LOMONOSOV MOSCOW STATE UNIVERSITY
SKOBELTSYN INSTITUTE OF NUCLEAR PHYSICS

Linearized solutions for U(1) gauged Q-balls

Gulamov I.E., Nugaev E.Y., Smolyakov M.N.

Setup

We consider the action, describing the simplest $U(1)$ gauge invariant four-dimensional scalar field theory, in the form

$$S = \int d^4x \left((\partial^\mu \phi^* - ieA^\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{1}$$

We use standard spherically symmetric ansatz for the fields describing a gauged Q-ball

$$\phi(t, \vec{x}) = \mathrm{e}^{i\omega t} f(r), \qquad f(r)|_{r\to\infty} \to 0, \qquad \left.\frac{df(r)}{dr}\right|_{r=0} = 0, \tag{2}$$

$$A_0(t, \vec{x}) = A_0(r), \qquad A_0(r)|_{r\to\infty} \to 0, \qquad \left.\frac{dA_0(r)}{dr}\right|_{r=0} = 0, \tag{3}$$

$$A_i(t, \vec{x}) \equiv 0, \qquad f(r) \in \mathbb{R}, A_0(r) \in \mathbb{R} \qquad f(r) > 0 \tag{4}$$

Gauged Q-balls with small back-reaction of the gauge field

The back-reaction of the gauge field is supposed to be small ($|g(r)| \ll \omega$, $|f(r) - f_0(r)| \ll f_0(r)$, where $f_0(r) = f_0(r, \omega)$ is a nongauged Q-ball solution in the case $e = 0$).

$$\varphi(r) = f(r) - f_0(r)$$

$$\Delta g - 2e^2 \omega f_0^2 = 0, \tag{5}$$

$$\Delta \varphi + \omega^2 \varphi + 2\omega g f_0 - \frac{1}{2} \frac{d^2 V}{df^2}\bigg|_{f=f_0} \varphi = 0, \tag{6}$$

where f_0 is defined as a solution to the equation

$$\omega^2 f_0 + \Delta f_0 - \frac{1}{2} \frac{dV}{df}\bigg|_{f=f_0} = 0 \tag{7}$$

The charge and the energy of gauged Q-balls

$$Q = Q_0 + \triangle Q = Q_0 + 4\pi \int\limits_0^\infty dr r^2 (2g f_0^2 + 4\omega f_0 \varphi) = Q_0(\omega) + \frac{dI(\omega)}{d\omega},$$

$$E = E_0 + \triangle E = E_0 + 4\pi\omega \int\limits_0^\infty dr r^2 (g f_0^2 + 4\omega f_0 \varphi) = E_0(\omega) + \omega \frac{dI(\omega)}{d\omega} - I(\omega),$$

$$I(\omega) = -16\,\pi e^2 \omega^2 \int\limits_0^\infty f_0^2(r,\omega) r \int\limits_0^r f_0^2(y,\omega) y^2 dy\, dr$$

where Q_0 and E_0 are charge and energy of nongauged solutions.

Explicit examples of gauged Q-balls. Model 1

Let us consider the model proposed in [G. Rosen, Phys. Rev. **183** (1969) 1186.] with the potential (in our notations)

$$V(\phi^*\phi) = -\mu^2\phi^*\phi\ln(\beta^2\phi^*\phi), \tag{8}$$

where μ and β are the model parameters. The spherically symmetric background (nongauged) solution for the Q-ball in this model takes the form

$$f_0(r) = \mu\xi \mathrm{e}^{-\frac{\omega^2}{2\mu^2}}\mathrm{e}^{-\frac{\mu^2 r^2}{2}}, \tag{9}$$

where $0 \leq \omega < \infty$ and $\xi = \frac{\mathrm{e}}{\beta\mu}$. The charge and the energy of the Q-ball look like

$$Q_0 = 2\pi^{\frac{3}{2}}\xi^2\frac{\omega}{\mu}\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \tag{10}$$

$$E_0 = 2\pi^{\frac{3}{2}}\xi^2\mu\left(\frac{\omega^2}{\mu^2}+\frac{1}{2}\right)\mathrm{e}^{-\frac{\omega^2}{\mu^2}}. \tag{11}$$

