Q-tubes in a piece-wise parabolic potential

Andrey Shkerin
Institute for Nuclear Research

QUARKS

Suzdal, 2014

Action

Consider (3+1)-dim complex scalar field with action

$$S = \int d^4x \left(\partial_\mu \Phi^* \partial^\mu \Phi - V\left(\Phi^* \Phi\right)\right)$$

Ansatz for Q-tube in cylindrical coordinates (r, φ, z) is $\Phi(r,\varphi,t) = F(r)\, e^{i\omega t} e^{in\varphi}$

Where $F(r)$ - smooth real function, with good behavior at origin and infinity,
ω - real parameter,
n - integer parameter

Conserved quantities:

$$E = \int d^3x T_{tt}, \quad T_{tt} = \left(\omega^2 + \frac{n^2}{r^2}\right) F^2 + \left(\frac{dF}{dr}\right)^2 + V\left(F^2\right)$$

$$Q = -i \int d^3x \left(\Phi^* \dot{\Phi} - \dot{\Phi}^* \Phi\right) = \int d^3x 2\omega F^2$$

$$\boxed{J = nQ}$$

Equation of motion:

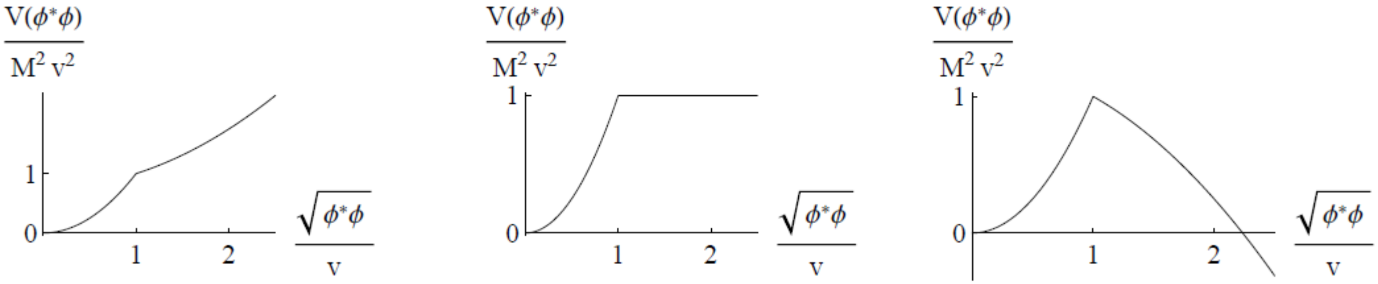
$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{n^2}{r^2}F + \omega^2 F = \frac{dV}{dF^2}F$$

Potential and solution

Choice the following expression for *V*:

$$V\left(|\Phi|^2\right) = M^2|\Phi|^2\theta\left(1-\frac{|\Phi|^2}{v^2}\right) + \left(m^2|\Phi|^2+\Lambda\right)\theta\left(\frac{|\Phi|^2}{v^2}-1\right)$$

Where Λ provides continuity at $|\Phi|^2 = v^2$: $\Lambda = v^2(M^2 - m^2)$



Equation of motion then:

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} + F\left(r^2\left(\omega^2-U\right)-n^2\right) = 0$$

