

Dependence of the atomic energy levels on a superstrong magnetic field with account of a finite nucleus radius

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based on the papers

Phys.Rev. D85 (2012) 044058; Phys.Rev. D87 (2013) 124035

QUARKS-2014

June 7, 2014

- Screening of the Coulomb potential
- Freezing of the atomic energy levels
- Critical nucleus charge with the account of screening
- Rising of the ground energy level
- Critical nucleus charge with the account of the finite nucleus radius

Magnetic fields in the Universe

In laboratory on the Earth: $B_{\text{lab}} \approx 3 \cdot 10^7$ G.

The Landau radius equals the Bohr radius for $B_a = e^3 m_e^2 \approx 2 \cdot 10^9$ G.

Neutron stars (radiopulsars): $B \sim 10^{13}$ G.

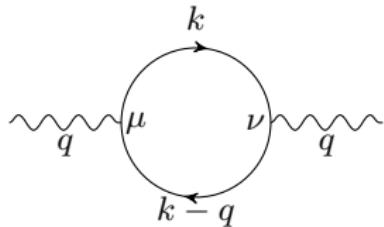
Critical magnetic field $B_0 = m_e^2/e \approx 4.4 \cdot 10^{13}$ G.

Magnetars: $B \sim 10^{15}$ G ($B \approx 2 \cdot 10^{15}$ G on the surface of SGR 1806-20).

Screening: $B \approx 3\pi m_e^2/e^3 \approx 6 \cdot 10^{16}$ G.

Vacuum polarization

$$G(k) = e^{-k_\perp^2/eB} (1 - i\gamma_1\gamma_2) \frac{\hat{k}_{0,3} + m_e}{\hat{k}_{0,3}^2 - m_e^2}$$



Integral over transverse momentum:

$$\int d^2 k_\perp e^{-k_\perp^2/eB} \sim eB$$

As a result we have:

$$\Pi_{\mu\nu} \simeq e^3 B \exp\left(-\frac{q_\perp^2}{eB}\right) \Pi_{\mu\nu}^{(2)}(q_\parallel)$$

Potential with the account of vacuum polarization

$$\Pi(q^2) = \frac{2e^3 B}{\pi} \left(1 - \frac{1}{\sqrt{t}\sqrt{1+t}} \ln \left(\sqrt{t} + \sqrt{1+t} \right) \right) \approx \frac{2e^3 B}{\pi} \frac{2t}{2t+3},$$

где $t \equiv -q_{\parallel}^2/4m_e^2$, $q_{\perp}^2 = 0$.

$$\Phi(\rho, z) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\vec{k}\vec{r}} \frac{4\pi e}{\vec{k}^2 + \Pi(k)}.$$

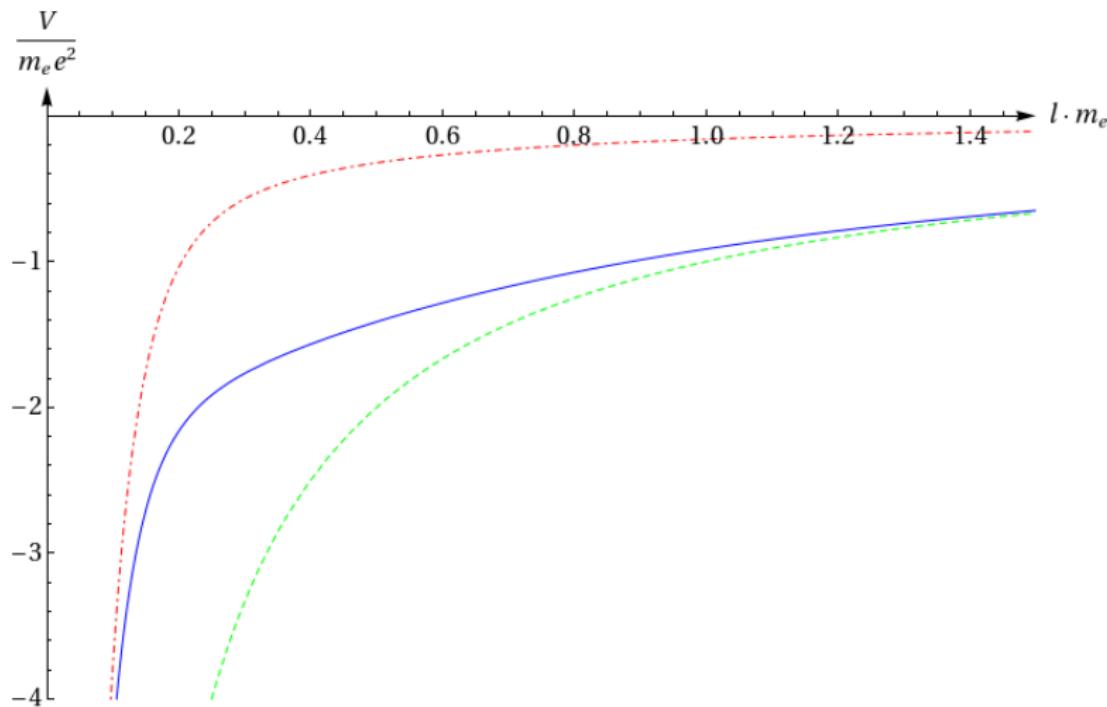
(Numerically — A.E.Shabad, V.V.Usov, PRL 98 (2007) 180403; PRD 77 (2008), 025001)