Explicit examples of gauged Q-balls. Model 1

$$\frac{I}{4\pi} = -\mu e^2 \frac{\sqrt{\pi}}{4\sqrt{2}} \xi^4 \left(\frac{\omega}{\mu}\right)^2 \mathrm{e}^{-\frac{2\,\omega^2}{\mu^2}}. \tag{12}$$

$$Q = Q_0 + \triangle Q = 2\pi^{\frac{3}{2}} \xi^2 \left(\tilde{Q}_0 + e^2 \xi^2 \triangle \tilde{Q}\right) = 2\pi^{\frac{3}{2}} \xi^2 \tilde{Q}, \tag{13}$$

$$E = E_0 + \triangle E = \mu\, 2\pi^{\frac{3}{2}} \xi^2 \left(\tilde{E}_0 + e^2 \xi^2 \triangle \tilde{E}\right) = \mu\, 2\pi^{\frac{3}{2}} \xi^2 \tilde{E} \tag{14}$$

$$\tilde{Q}_0 = \tilde{\omega} \mathrm{e}^{-\tilde{\omega}^2} \qquad \tilde{E}_0 = \left(\tilde{\omega}^2 + \frac{1}{2}\right) \mathrm{e}^{-\tilde{\omega}^2}, \tag{15}$$

$$\triangle \tilde{Q} = \left(\sqrt{2}\,\tilde{\omega}^3 - \frac{\tilde{\omega}}{\sqrt{2}}\right) \mathrm{e}^{-2\,\tilde{\omega}^2}, \qquad \triangle \tilde{E} = \left(\sqrt{2}\tilde{\omega}^4 - \frac{\tilde{\omega}^2}{2\sqrt{2}}\right) \mathrm{e}^{-2\,\tilde{\omega}^2}, \tag{16}$$

Validity criteria

$$\alpha(\omega) = e^2 \xi^2 \tilde{\omega}^2 \mathrm{e}^{-\tilde{\omega}^2} \ll 1.$$

Explicit examples of gauged Q-balls. Model 1

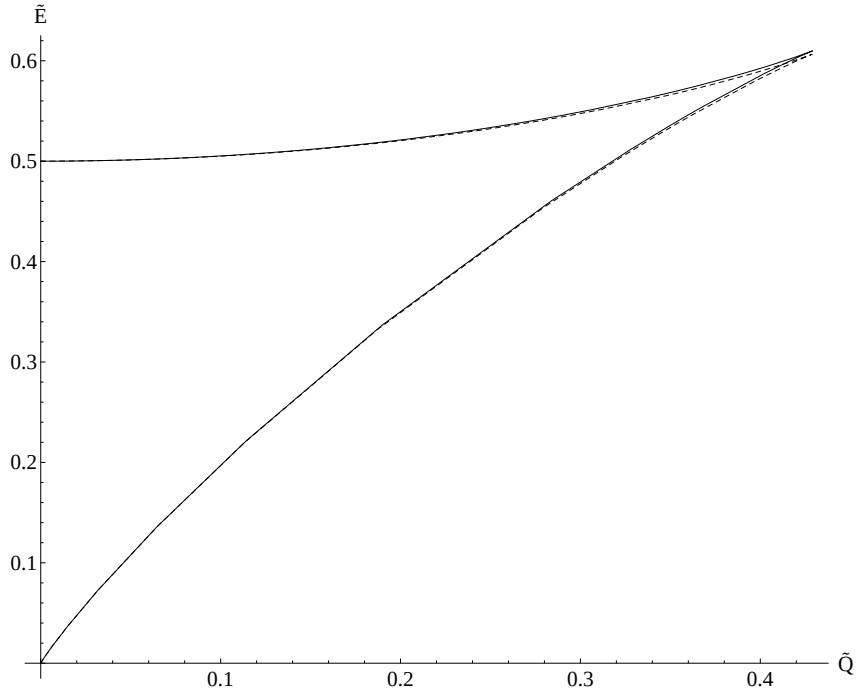


Figure: $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. Here, $e^2\xi^2 = 0.05$ and $0 \leq \tilde{\omega} \leq 10$, where $\tilde{\omega} = \frac{\omega}{\mu}$.