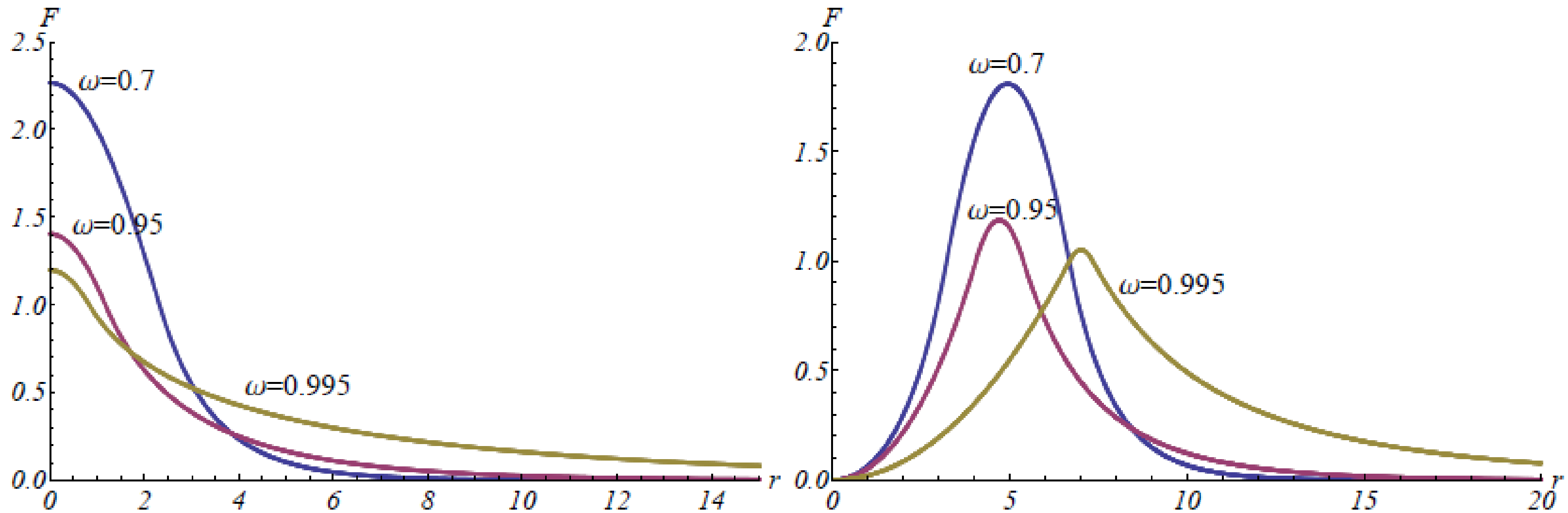
$U = M^2$ for $F^2 < v^2$

$U = m^2$ for $F^2 > v^2$

Its solutions are Bessel functions in each region of constant *U*.

They are fixed by requirements of regularity at origin and infinity and smoothness of *F*.

Structure of solutions



$n = 0$

$$r > r_0, \quad U = M^2: \quad F(r) \sim K_0\left(\sqrt{M^2 - \omega^2}\, r\right)$$

$$r < r_0, \quad U = m^2: \quad F(r) \sim J_0\left(\sqrt{\omega^2 - m^2}\, r\right)$$

We obtain constraints on ω:

$$m < \omega < M, \quad m^2 \geq 0$$

$$0 \leq \omega < M, \quad m^2 < 0$$

$n > 0$

$$r > r_2, \quad U = M^2: \quad F(r) \sim K_n\left(\sqrt{M^2 - \omega^2}\, r\right)$$

$$r < r_1, \quad U = M^2: \quad F(r) \sim I_n\left(\sqrt{M^2 - \omega^2}\, r\right)$$

$$r_1 < r < r_2, \quad U = m^2:$$

$$F(r) = C_2 J_n\left(\sqrt{\omega^2 - m^2}\, r\right) + C_3 Y_n\left(\sqrt{\omega^2 - m^2}\, r\right)$$

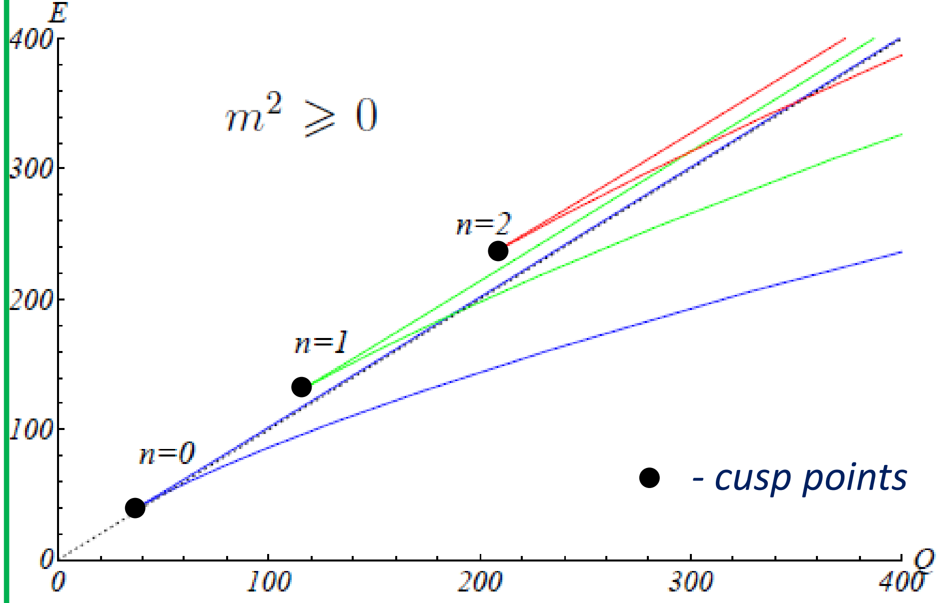
Asymptotes of the solution:

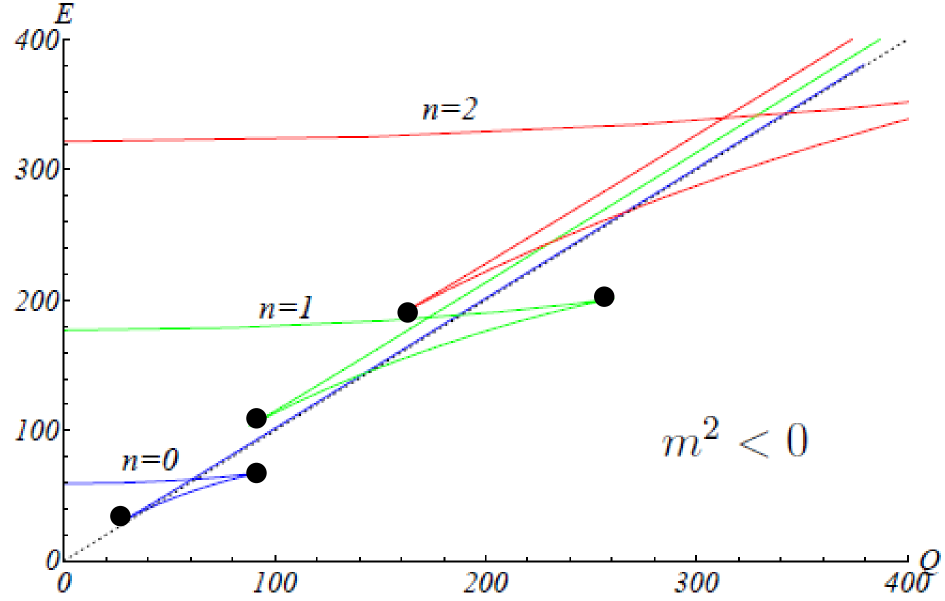
$$F(0) = F'(0) = \ldots = F^{(n-1)}(0) = 0,$$

$$F(r) \to 0, \quad r \to \infty$$

Properties of solutions

E(Q)-plots for the different values of *n* , $M = v = 1$





$\omega \to m \quad E \sim Q^{3/4}$

$\omega \to M \quad E \sim Q$

$\omega = 0$ – static solution with $Q = 0$

$\omega \to M \quad E \sim Q$

$$\boxed{\frac{dE}{d\omega} = \omega \frac{dQ}{d\omega}}$$

This property is very known to hold for Q-ball solutions too.

Classical stability

Equation of motion following from the general action (we suppose Φ does not depend on *z*):

$$r^2\frac{d^2\Phi}{dr^2}+r\frac{d\Phi}{dr}+\frac{d^2\Phi}{d\phi^2}-r^2\frac{d^2\Phi}{dt^2}=r^2\Phi\frac{dV}{d|\Phi|^2}$$

Search solution in the form: $\Phi=\Phi_0+h$, where Φ_0 - Q-tube solution,

$h=h(r,\varphi,\mathrm{t})$ - small perturbation

Choice the ansatz for perturbations in the form: $h=e^{i\omega t+in\phi}\sum_{l=0}^{\infty}\left(c_1^l e^{i(\alpha t+l\phi)}+c_2^{l*}e^{-i(\alpha^* t+l\phi)}\right)$

