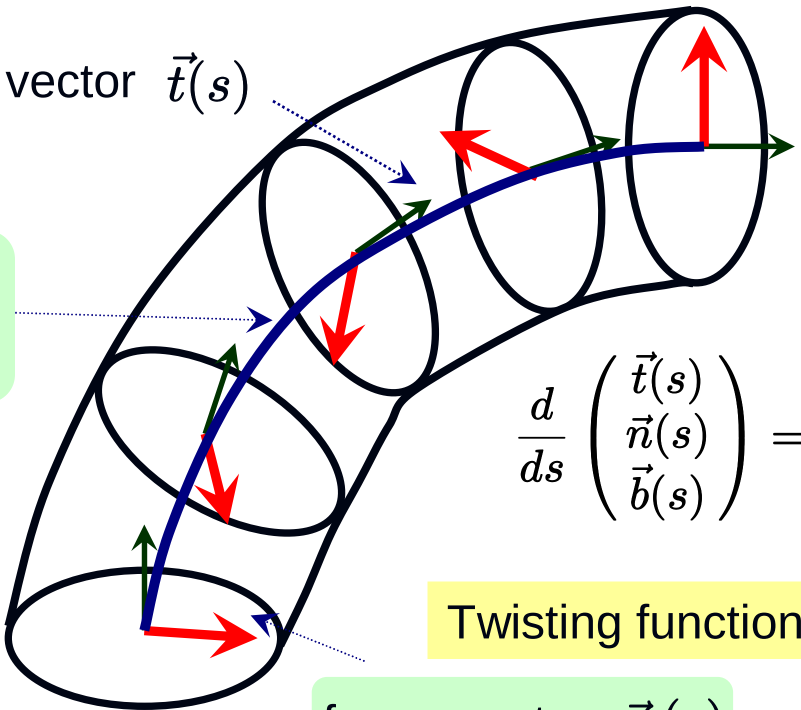
D. Harland, J Jakka, Ya Shnir and J.M. Speight
J. Phys. A46 225402 (2013)

Tubular coordinates:

- Arclength parameter $s \in [0, L]$
- Polar coordinates ρ, θ in the disk

- Torsion $\tau(s)$
- Curvature $\kappa(s)$

Tangent vector $\vec{t}(s)$



Position curve $\gamma(s) \in \mathbb{R}^3$

Preimage of $\phi = (0,0,1)$

frame vector $\vec{m}(s)$

Frenet frame

$$\vec{m}(s) = \vec{n}(s)\sin\alpha(s) + \vec{b}(s)\cos\alpha(s)$$

$$\frac{d}{ds}\begin{pmatrix}\vec{t}(s)\\ \vec{n}(s)\\ \vec{b}(s)\end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{t}(s)\\ \vec{n}(s)\\ (s)\end{pmatrix}$$

Twisting function: $\alpha(s) = \vec{t}\cdot\vec{m}' \times \vec{m}$

$$\alpha(L) = \alpha(0) + 2\pi N$$

Faddeev-Skyrme effective energy functional:

$$E = \int \left(A + B\kappa^2 + C(\alpha' - \tau)^2\right) ds$$

Elastic rod model

$$E = \int \left(A + B\kappa^2 + C(\alpha' - \tau)^2 \right) ds$$

- Stretching energy
- Bending energy
- Twisting energy

Constraints: rods do not intersect themselves;
the radius of the curvature $1/\kappa$ cannot be less than ρ

The model contains 3 parametes: A,B,C

$$\rho = \sqrt{\frac{B+C}{A}}; \quad C/B = 0.85$$

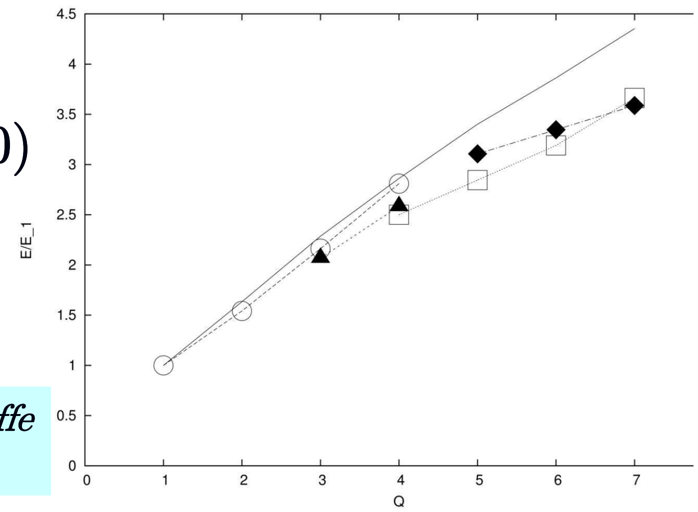
Example: Axially-symmetric hopfions:

$$\mathcal{A}_{m,n} \longrightarrow \vec{m}(s) = (\cos(2\pi sN/L); -\sin(2\pi sN/L), 0)$$

$$\alpha = \frac{2\pi sN}{L}; \qquad E = 4\pi\sqrt{1 + CN^2}$$

$$Q \equiv N$$

D. Harland, J.M. Speight and P.M. Sutcliffe
Phys. Rev.D83 (2011) 065008



Isospinning Hopfions

R.Battye and M. Haberichter
Phys.Rev. D87 (2013) 105003

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$\phi : S^1_\omega \times \mathbb{R}^3 \to S^2$

$\phi_1 + i\phi_2 \to (\phi_1 + i\phi_2)e^{i\omega t}$

- **Kinetic energy:** $T[\phi] = \frac{1}{2}\int_{\mathbb{R}^3}\left\{|\dot{\phi}|^2 + |\phi^*(\iota_{\dot{\phi}}\Omega)|^2\right\} d^3x$

- **Potential energy:** $V[\phi] = \frac{1}{2}\int_{\mathbb{R}^3}\left\{|d\phi|^2 + |\phi^*\Omega)|^2 + U(\phi)\right\} d^3x$

- **Action:** $S[\phi] = \int_{S^1_\omega}(T - V)$

What is the degree Q isospinning hopfion of given angular frequency ω ?

Reduction to the stationary problem: $\phi(x,t) = R(\omega t)\psi(x), \quad \psi : \mathbb{R}^3 \to S^2$

$$S[R(\omega t)\psi(x)] = \frac{2\pi}{\omega}\left\{\frac{1}{2}\omega^2\int_{\mathbb{R}^3}\left(|\psi_\infty \times \psi|^2 + |d(\psi_\infty \cdot \psi)|^2\right) - V[\psi]\right\} d^3x$$

Variational problem

Pseudoenergy functional:

$$F_\omega[\psi] = \int_{\mathbb{R}^3} \left\{ \frac{1}{2}\left(|d\phi|^2 - \omega^2 |d(\psi_\infty \cdot \phi)|^2\right) + \frac{1}{2}|\psi^* \Omega|^2 + \left(U(\psi) - \frac{1}{2}\omega^2 |\psi_\infty \times \psi|^2 \right) \right\} d^3x$$

Dirichlet energy

Effective potential $U_\omega(\psi)$

Deformed S^2 metric:

$$\langle X, Y \rangle_\omega = X \cdot Y - \omega^2 (\psi_\infty \cdot X)(\psi_\infty \cdot Y)$$

$$U_\omega(\psi) = \frac{1}{2}(\mu^2 - \omega^2)(1 - \psi_3^2)^2$$

- **Isospin:**
- **Moment of inertia**

$$J = \omega \Lambda[\psi] = \omega \int_{\mathbb{R}^3} \{ |\psi_\infty \times \psi|^2 + |d(\psi_\omega \cdot \psi|^2) \} \, d^3x$$

Two equivalent variational problems:

I For fixed ω extremize

$$F_\omega[\psi] = V[\psi] - \frac{1}{2}\omega^2 \Lambda[\psi]$$

II For fixed J extremize

$$E[\psi] = V[\psi] + T[\psi] = V[\psi] + \frac{J^2}{2\Lambda[\psi]}$$

D. Harland, J Jakka, Ya Shnir and J.M. Speight
J. Phys. A46 225402 (2013)

R.Battye and M. Haberichter
Phys.Rev. D87 (2013) 105003

Numerical Results

P.M. Sutcliffe, Proc. Roy. Soc. Lond. A 463 (2007) 3001.