Explicit examples of gauged Q-balls. Model 1

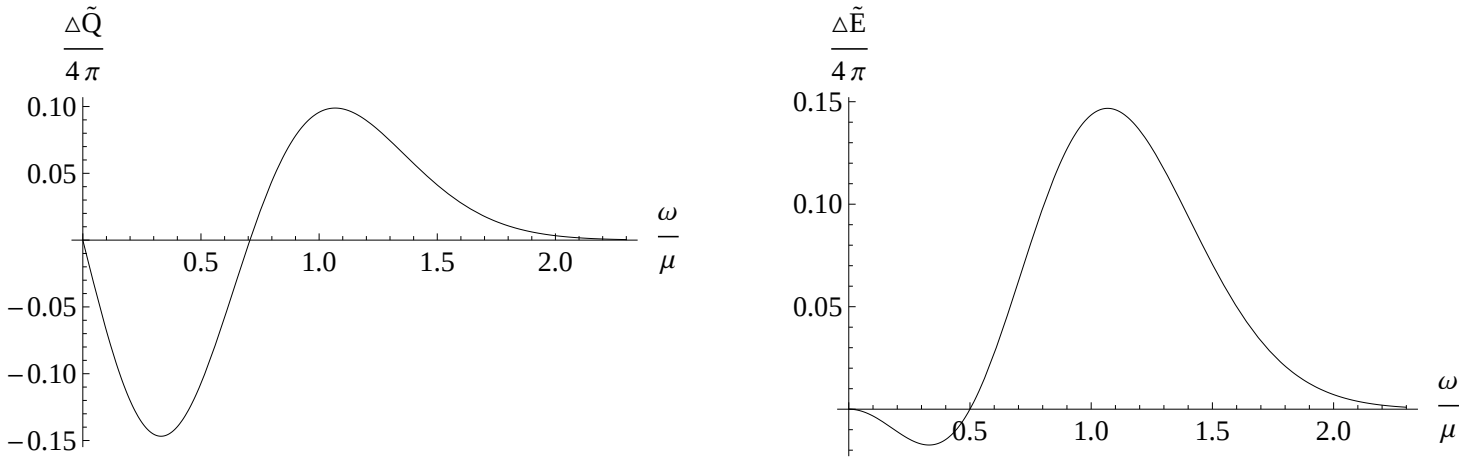


Figure: $\triangle\tilde{Q}$ (left plot) and $\triangle\tilde{E}$ (right plot) for $0 \leq \tilde{\omega} \leq 2.3$.

Model 1

This result was obtained at [V. Dzhunushaliev and K. G. Zloshchastiev, Central Eur. J. Phys. **11** (2013) 325.] first time

$$g(r) = \mu\alpha_1\,\Phi_g(\omega)F_g(r), \tag{17}$$

$$\varphi(r) = \mu\alpha_1\xi\,\Phi_\varphi(\omega)F_\varphi(r), \tag{18}$$

where

$$\Phi_g(\omega) = \frac{\sqrt{\pi}}{2}\frac{\omega}{\mu}\,\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \tag{19}$$

$$F_g(r) = -\frac{1}{\mu\,r}\mathrm{erf}(\mu\,r), \tag{20}$$

$$\Phi_\varphi(\omega) = \sqrt{\pi}\left(\frac{\omega}{\mu}\right)^2\mathrm{e}^{-\frac{3\omega^2}{2\mu^2}}, \tag{21}$$

$$F_\varphi(r) = \mathrm{e}^{-\frac{3\mu^2r^2}{2}}\left(\frac{1}{4\sqrt{\pi}} + \frac{1}{4}\,\mathrm{e}^{\mu^2r^2}\left(\mu r + \frac{1}{2\mu r}\right)\mathrm{erf}(\mu r)\right). \tag{22}$$

Here $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int\limits_0^z \mathrm{e}^{-t^2}dt$.

Model 2

The piecewise scalar field potential was mentioned at [Phys.Rev.D87:085043,2013]

$$V(\phi^*\phi) = M^2\phi^*\phi\,\theta\left(1-\frac{\phi^*\phi}{v^2}\right)+\left(m^2\phi^*\phi+v^2(M^2-m^2)\right)\theta\left(\frac{\phi^*\phi}{v^2}-1\right),$$

where $M^2>0$, $M^2>m^2$, and θ is the Heaviside step function with the convention $\theta(0)=\frac{1}{2}$.