Q-tube background

Functions of *r*

$\alpha=-i\gamma+\gamma',\qquad \gamma,\gamma'\in\mathbb{R}.$

- It separates the variables,
- It changes the angular momentum of the tube

Linearized equations then:

$$r^2\frac{d^2c_1}{dr^2}+r\frac{dc_1}{dr}-c_1\left[r^2\left((\gamma+i(\omega+\gamma'))^2+U\right)+(n+l)^2\right]=$$
$$=\frac{r^2F^2}{v^2}(m^2-M^2)\,\delta\left(\frac{F^2}{v^2}-1\right)(c_1+c_2),$$

$$r^2\frac{d^2c_2}{dr^2}+r\frac{dc_2}{dr}-c_2\left[r^2\left((\gamma-i(\omega-\gamma'))^2+U\right)+(n-l)^2\right]=$$
$$=\frac{r^2F^2}{v^2}(m^2-M^2)\,\delta\left(\frac{F^2}{v^2}-1\right)(c_1+c_2),$$

Except for the points $F^2=v^2$ they are Bessel equations!

Existence of solutions

Consider the tube with *n=0.*

After imposing the regularity requirements at the origin and infinity we rest with

$C_1 c_{1,left}$, $C_3 c_{2,left}$ - Solutions at $r < r_0$

$C_2 c_{1,right}$, $C_4 c_{2,right}$ - Solutions at $r > r_0$

Matching at the point $r = r_0$ leads to*:*

$$
\begin{aligned}
C_1 c_{1,left}(r_0) - C_2 c_{1,right}(r_0) &= 0,\\
C_3 c_{2,left}(r_0) - C_4 c_{2,right}(r_0) &= 0,\\
C_2 c'_{1,right}(r_0) - C_1 c'_{1,left}(r_0) - A\left(C_1 c_{1,left}(r_0) + C_3 c_{2,left}(r_0)\right) &= 0,\\
C_4 c'_{2,right}(r_0) - C_3 c'_{2,left}(r_0) - A\left(C_1 c_{1,left}(r_0) + C_3 c_{2,left}(r_0)\right) &= 0,\\
A = \frac{v\left(m^2 - M^2\right)}{2|F'(r_0)|}.
\end{aligned}
$$

Equations on the constants $C_{1,2,3,4}$

Growing modes exists if the determinant of this system, $\Delta(\gamma, \gamma'; l)$ equals to 0 for some $\gamma > 0, \gamma', l$.

If *n>0*, the number of solutions of the perturbation equations is doubled as well as the number of equations in the system.

Perturbations of tubes with *n=0*

Fix background Q-tube.

Search for the solutions of the equation $\Delta(\gamma,\gamma';l)=0$ with $\gamma>0$, keeping $\gamma'=0$.

$$c_{1,2,left} = J_{\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - m^2}\right),$$
$$c_{1,2,right} = H^{(1)}_{\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - M^2}\right)$$

Asymptotes of Δ: $\gamma \to \infty$

$$\gamma \gg |m|, M$$

$$c_{1,2,left} = e^{\pm\frac{i\pi l}{2}} I_l(r\gamma) + O(\sqrt{\gamma}),$$
$$c_{1,2,right} = \frac{2i}{\pi} e^{\mp\frac{i\pi l}{2}} K_l(r\gamma) + O(\sqrt{\gamma})$$

$$Re\Delta(\gamma) = \frac{1}{r_0^2}\left(\frac{|A|}{\gamma} - 1\right), \quad Im\Delta(\gamma) = 0.$$

$|A| \sim |F'(r_0)|^{-1} \to \infty$ when $\omega \to M$

Upper branches of *E(Q)*-plots are unstable with

$\gamma \to \infty$ when $\omega \to M$

This result does not depend on *l*.