(Analytically — B.Machet, M.I.Vysotsky, PRD 83 (2011), 025022; SG, B.Machet, M.I.Vysotsky, PRD 85 (2012) 044058)

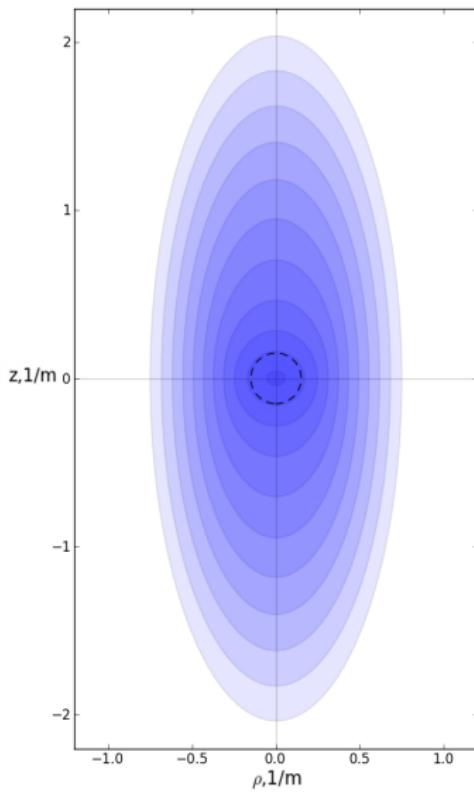
$$\begin{aligned} \frac{1}{|z|} &\rightarrow \frac{1}{|z|} \left(1 - e^{-|z|\sqrt{6m_e^2}} + e^{-\mu|z|} \right) \\ \frac{1}{\rho} &\rightarrow \frac{1}{\rho} \left(e^{-\mu\rho} + \frac{\sqrt{6m_e^2}}{\mu} \right) \end{aligned}$$

where $\mu \equiv \sqrt{6m_e^2 + (2e^3 B/\pi)}$.

$\Phi(0, z)$ and $\Phi(\rho, 0)$ ($B = 5 \cdot 10^4 B_0$)



Equipotential lines for $B = 10^4 B_0$



Energy levels of hydrogen atom with the account of screening (Schrödinger)

(B.M. Karnakov, V.S. Popov, JETP, 97, 890)

Karnakov–Popov equation:

$$\ln \left(\frac{B}{B_a} \right) = \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|)$$

where λ defines corresponding energy $E_m = -(m_e e^4 / 2) \lambda^2$

(B.Machet, M.I.Vysotsky, Phys. Rev. D 83 (2011), 025022)

With the account of screening for ground energy level we get:

$$\ln \left(\frac{B/B_a}{1 + \frac{e^6}{3\pi} \frac{B}{B_a}} \right) = \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + \ln 2 + 3\gamma$$

For $B \rightarrow \infty$ we get $\lambda_\infty \approx 11.21$ or $E_0^\infty = -1.7 \text{ keV}$.