$$f_0(r<R)=f_0^{<}(r)=v\frac{R\sin\left(\sqrt{\omega^2-m^2}\,r\right)}{r\sin\left(\sqrt{\omega^2-m^2}\,R\right)}, \tag{23}$$

$$f_0(r>R)=f_0^{>}(r)=v\frac{R\mathrm{e}^{-\sqrt{M^2-\omega^2}r}}{r\mathrm{e}^{-\sqrt{M^2-\omega^2}R}}, \tag{24}$$

$$R=R(\omega)=\frac{1}{\sqrt{\omega^2-m^2}}\left(\pi-\arctan\left(\frac{\sqrt{\omega^2-m^2}}{\sqrt{M^2-\omega^2}}\right)\right). \tag{25}$$

Model 2

$$
\begin{aligned}
g(r<R) = g_<(r) &= C_1\left(\ln(\omega r) - \mathrm{Ci}(2\omega r) + \frac{\sin(2\omega r)}{2\omega r}\right) + C_2, \\
g(r>R) = g_>(r) &= \frac{C_3}{r} + C_4\left(\frac{\mathrm{e}^{-2\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r} - \mathrm{E}_1(2\sqrt{M^2-\omega^2}\,r)\right),
\end{aligned}
$$

where

$$
\mathrm{Ci}(y) = -\int_y^\infty \frac{\cos(t)}{t}\,dt,
$$

$$C_1 = C_1(\omega) = e^2 v^2 \omega R^2 \frac{1}{\sin^2(\omega R)}, \tag{26}$$

$$C_2 = C_2(\omega) = -e^2 v^2 \omega R^2 \left(2\, \mathrm{e}^{2\sqrt{M^2-\omega^2}R} \mathrm{E}_1(2\sqrt{M^2-\omega^2}R) + \frac{-\mathrm{Ci}(2\omega R) + \ln(\omega R) + 1}{\sin^2(\omega R)} \right), \tag{27}$$

$$C_3 = C_3(\omega) = -e^2 v^2 \omega R^2 \left(\frac{M^2}{\omega^2\sqrt{M^2-\omega^2}} + \frac{R}{\sin^2(\omega R)} \right), \tag{28}$$

$$C_4 = C_4(\omega) = e^2 v^2 \omega R^2 \left(2\, \mathrm{e}^{2\sqrt{M^2-\omega^2}R} \right), \tag{29}$$

$$
\begin{aligned}
\varphi(r<R) &= B\frac{\sin(\omega r)}{r} + \frac{\sin(\omega r)}{\omega r}\int\limits_0^r G_<(t)\cos(\omega t)dt \\
&- \frac{\cos(\omega r)}{\omega r}\int\limits_0^r G_<(t)\sin(\omega t)dt, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\varphi(r>R) &= A\frac{\mathrm{e}^{-\sqrt{M^2-\omega^2}\,r}}{r} - \frac{\mathrm{e}^{\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r}\int\limits_r^\infty G_>(t)\mathrm{e}^{-\sqrt{M^2-\omega^2}\,t}dt \\
&- \frac{\mathrm{e}^{-\sqrt{M^2-\omega^2}\,r}}{2\sqrt{M^2-\omega^2}\,r}\int\limits_R^r G_>(t)\mathrm{e}^{\sqrt{M^2-\omega^2}\,t}dt, \qquad (31)
\end{aligned}
$$

$$G_<(r) = -2\omega r g_<(r) f_0^<(r), \tag{32}$$

$$G_>(r) = -2\omega r g_>(r) f_0^>(r), \tag{33}$$

$$B = B(\omega) = \frac{1}{D} F_1 \frac{\mathrm{e}^{\sqrt{M^2-\omega^2}R}}{\sin(\omega R)} - \frac{F_2}{\omega} + \frac{F_3}{\omega^2 R},$$

$$A = A(\omega) = \frac{\mathrm{e}^{\sqrt{M^2-\omega^2}R}}{D} \left(F_1 \mathrm{e}^{\sqrt{M^2-\omega^2}R} \left(1 + \frac{D}{2\sqrt{M^2-\omega^2}} \right) + F_3 \frac{M^2 \sin(\omega R)}{\omega^2} \right),$$

$$D = D(\omega) = \frac{M^2 R}{1 + R\sqrt{M^2-\omega^2}}, \tag{34}$$

$$F_1 = F_1(\omega) = \int_R^\infty G_>(t) \mathrm{e}^{-\sqrt{M^2-\omega^2}t} dt, \tag{35}$$

