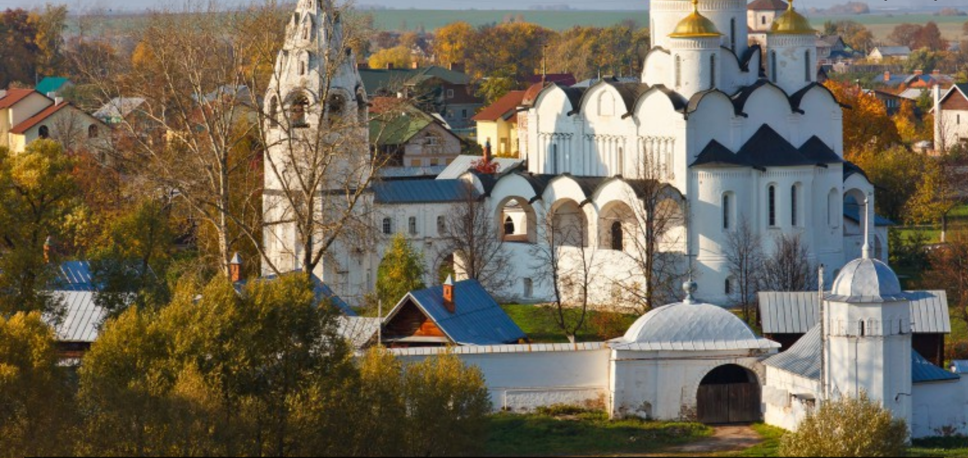
Isospinning Skyrmions

Mareike Haberichter (SMSAS)

University of Kent

QUARKS-2014
18th International Seminar on High Energy Physics
Suzdal, Russia

In Collaboration with:
Richard Battye (UoM)
Steffen Krusch (UoK)



Plan

1. Review of the Skyrme Model
2. Classically & Isospinning Skyrmions
3. Weaknesses of the Rigid Body Approach
4. Numerical Results on isospinning Skyrme solitons beyond the rigid body approach
5. Conclusions

The Classical $SU(2)$ Skyrme Model

University of Kent

The Skyrme Lagrangian expressed in terms of the $SU(2)$ matrix $U(t, \mathbf{x})$ and the $su(2)$-valued right-handed chiral current $R_\mu = (\partial_\mu U)\, U^\dagger$

$m = 2m_\pi/(F_\pi e)$

$$L_{\text{Sky}} = \frac{F_\pi}{4e} \int \left\{ -\frac{1}{2}\mathrm{Tr}\left(R_\mu R^\mu\right) + \frac{1}{16}\mathrm{Tr}\left([R_\mu, R_\nu][R^\mu, R^\nu]\right) + \mathrm{m}^2 \mathrm{Tr}\left(U - \mathbb{1}_2\right) \right\} \mathrm{d}^3\mathrm{x}\,.$$

Finite energy configuration: $U(\mathbf{x}) \to \mathbb{1}_2$ for $|\mathbf{x}| \to \infty \Rightarrow$ $\boxed{U : S^3 \mapsto SU(2) \cong S^3}$

$$\Rightarrow \boxed{B \in \mathbb{Z} = \pi_3\left(SU(2)\right)}$$

Topological charge:

$$B = -\frac{1}{24\pi^2} \int \epsilon_{ijk} \mathrm{Tr}\left(R_i R_j R_k\right) \mathrm{d}^3\mathrm{x}.$$

9-dim symmetry group:

$$\mathbb{R}^3 \times SO(3)^J \times SO(3)^I.$$

Translations and rotations in $\mathbb{R}^3$: $\quad \mathbf{x} \to D(A_2)(\mathbf{x} - \mathbf{X}),$

Isospin transformations: $\quad U \to A_1 U A_1^\dagger.$

Skyrmions & Nuclei

We are interested in modelling light atomic nuclei by classically (iso)spinning Skyrmion solutions.