$\gamma \to 0$

We know some solutions of perturbation equations which have $\gamma = 0$

U(1) invariance $\longrightarrow$ $h \sim i\Phi_0, \quad l = 0$

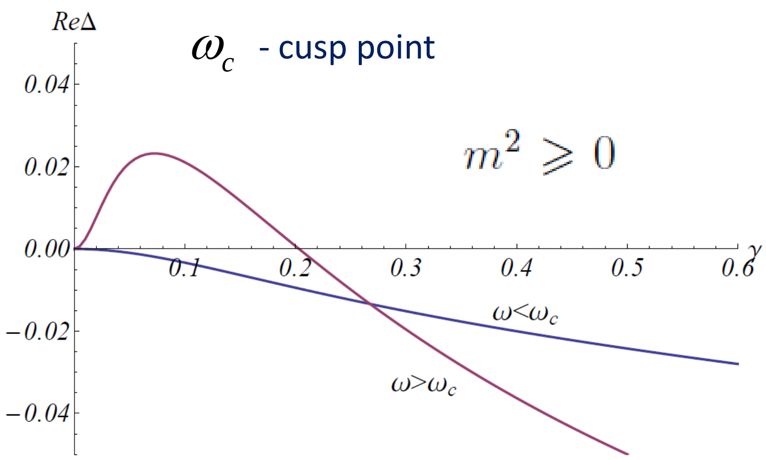
Breaking of translational symmetry

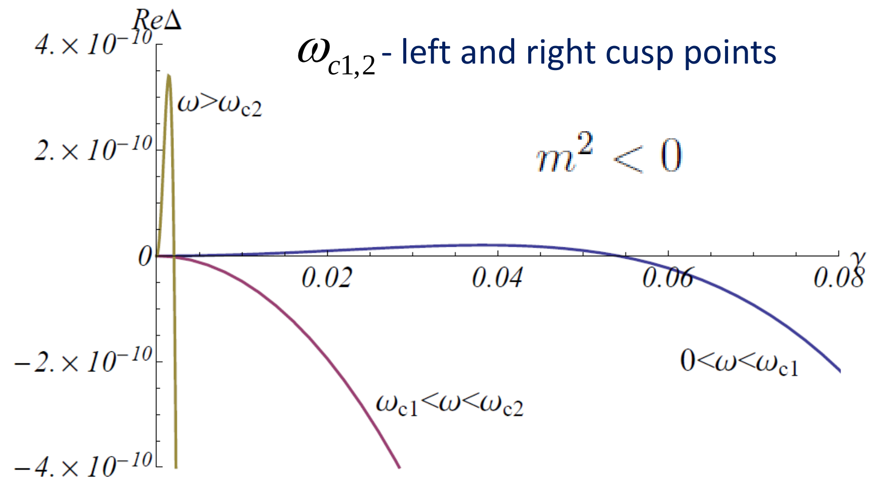
$\longrightarrow$ $h \sim \Phi^l_{0,r}, \quad l = 1$

$$\Delta(0,0;0) = \Delta(0,0;1) = 0$$

Perturbations of tubes with *n=0*

$\Delta(\gamma,0;0)$ for the different values of ω





We see that $\operatorname{sign} \operatorname{Re} \Delta_{\gamma}^{ll}(0,0;0) = \operatorname{sign} \partial^2 E / \partial \mathrm{Q}^2, \quad \operatorname{Im} \Delta(\gamma,0;0) = 0, \; \forall \gamma$

Since $\operatorname{Re} \Delta(\gamma,0;0) < 0, \;\; \gamma \to \infty$, the growing modes with $\gamma' = l = 0$ exists for the Q-tubes living on the upper and left (if any) branches of *E(Q)*-plots

They are unstable!

In addition, we have no any new roots of the equation $\Delta(\gamma,0;l) = 0$ for any *l>0.*