$$F_2 = F_2(\omega) = \int_0^R G_<(t) \cos(\omega t) dt, \tag{36}$$

$$F_3 = F_3(\omega) = \int_0^R G_<(t) \sin(\omega t) dt. \tag{37}$$

Model 2

The charge and the energy of the Q-ball look like

$$Q_0 = 4\pi R^2 \omega v^2 \left(\frac{(M^2 - m^2)(R\sqrt{M^2 - \omega^2} + 1)}{(\omega^2 - m^2)\sqrt{M^2 - \omega^2}} \right), \tag{38}$$

$$E_0 = \omega Q_0 + 4\pi \frac{R^3 v^2 (M^2 - m^2)}{3}. \tag{39}$$

$$\frac{I}{4\pi} = e^2\omega^2 \left[a^4 \left(\frac{\sin(2\sqrt{\omega^2-m^2}\,R)}{2\sqrt{\omega^2-m^2}} - R + \frac{\mathrm{Si}(2\sqrt{\omega^2-m^2}\,R)}{2\sqrt{\omega^2-m^2}} - \frac{\mathrm{Si}(4\sqrt{\omega^2-m^2}\,R)}{4\sqrt{\omega^2-m^2}} \right) \right.$$
$$- 4b^2 \left(a^2 \left(\frac{R}{2} - \frac{\sin(2\sqrt{\omega^2-m^2}\,R)}{4\sqrt{\omega^2-m^2}} \right) + \frac{b^2 \mathrm{e}^{-2\sqrt{M^2-\omega^2}\,R}}{2\sqrt{M^2-\omega^2}} \right) \mathrm{E}_1(2\sqrt{M^2 - \omega^2}\,R)$$
$$\left. + \frac{2b^4}{\sqrt{M^2-\omega^2}} \mathrm{E}_1(4\sqrt{M^2 - \omega^2}\,R) \right], \tag{40}$$