Input: Rational map ansatz

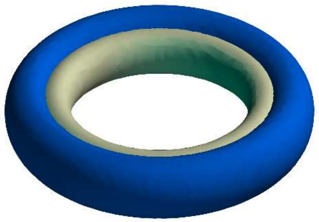
$$(Z_1, Z_0) = \left((x_1 + ix_2)\frac{\sin f(r)}{r},\quad \cos f(r) + ix_3\frac{\sin f(r)}{r}\right),\quad f(0) = \pi,\ f(\infty) = 0$$

$$W = \frac{\psi_1 + i\psi_2}{1 + \psi_3} = \frac{Z_1^\alpha Z_0^\beta}{Z_1^a + Z_0^b};\qquad \vec{\phi} = \frac{1}{1 + |W|^2}\left(W + \bar{W},\ i(\bar{W} - W),\ 1 - |W|^2\right)$$

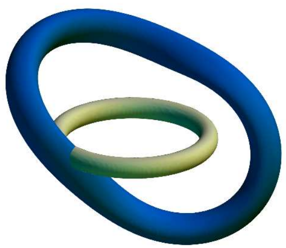
$W : S^3 \to \mathbb{CP}^1$

The Hopf charge: $Q = \alpha b + a\beta$

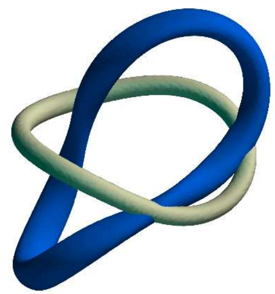
- **Degrees one, two and three:**



$1\mathcal{A}_{1,1}$



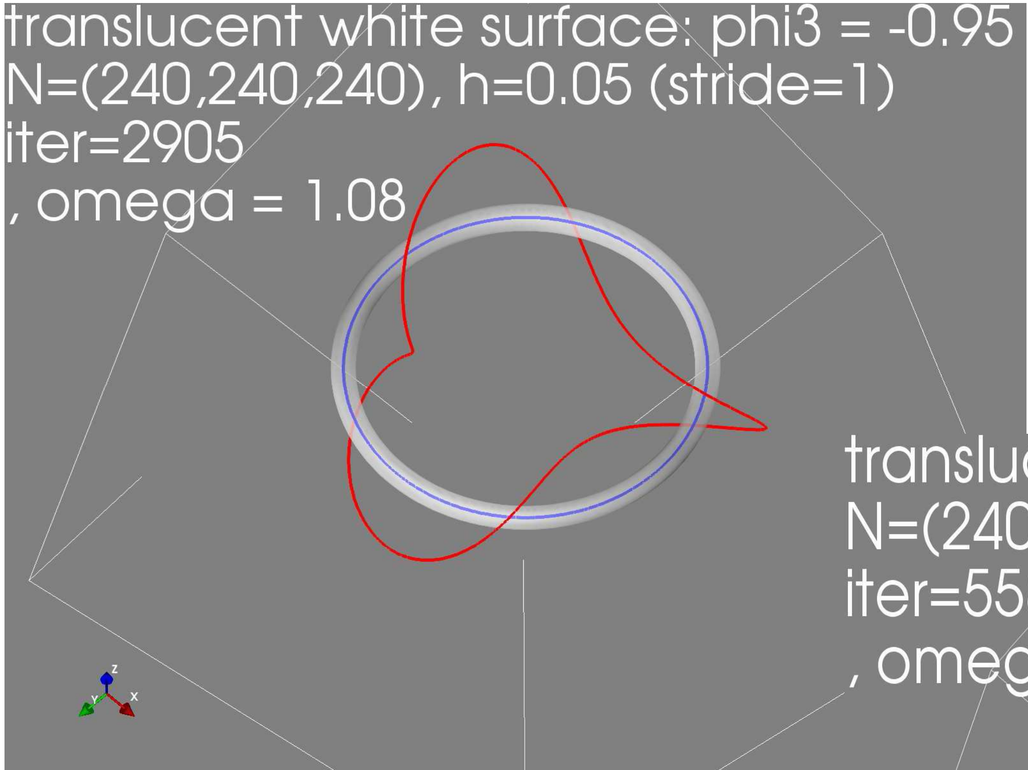
$2\mathcal{A}_{2,1} \to 2\widetilde{\mathcal{A}}_{2,1}$