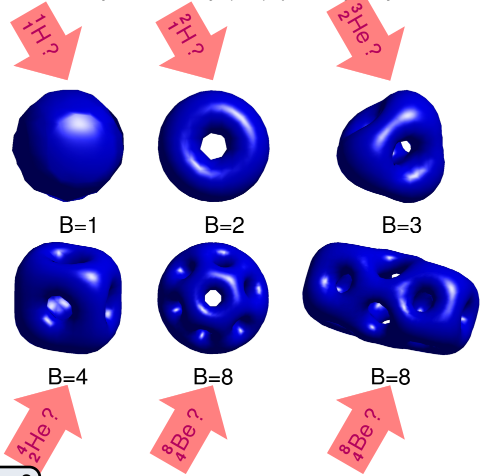
Skyrmions:

Skyrmions are topologically stabilsed solutions in a field theory of pions and can be used to model atomic nuclei. Their topologically conserved charge B can be identified with the mass or atomic number.

Nuclear states can be characterized by spin and isospin quantum numbers.

↓

What happens if you (iso)spin a Skyrmion?



Classically Spinning & Isospinning Skyrmions

University of Kent

Rotation & Isorotation of a static soliton configuration $U_0(\mathbf{x})$:

$$\widehat{U}(\mathbf{x}, t) = A_1(t) U_0[D(A_2(t))(\mathbf{x})] A_1^\dagger(t), \quad A_1, A_2 \in SU(2)$$

Classical soliton mass: $R_i = (\partial_i U_0)\, U_0^\dagger$

$$M = \int \left\{ -\frac{1}{2}\mathrm{Tr}\,(R_i R_i) - \frac{1}{16}\mathrm{Tr}\left([R_i, R_j][R_i, R_j]\right) + m^2 \mathrm{Tr}\,(\mathbb{1}_2 - U) \right\} \mathrm{d}^3 x \,,$$

Kinematical contribution:

$$T = \frac{1}{2} a_i \boxed{U_{ij}} a_j - a_i \boxed{W_{ij}} b_j + \frac{1}{2} b_i \boxed{V_{ij}} b_j \,.$$

Isorotation

Rotation & Isorotation

Rotation

Body-fixed angular velocities:

$$a_j = -i\mathrm{Tr}\left(\tau_j A_1^\dagger \dot{A}_1\right), \qquad b_j = i\mathrm{Tr}\left(\tau_j \dot{A}_2 A_2^\dagger\right).$$

Inertia tensors

University of Kent

The inertia tensors U_{ij}, V_{ij} and W_{ij} are explicitly given by

$$U_{ij} = -\int \mathrm{Tr}\left(T_i T_j + \frac{1}{4}[R_k, T_i][R_k, T_j]\right) \mathrm{d}^D x \, ,$$

$$V_{ij} = -\int \epsilon_{ilm}\epsilon_{jnp} x_l x_n \mathrm{Tr}\left(R_m R_p + \frac{1}{4}[R_k, R_m][R_k, R_p]\right) \mathrm{d}^D x \, ,$$

$$W_{ij} = \int \epsilon_{jlm} x_l \mathrm{Tr}\left(T_i R_m + \frac{1}{4}[R_k, T_i][R_k, R_m]\right) \mathrm{d}^D x \, ,$$

where $R_k = (\partial_k U_0)\, U_0^\dagger$ is the right-invariant $su(2)$ current and

$$T_i = \frac{i}{2}\,[\tau_i, U_0]\, U_0^\dagger ,$$

is another $su(2)$ current. The momenta conjugate to a_i and b_i:

$$K_i = \frac{\partial T}{\partial a_i} = U_{ij} a_j - W_{ij} b_j \, ,$$

$$L_i = \frac{\partial T}{\partial b_i} = -W_{ij}^T a_j + V_{ij} b_j \, .$$

Spherically Symmetric Deforming Skyrmions

University of Kent

$O(3)$ Symmetry:

$$U(\mathbf{x}) = \exp\left\{ i f(r) \hat{x}\cdot\boldsymbol{\tau} \right\},$$

$L(L+1)$

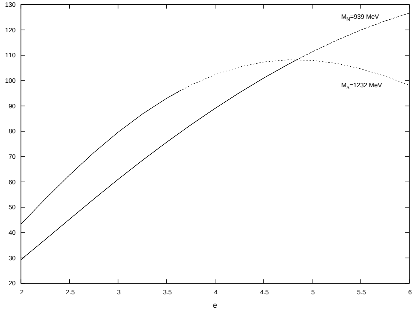
$$U_{ij} = V_{ij} = W_{ij} = V\delta_{ij} \Rightarrow E_{\mathrm{Sky}} = M + \frac{1}{2}\frac{\mathbf{L}^2}{V}.$$

Minimization of M:
Infinite tower of rotational states

$\Updownarrow$

Minimization of E_{Sky}:
Existence of a $\boldsymbol{L_{max}}$

$$|\mathbf{L}| \leq \sqrt{\frac{3}{2}} Vm \Leftrightarrow \omega \leq \sqrt{\frac{3}{2}} m.$$



F_π plotted as a function of e, for which E_{Sky} is equal to the nucleon mass and Δ-mass. The curves cross at $\mathbf{e = 4.84}$ and $\mathbf{F_\pi = 108.125\,MeV}$ ($m_\pi = 138\,\mathrm{MeV}$).

Battye, Krusch, Sutcliffe (05)

Axially Symmetric Deforming Skyrmions

Axially symmetric Ansatz:

$$U = \cos f + i\boldsymbol{\tau}\cdot\mathbf{n_R}\sin f,$$

$$\mathbf{n_R} = (\sin g \cos n\phi, \sin g \sin n\phi, \cos g)\,,$$

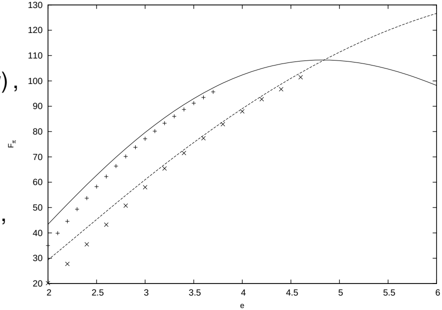
where $f \equiv f(\rho, z)$, $g \equiv g(\rho, z)$.
Inertia Tensors:

$$U_{11} = U_{22},\ V_{11} = V_{22},\ W_{11} = W_{22} = 0,$$

$$U_{33} = \frac{1}{n} W_{33} = \frac{1}{n^2} V_{33}.$$

Existence of a $\boldsymbol{L_{max}}$

$$|\mathbf{L}| \leq Vm \Leftrightarrow \omega \leq m.$$



F_π plotted as a function of e, for which E_{Sky} is equal to the nucleon mass and Δ-mass.).

Battye, Krusch, Sutcliffe (05)
Houghton, Magee (06)
Fortier, Marleau (08)

Rigid Body Quantization – Predicted Ground States

University of Kent

Lower Charge Skyrmions:

B	$\|J\rangle\|I\rangle_0$	$\|J\rangle\|I\rangle_1$	$\|J\rangle\|I\rangle_2$	Experiment	$\|J\rangle\|I\rangle_{Exp.}$	Match
1	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^1_1H$	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	✓
2	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^2_1H$	$\|1\rangle\|0\rangle$	✓
3	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^3_2He$	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	✓
4	$\|0\rangle\|0\rangle$	$\|4\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^4_2He$	$\|0\rangle\|0\rangle$	✓

Rigid Body Quantization – Predicted Ground States

University of Kent

Lower Charge Skyrmions:

B	$\vert J\rangle\vert I\rangle_0$	$\vert J\rangle\vert I\rangle_1$	$\vert J\rangle\vert I\rangle_2$	**Experiment**	$\vert J\rangle\vert I\rangle_{Exp.}$	**Match**
1	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{1}{2}\rangle\vert\frac{3}{2}\rangle$	^1_1H	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	✓
2	$\vert 1\rangle\vert 0\rangle$	$\vert 3\rangle\vert 0\rangle$	$\vert 0\rangle\vert 1\rangle$	^2_1H	$\vert 1\rangle\vert 0\rangle$	✓
3	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{5}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{3}{2}\rangle$	^3_2He	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	✓
4	$\vert 0\rangle\vert 0\rangle$	$\vert 4\rangle\vert 0\rangle$	$\vert 0\rangle\vert 1\rangle$	^4_2He	$\vert 0\rangle\vert 0\rangle$	✓

Higher Charge Skyrmions:

B	$\vert J\rangle\vert I\rangle_0$	$\vert J\rangle\vert I\rangle_1$	$\vert J\rangle\vert I\rangle_2$	**Experiment**	$\vert J\rangle\vert I\rangle_{Exp.}$	**Match**
5	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{1}{2}\rangle\vert\frac{3}{2}\rangle$	^5_2He	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	✗ $^5_2He^*$
5*	$\vert\frac{5}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{7}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{3}{2}\rangle$	^5_2He	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	✗ $^5_2He^*$
6	$\vert 1\rangle\vert 0\rangle$	$\vert 3\rangle\vert 0\rangle$	$\vert 1\rangle\vert 1\rangle$	^6_3Li	$\vert 1\rangle\vert 0\rangle$	✓
7	$\vert\frac{7}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{13}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{3}{2}\rangle$	^7_3Li	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	✗ $^7_3Li^{**}$
8	$\vert 0\rangle\vert 0\rangle$	$\vert 2\rangle\vert 0\rangle$	$\vert 0\rangle\vert 1\rangle$	^8_4Be	$\vert 0\rangle\vert 0\rangle$	✓
9	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{1}{2}\rangle\vert\frac{3}{2}\rangle$	^9_4Be	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	✗ $^9_4Be^{**}$
9*	$\vert\frac{1}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{5}{2}\rangle\vert\frac{1}{2}\rangle$	$\vert\frac{3}{2}\rangle\vert\frac{3}{2}\rangle$	^9_4Be	$\vert\frac{3}{2}\rangle\vert\frac{1}{2}\rangle$	✗ $^9_4Be^{**}$

Krusch (03)

Classically Isospinning Skyrmions

University of Kent

We collect sigma field and triplet of pion fields together in a four component unit vector $\boldsymbol{\phi} = (\sigma, \boldsymbol{\pi})$:

- Classical Skyrmion mass:

$$M = \int (\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}) + \frac{1}{2}\left[(\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi})^2 - \left(\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi}\right)^2 \right] + 2m^2 (1 - \sigma)\, \mathrm{d}^3 x\,,$$

- Moment of inertia:

$$U_{ij} = 2 \int \left(\phi_d \phi^d \delta_{ij} - \phi_i \phi_j\right) (1 + \partial_k \boldsymbol{\phi} \cdot \partial_k \boldsymbol{\phi}) - \epsilon_{ide}\epsilon_{jfg} \left(\phi^d \partial_k \phi^e\right) \left(\phi^f \partial_k \phi^g\right)\, \mathrm{d}^3 x\,.$$

Uniformly isospinning soliton solutions in Skyrme models are obtained by solving one of the following, precisely equivalent variational problems for ϕ:

1. Extremize the pseudoenergy functional $F_\omega(\phi) = -L$ for fixed $|\boldsymbol{\omega}|$,
2. Extremize the Hamiltonian $H = M_B + \frac{1}{2} K_i U_{ij}^{-1} K_j$ for fixed isospin $K_i = U_{ij}\omega_j$.

The Rational Map Approximation

University of Kent

- Rational map:

  $R(z) = \frac{p(z)}{q(z)}$, $\boxed{R : S^2 \to S^2}$.

- Skyrme field Ansatz:

$$U(r, z) = \exp\left[\frac{if(r)}{1 + |R|^2}\begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix}\right],$$

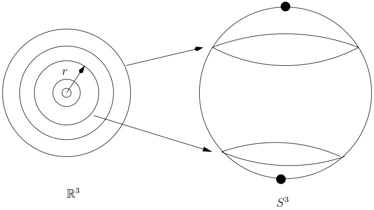
  where

  $B = \max[\deg(p), \deg(q)]$.