Model 2

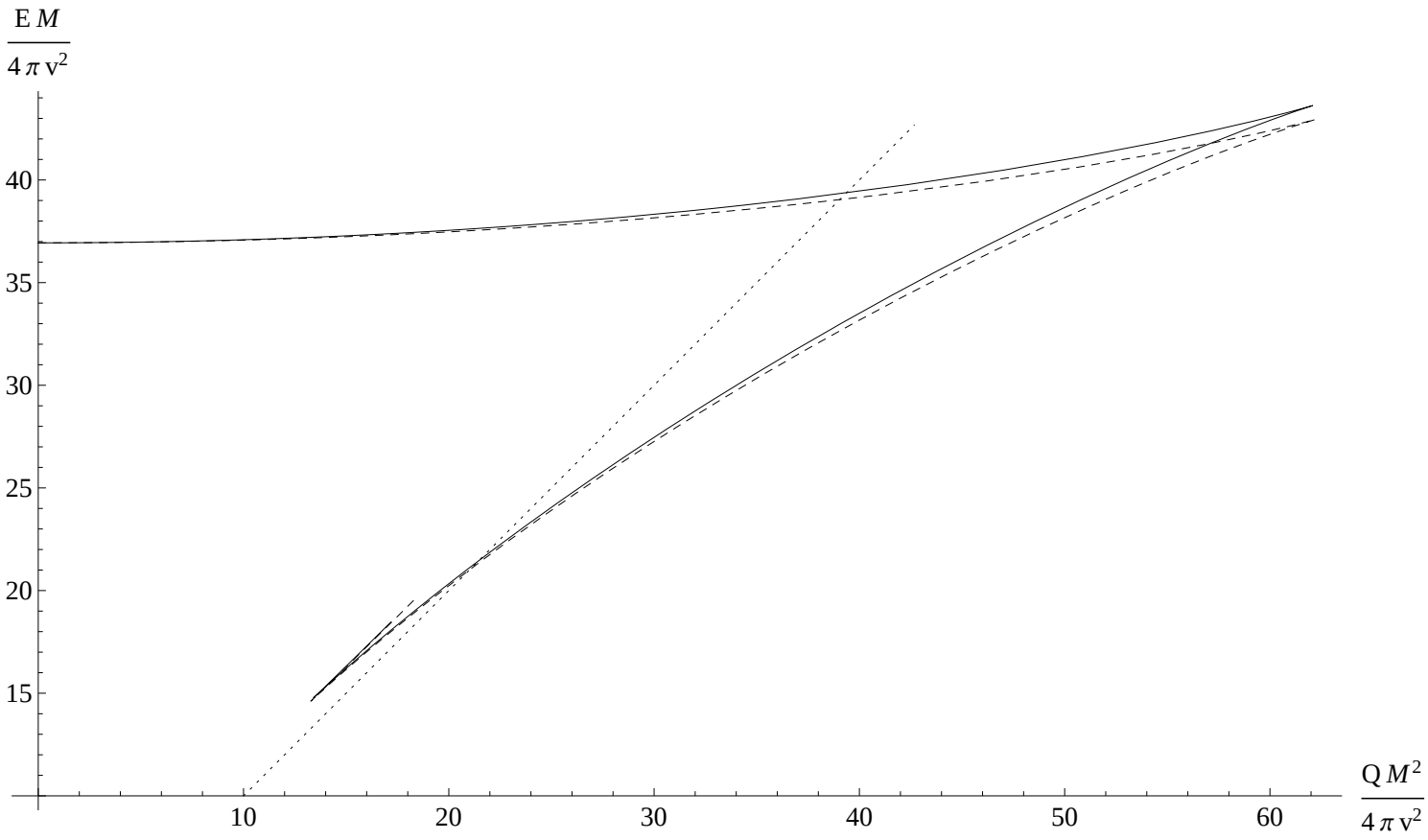


Figure: $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. The dotted line stands for free scalar particles of mass M at rest. Here, $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = \frac{e^2 v^2}{M^2} = 0.001$, and $0 \leq \tilde{\omega} \leq 0.99$.

Model 2

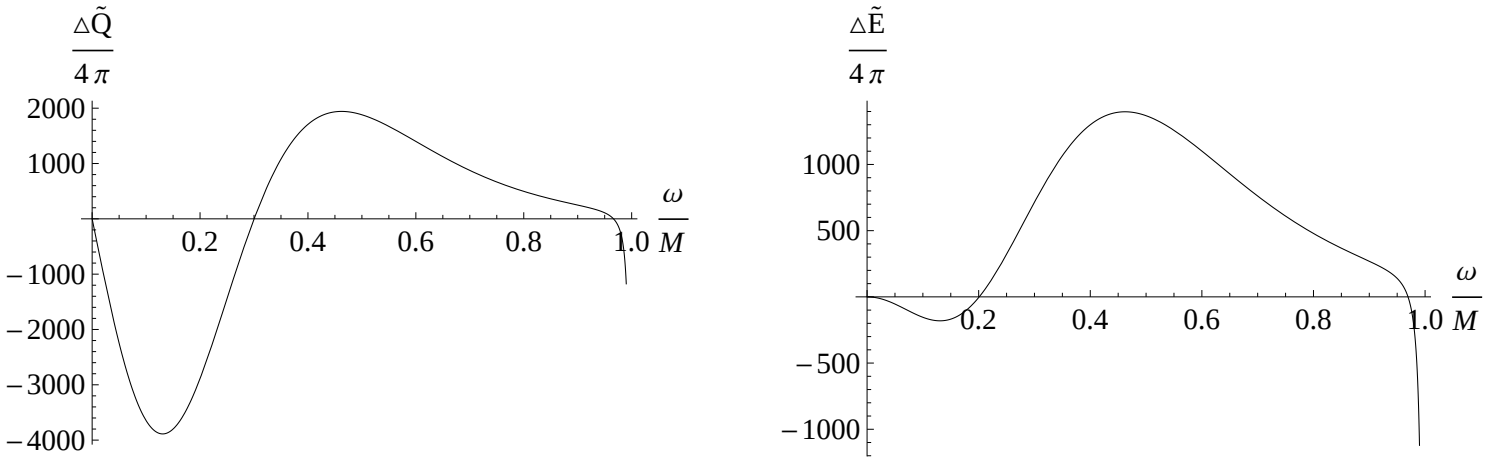


Figure: $\triangle\tilde{Q}$ (left plot) and $\triangle\tilde{E}$ (right plot) for $m^2 < 0$, $\frac{|m|}{M} = 0.6$ and $0 \leq \tilde{\omega} \leq 0.99$.