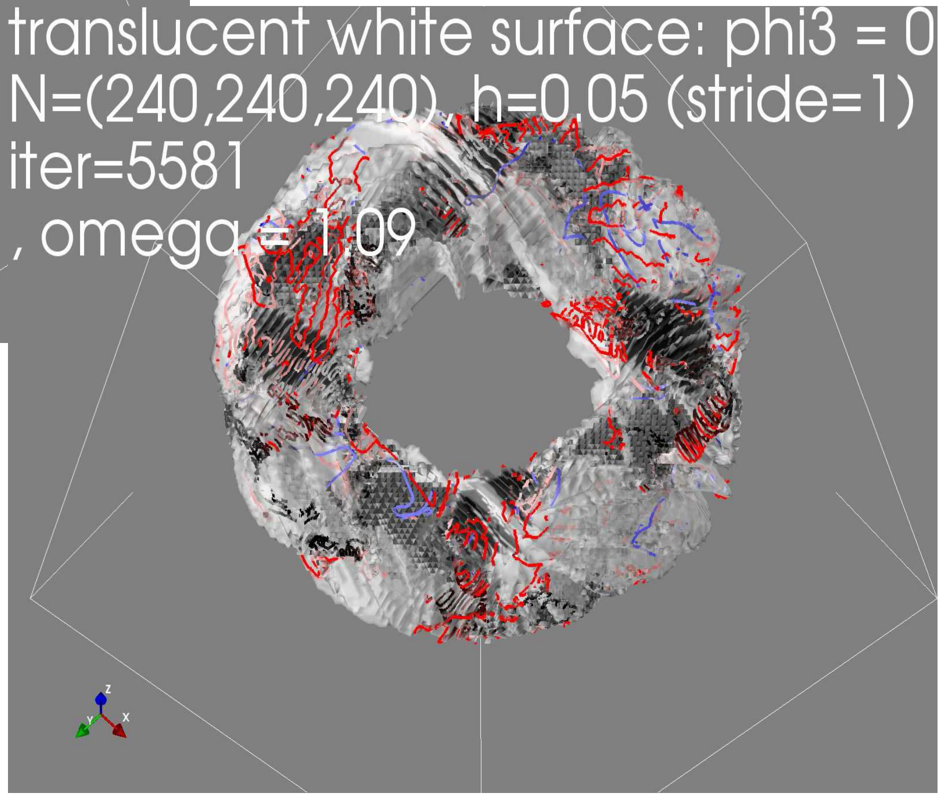
$3\widetilde{\mathcal{A}}_{3,1}$

Note: To fit the numerical results the parameters of the rod model are taken as $(4\pi)^2 AC = 1.1;\quad \frac{A}{C} = 3.5$