- Radial and angular integrals decouple:

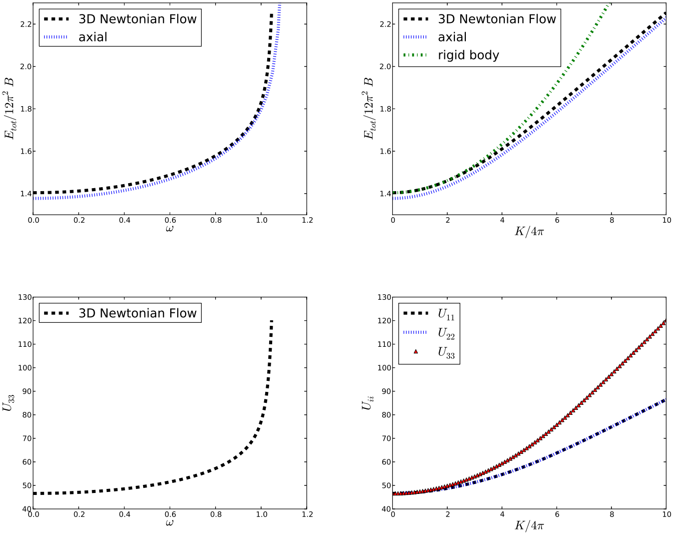
$$E = \frac{1}{3\pi}\int\left(r^2 f'^2 + 2B\left(f'^2 + 1\right)\sin^2 f + \mathcal{I}\frac{\sin^4 f}{r^2}\right)\mathrm{d}r,$$

$$\mathcal{I} = \frac{1}{4\pi}\int\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^4 \frac{2i\,\mathrm{d}z\mathrm{d}\bar{z}}{(1 + |z|^2)^2}.$$

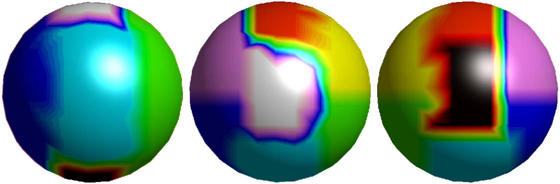


B	G	$R(z)$
1	$O(3)$	z
2	$D_{\infty h}$	z^2
3	T_d	$\frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}$
4	O_h	$\frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$
8	D_{6d}	$\frac{z^6 - ia}{z^2(iaz^6 - 1)}$
8	D_{4h}	$\frac{z^8 + bz^6 - az^4 + bz^2 + 1}{z^8 - bz^6 - az^4 - bz^2 + 1}$ +Pert.

$B = 1$, $\widehat{K} = (0, 0, 1)$, $\mu = 1$



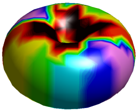
Colour Scheme



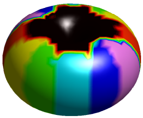
$\hat{\pi}_1 = \hat{\pi}_2 = 0$ & $\hat{\pi}_3 = +1$ **black** $\Leftrightarrow$ -1 *white*

$$\hat{\pi}_1 + i\hat{\pi}_2 = \begin{cases} 1 & \text{red} \Leftrightarrow -1 \quad \text{cyan} \\ e^{2\pi i/3} & \text{blue} \Leftrightarrow e^{5\pi i/3} \quad \text{yellow} \\ e^{4\pi i/3} & \text{green} \Leftrightarrow e^{\pi i/3} \quad \text{magenta} \end{cases}$$