Model 2

$$max\left(\frac{|g(0)|}{\omega}, \frac{|2\triangle E - \omega\triangle Q|}{2\omega Q_0}, \frac{|g(0)|}{f_0(r_i)}\frac{df_0(r_i)}{d\omega}\right) \ll 1$$

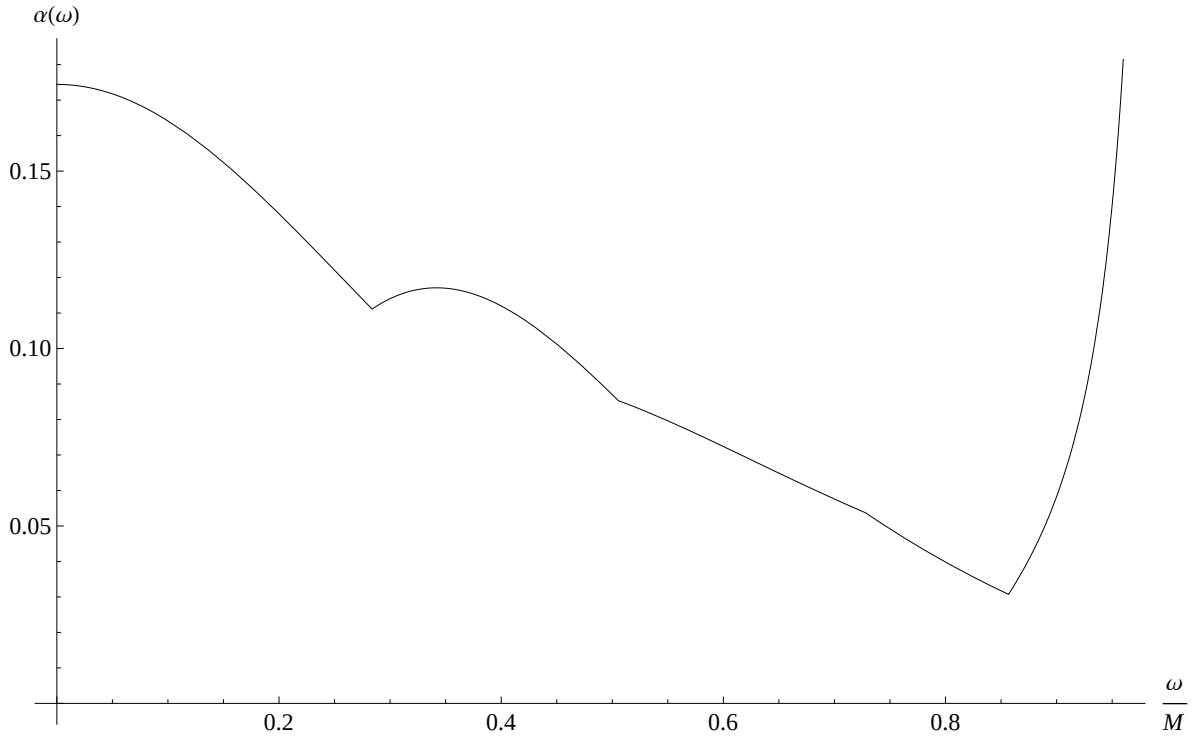


Figure: $\alpha(\omega)$ for $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = 0.001$, and $0 \leq \tilde{\omega} \leq 0.96$.