Instability of the isospinning Hopfions

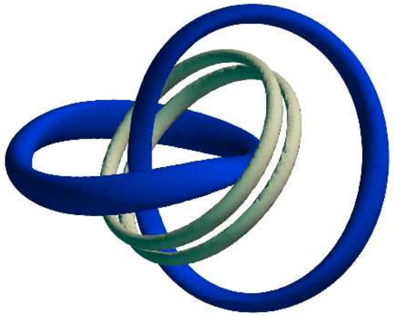


$$\omega_1 \sim 1; \quad \omega_2 < \mu$$



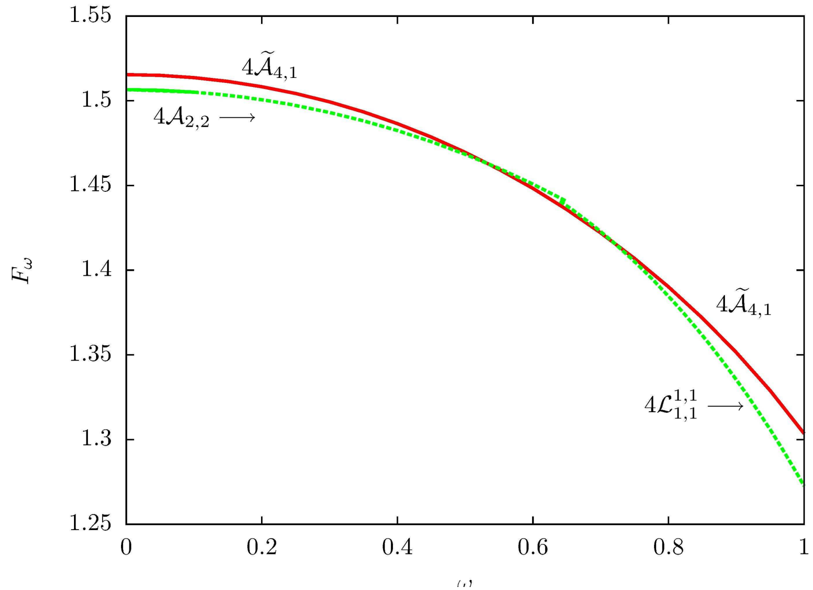
$$3\widetilde{\mathcal{A}}_{3,1}$$

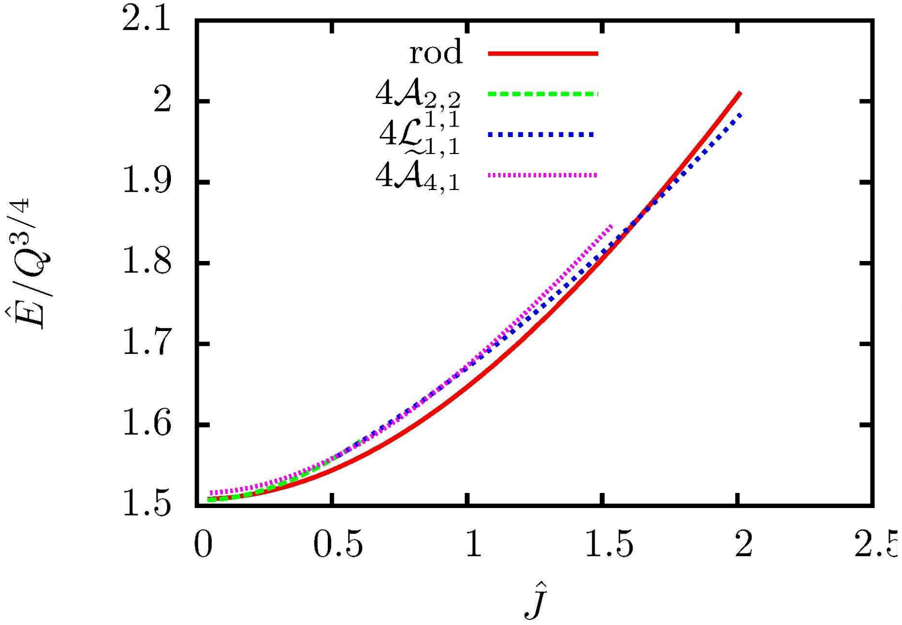
Q=4

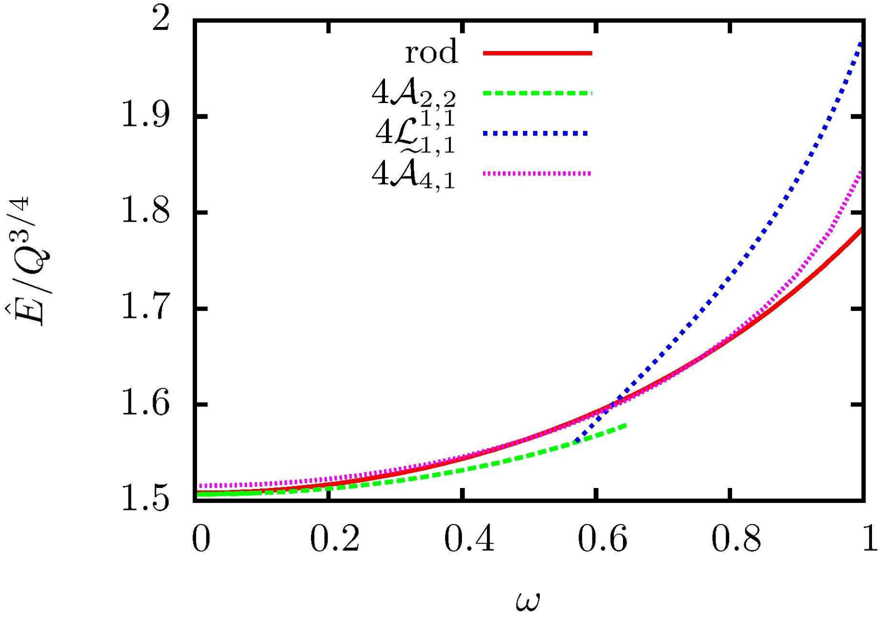


$4\mathcal{A}_{2,2} \rightarrow 4\mathcal{L}^{1,1}_{1,1}$

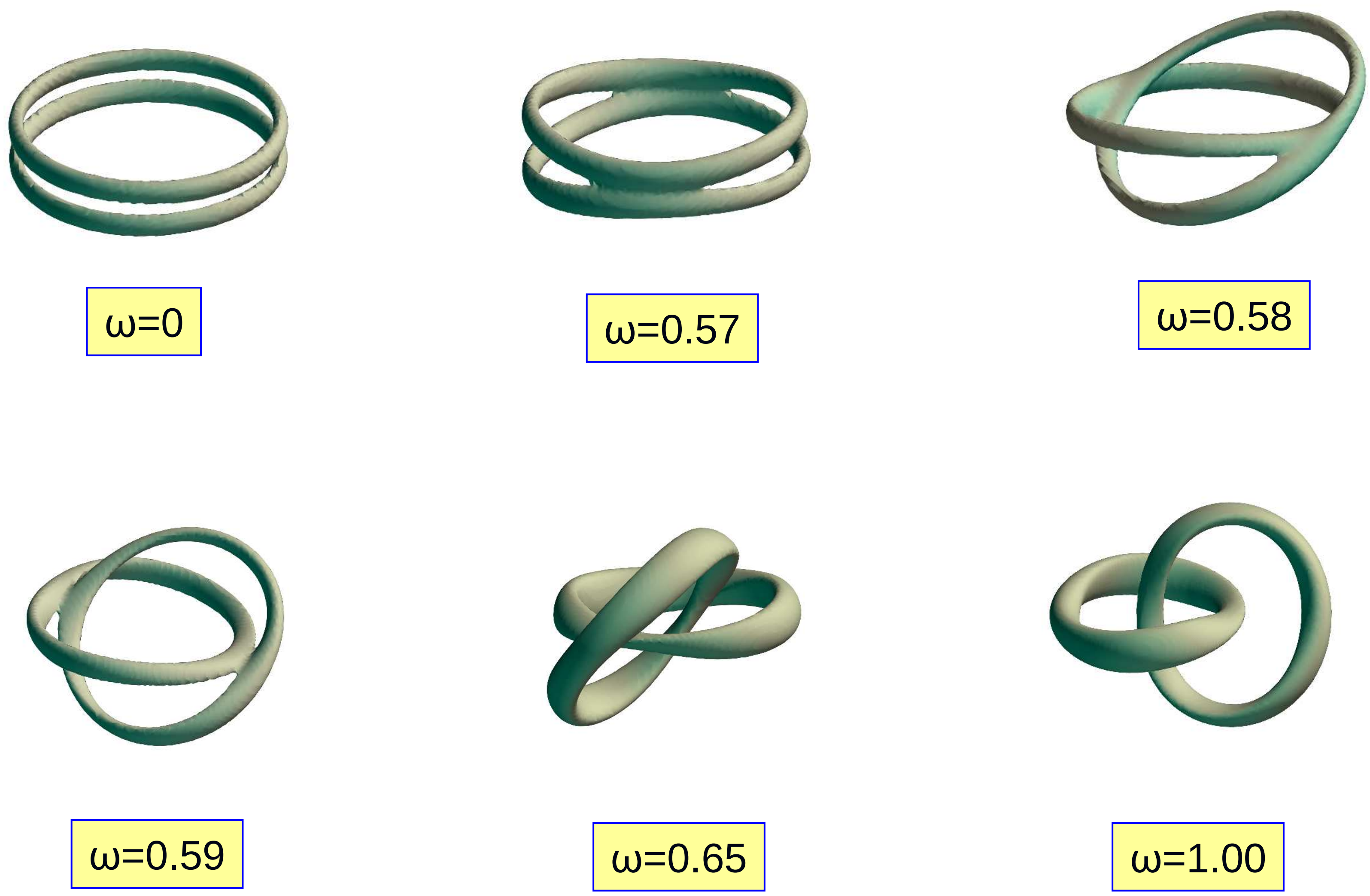
$4\widetilde{\mathcal{A}}_{4,1}$



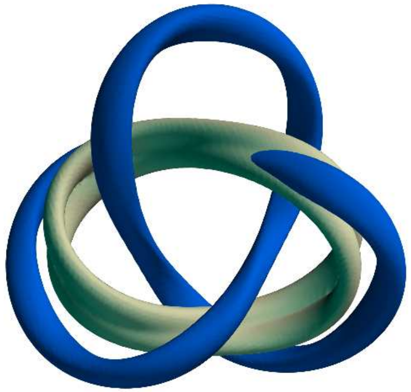


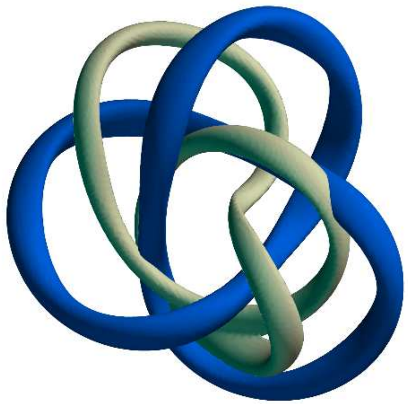


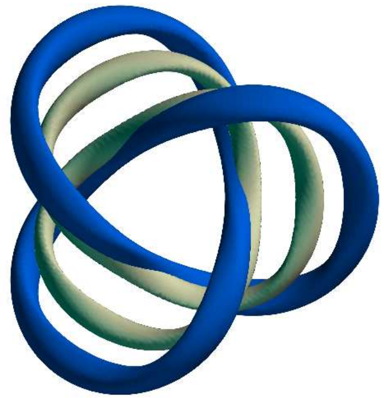