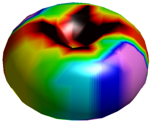
$B = 2$, $\widehat{K} = (0, 0, 1)$



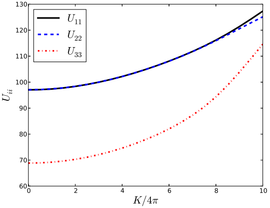
$\mu = 1$

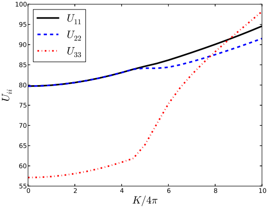


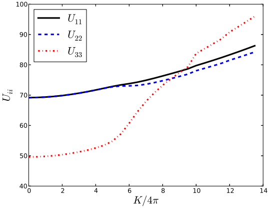
$\mu = 1.5$



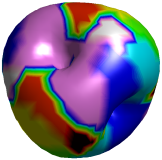
$\mu = 2$



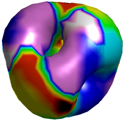




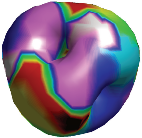
$B = 3$, $\widehat{K} = (0, 0, 1)$



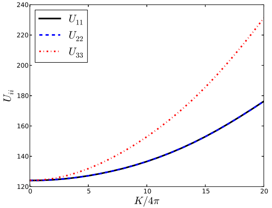
$\mu = 1$

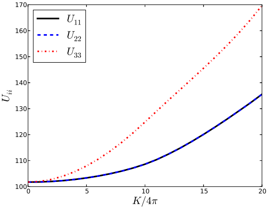


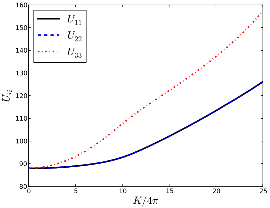
$\mu = 1.5$



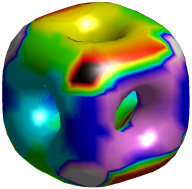
$\mu = 2$



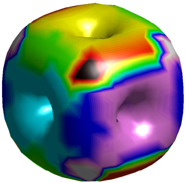




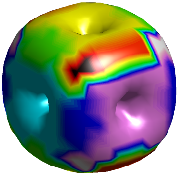
$B = 4, \widehat{K} = (0, 1, 0)$



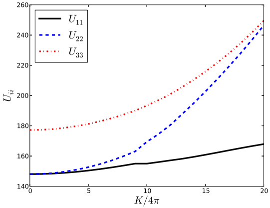
$\mu = 1$

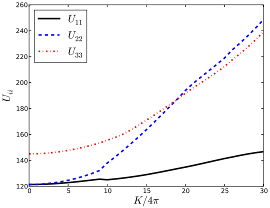


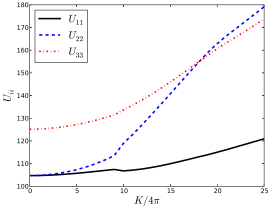
$\mu = 1.5$



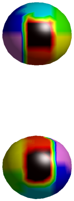
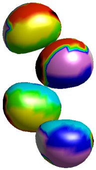
$\mu = 2$



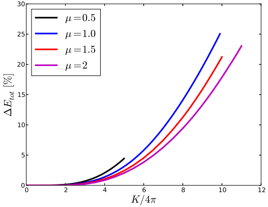




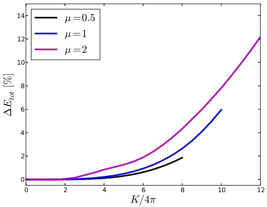
Compare: Skyrmion Dynamics

$B = 2$	$B = 3$	$B = 4$
		
Head on collision	Twisted line scattering	Scattering of four Skyrmions (D_4 symmetry)

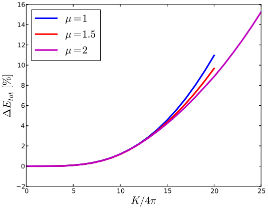
Deviations from the rigid body approximation