Model 2

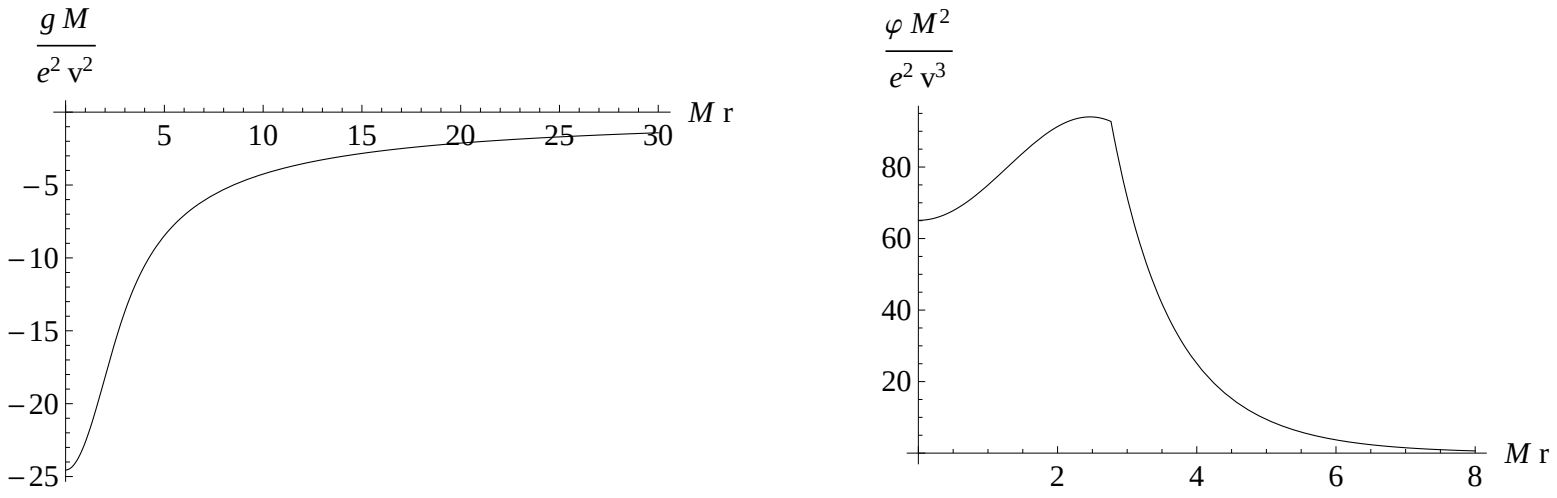
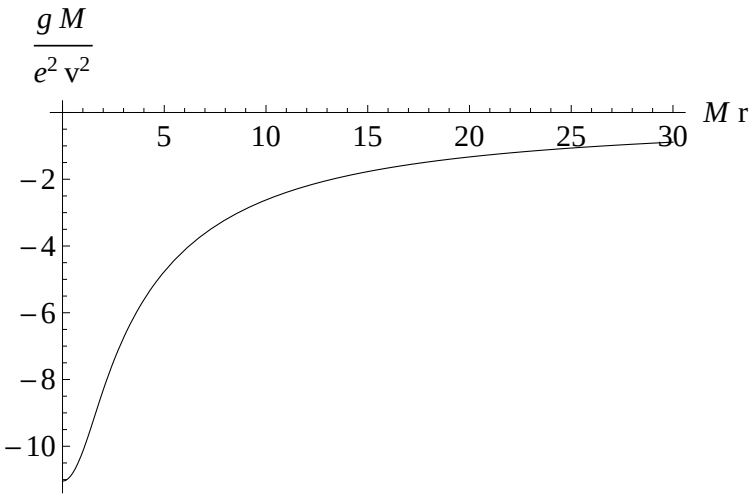


Figure: Solutions for the fields $g(r)$ (left plot) and $\varphi(r)$ (right plot). Here, $m = 0$ and $\tilde{\omega} = 0.8$.

Model 2



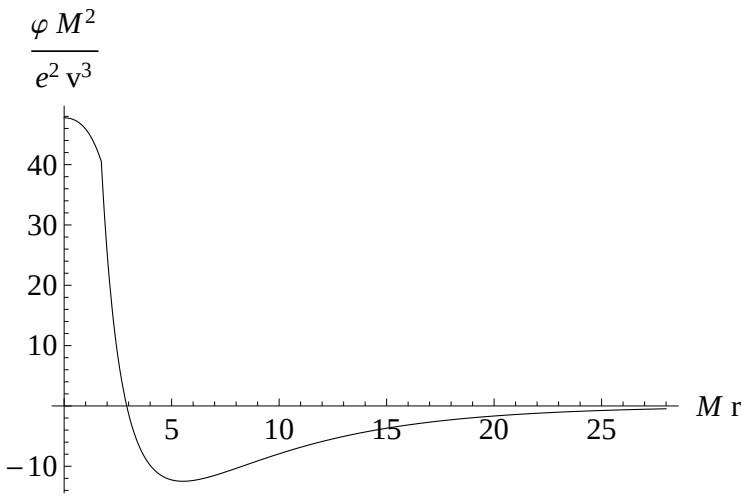


Figure: Solutions for the fields $g(r)$ (left plot) and $\varphi(r)$ (right plot). Here, $m = 0$ and $\tilde{\omega} = 0.99$.

Thank you for attention!