Critical charge in the absence of magnetic field

Pointlike Coulomb center with charge $Z > 137$ can not exist.

1945-1970

But if the finite size of a nucleus is taken into account then Dirac equation can be solved for $Z > 137$. At $Z = Z_{cr} \approx 172$ ground energy level reaches lower continuum ($\varepsilon = -m_e$) and spontaneous electron-positron pair production occurs. Electrons occupy ground level and one can observe two free positrons.

Critical charge in a magnetic field

V.N. Oraevskii, A.I. Rez, V.B. Semikoz, JETP, Vol. 45, No 3, p. 428 (March 1977)

Solution found for $B \gg m^2 e^3 Z^2$; $B \gg \frac{m^2}{e(Ze^2)^2}$

$$Ze^2 \ln \left(2 \frac{\sqrt{m_e^2 - \varepsilon^2}}{\sqrt{eB}} \right) + \arctan \left(\sqrt{\frac{m_e + \varepsilon}{m_e - \varepsilon}} \right) + \arg \Gamma \left(-\frac{Ze^2 \varepsilon}{\sqrt{m_e^2 - \varepsilon^2}} + iZe^2 \right) \\ - \arg \Gamma(1 + 2iZe^2) - \frac{Ze^2}{2} (\ln 2 + \gamma) = \frac{\pi}{2} + n\pi ,$$

for ground energy level $n = 0$ for $\varepsilon > 0$ and $n = 1$ for $\varepsilon < 0$.

Critical charge:

$$\frac{B}{B_0} = 2(Z_{cr}e^2)^2 \exp \left(-\gamma + \frac{\pi - 2 \arg \Gamma(1 + 2iZ_{cr}e^2)}{Z_{cr}e^2} \right).$$

$$B/B_0 \quad 10^2 \quad 10^3 \quad 2 \cdot 10^4$$

$$Z_{cr} \quad 96 \quad 61 \quad 41$$

Used method

For the fields $B \gg m^2 e^3 Z^2$ ($a_H = \frac{1}{\sqrt{eB}} \ll \frac{1}{m_e Ze^2} = a_B$)

$$(\alpha(\mathbf{p} + \mathbf{eA}) + V + \beta)\psi = \varepsilon\psi$$



$$\frac{d^2\chi}{dz^2} + 2m_e(E - U)\chi = 0 ,$$

$$E = \frac{\varepsilon^2 - m_e^2}{2m_e},$$

$$U = \frac{\varepsilon}{m_e}\bar{V} - \frac{1}{2m_e}\bar{V}^2 + \frac{\bar{V}''}{4m_e(\varepsilon + m_e - V)} + \frac{3/8(\bar{V}')^2}{m_e(\varepsilon + m_e - V)^2} ,$$

$$\bar{V} = \frac{1}{a_H^2} \int_0^\infty V(\sqrt{\rho^2 + z^2}) \exp\left(-\frac{\rho^2}{2a_H^2}\right) \rho d\rho$$

Results for $Z = 1$ with the account of screening

$$\bar{V}(z) = -\frac{Ze^2}{a_H} \left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}|z|} \right] \sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{2a_H^2}\right) \operatorname{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right)$$