Perturbations of tubes with *n>0*

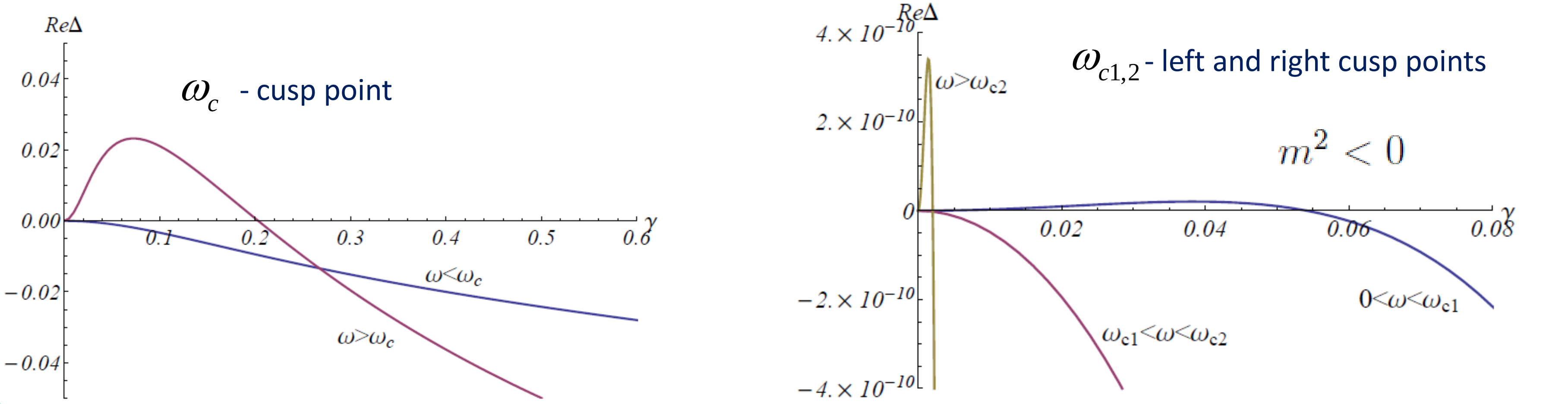
The set of solutions is now

$$c_{1,2,left} = J_{n\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - M^2}\right), \quad r < r_1 \qquad c_{1,2,middle1} = J_{n\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - m^2}\right),$$

$$c_{1,2,right} = H^{(1)}_{n\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - M^2}\right), \quad r > r_2 \qquad c_{1,2,middle2} = H^{(1)}_{n\pm l}\left(r\sqrt{(\omega \mp i\gamma)^2 - m^2}\right) \qquad r_1 < r < r_2$$

The following analysis lies closely to the case *n=0*.

$\Delta(\gamma,0;0)$ for the different values of ω, $\quad n = 2$



Again, $\operatorname{sign} \operatorname{Re} \Delta^{ll}_{\gamma}(0,0;0) = \operatorname{sign} \partial^2 E / \partial Q^2, \quad \operatorname{Im} \Delta(\gamma,0;0) = 0,\ \forall \gamma \quad \operatorname{Re} \Delta(\gamma,0;0) < 0,\ \gamma \to \infty$

The upper and the left (in any) branches of Q-tubes are unstable against perturbations with $\gamma' = l = 0$

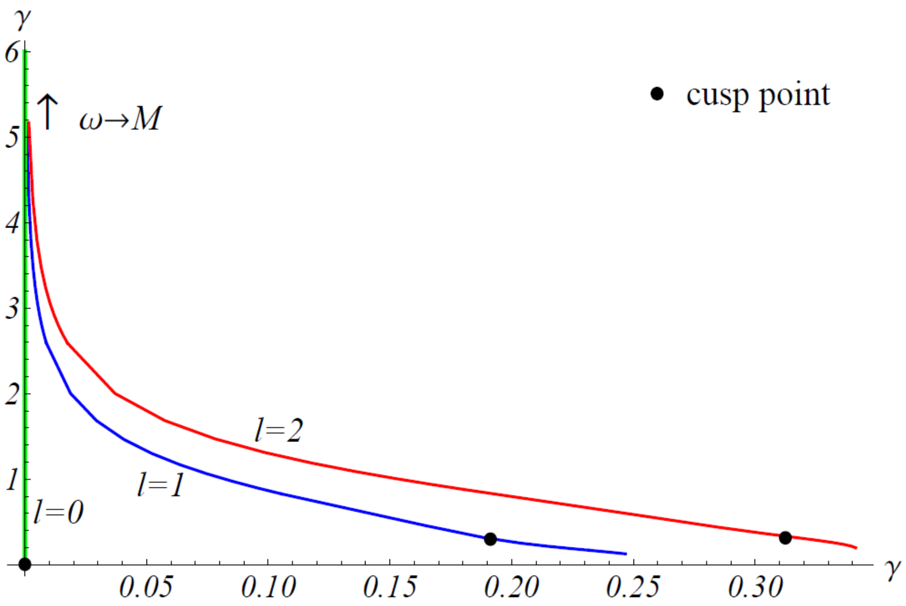
We still have no new roots of the equation $\Delta(\gamma,0;l) = 0$ for any *l>0.*