$4\mathcal{A}_{2,2} \longrightarrow 4\mathcal{L}^{1,1}_{1,1}$



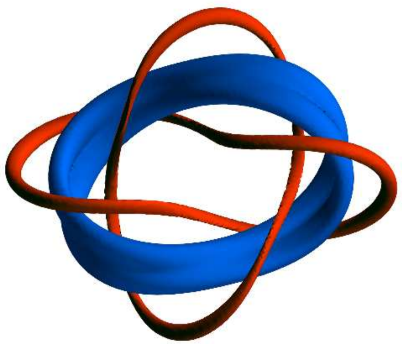
Q=8

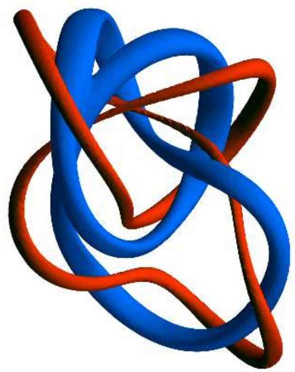


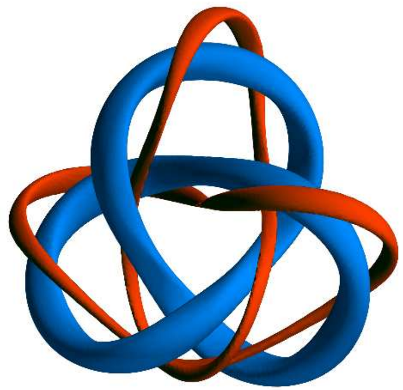




$8\widetilde{\mathcal{A}}_{4,2}$ $8\mathcal{L}^{3,3}_{1,1}$ $8\mathcal{K}_{3,2}$







Summary and Outlook

- Evolution of the isospinning solitons is considered beyond the rigid body approximation, the new approach is related with minimisation of the pseudoenergy functional
- There are two critical frequencies, the pseudoenergy functional is unbounded from below if

  $\omega_2 > \mu$ - instability w.r.t. radiation

  $\omega_1 > 1$ - the isospinning configurations are destabilized via the nonlinear velocity terms generated by the Skyrme term
- Various transitions between the isorotating hopfions are observed
- Modified elastic rod model works well
- Gauged Hopfions are coupled to the magnetic fluxes, the quantization of the fluxes matches the topology of the Hopfion
- Various symmetry-breaking bifurcations in the dual core baby Skyrme model are investigated