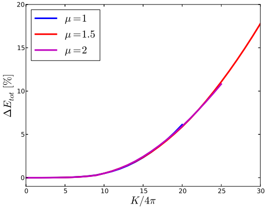
(h) $B = 1$



(i) $B = 2$



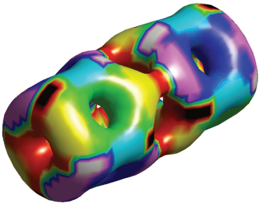
(j) $B = 3$



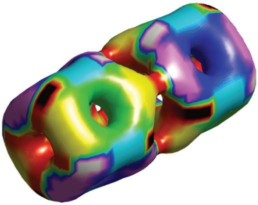
(k) $B = 4$

$B = 8$, D_{4h} & D_{6d}, $m = 1$

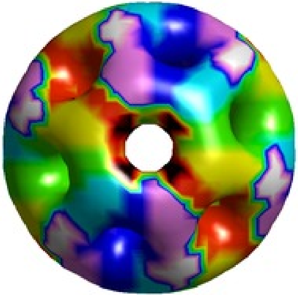
$\widehat{\mathbf{K}} = (0, 1, 0)$



$\widehat{\mathbf{K}} = (0, 0, 1)$



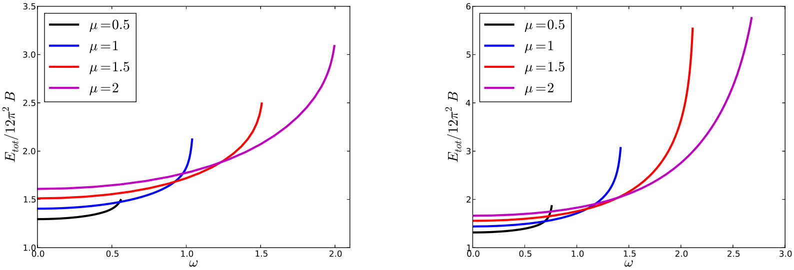
$\widehat{\mathbf{K}} = (0, 0, 1)$



Break-up of isospinning $B = 8$ Skyrmion solutions into charge-4 sub-units, the Skyrme model analogue of α-particles.

$B = 1$ **Skyrmion: Critical frequencies**

University of Kent



$V = 2\mu^2 (1 - \sigma)$ $\qquad$ $V = 2\mu^2(1 - \sigma^2)$

Figure : Total energy E_{tot} and isospin K for $B = 1$ soliton solutions in the standard (left) and "new" (right) Skyrme model as function of angular frequency ω. The mass parameter takes the values $\mu = 0.5, 1, 1.5, 2$.

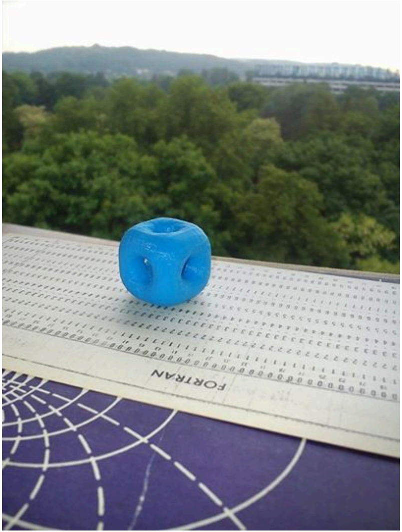
Summary & Outlook

- Classically spinning Skyrmions can be used to model spin and isospin states of nuclei.
- Skyrmions deform as they rotate and isorotate. These “centrifugal” deformations have to be taken into account when approximating nuclei by spinning Skyrmion solutions.

Future Work & Open Questions:

- How well can we approximate nuclear states by classically spinning and isospinning Skyrmion solutions?
- What are the preferred body-fixed axes of rotation? How is the Skyrmion orientated as it spins?
- How to calibrate the Skyrme model?
- Here we only considered isorotations. We neglected completely the kinetic contributions of the inertia tensors V_{ij} and W_{ij} to the total energy!
- As a first step we want to investigate how accurately the observed isopin states, excitation energies and energy spectra of *He-6, Be-10, C-14, O-18 and Ne-22* can be matched by isospinning, deforming Skyrmion solutions.

Thank you for listening!



3D printed $B = 4$ Skyrmion solution.