Perturbations with $\gamma' \neq 0$

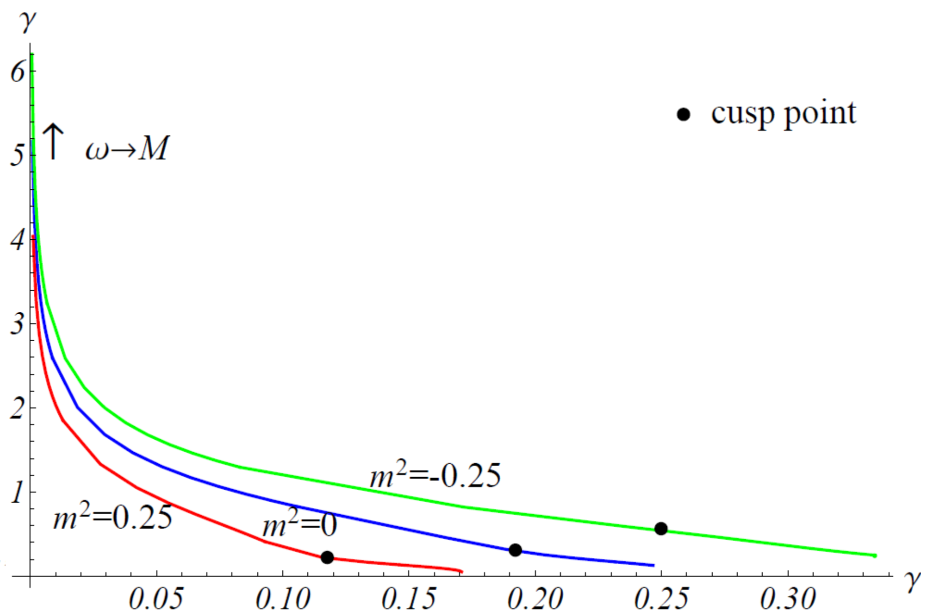
OK, we have investigated stability of Q-tubes against perturbations with $\gamma' = 0$.

All solutions have $l = 0$, i.e. they do not change the angular momentum of the initial Q-tube.

What about perturbations with $\gamma' \neq 0$?



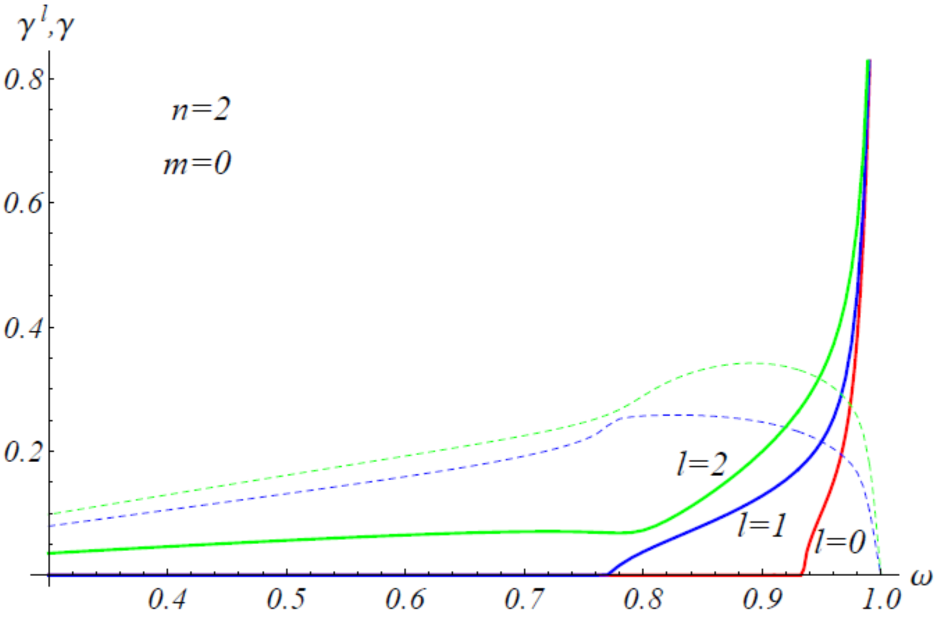
Family of roots of the equation $\Delta(\gamma, \gamma'; l) = 0$ for the different values of *l*.
Shown the example with *n=2, m=0, M=v=1.*



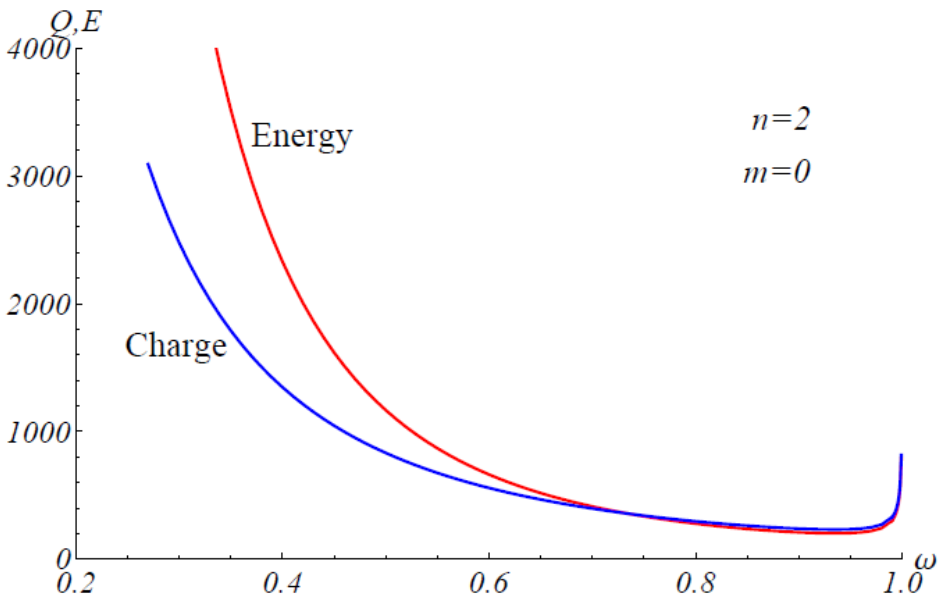
Family of roots of the equation $\Delta(\gamma, \gamma'; l) = 0$ for the different values of *m*.
Shown the example with *n=2, l=1, M=v=1.*

Transitions between tubes with different *n*?

Perturbations with $\gamma' \neq 0$



Roots of $\Delta(\gamma,\gamma';l)=0$ for the different values of *l*, in dependence on ω .



E and *Q* in the same range of ω.

Comparing the energy scale of perturbations γ, γ' to the energy density of Q-tube *E*, we are sure about validity of classical consideration.

Conclusion

- Using the piece-wise parabolic potential, we constructed analytical soliton solutions and investigated their properties that turned out to be similar to those of Q-balls.

- Using the piece-wise parabolic potential, we obtained linearized equations of motion. Solving them, we found solutions, responsible for classical instability of Q-tubes. We found that:

  - All Q-tubes, living on the upper (and the left, if any) branches of *E(Q)*-dependences, are unstable against perturbations with $l = 0, \gamma' = 0$, and the lower branches are stable against them;
  - All Q-tubes with $n > 0$, including those living on the lower branches, become unstable against perturbations with $l > 0, \gamma' \neq 0$;

- We could not find instabilities in the sector of Q-tubes with $n = 0$ which have $l > 0$.
  In particular, we did not find growing modes under the lower branch of $n = 0$ tubes. It may signalize the validity of the stability criterion $\partial^2 E / \partial Q^2 > 0$ and true stability for nonrotating tubes.

Thank you for attention!