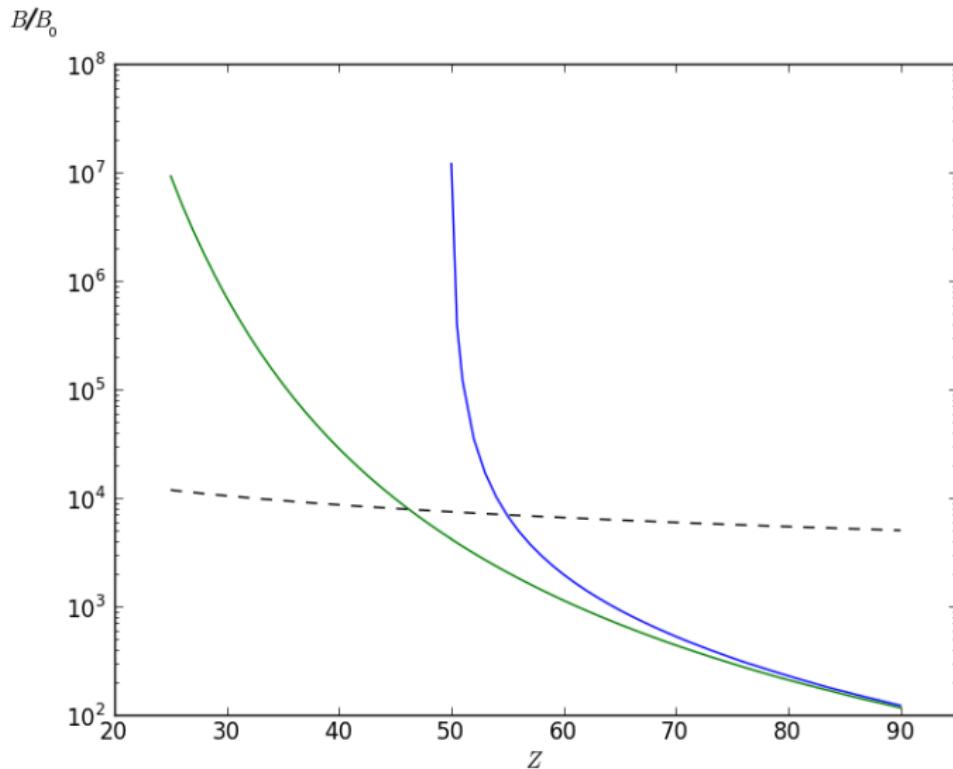
For λ we get:

$\frac{B}{B_0}$	Machet–Vysotsky formula (Schrödinger)	Numerical results (Schrödinger)	Numerical results (Dirac)
10^0	5.736	5.735	5.734
10^1	7.367	7.371	7.368
10^2	9.082	9.095	9.090
10^3	10.53	10.584	10.575
10^4	11.11	11.231	11.219
10^5	11.20	11.357	11.346
10^6	11.21	11.381	11.369
10^7	11.21	11.386	11.375
10^8	11.21	11.387	11.376

Results for $Z = 40$

$\frac{B}{B_0}$	ε/m_e (ORS equation)	ε/m_e (Numerical results)	ε/m_e (Numerical results with the account of screening)
10^0	0.819	0.850	0.850
10^1	0.653	0.667	0.667
10^2	0.336	0.339	0.346
10^3	-0.158	-0.159	-0.0765
10^4	-0.758	-0.759	-0.376
$2 \cdot 10^4$	-0.926	-0.927	-0.423
...	at $B/B_0 \approx 2.85 \cdot 10^4$, $\varepsilon = -m_e$...
10^5	—	—	-0.4887
10^6	—	—	-0.5241
10^7	—	—	-0.5351
10^8	—	—	-0.5386
10^9	—	—	-0.5397

Critical nucleus charge with the account of screening



Qualitative consideration

Matching the solutions which is valid at $z \ll a_B$ and $z \gg 1/m_e$, we get:

$$2 \ln \frac{z_0}{a_B} + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2 \ln 2 = 2 \int_0^{z_0} dz \int \frac{\Phi(\rho, z)}{e} |R_{00}(\rho)|^2 d^2\rho \equiv I,$$

where $E \equiv - (m_e e^4 / 2) \lambda^2$, $a_B \equiv 1/(m_e e^2)$, $\gamma = 0.5772\dots$

For the Coulomb potential:

$$I \approx \ln \frac{z_0^2}{a_H^2}$$

For the screened potential ($\mu = \sqrt{6m_e^2 + (2e^3 B / \pi)}$, $\mu \gg m_e$):

$$I \approx \ln \frac{z_0^2}{1/m_e^2} + \ln \frac{1}{\mu^2 a_H^2}$$

Qualitative consideration

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$$2 \ln \frac{z_0}{a_B} + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2 \ln 2 = 2 \int_0^{z_0} dz \int \frac{\Phi(\rho, z)}{e} |R_{00}(\rho)|^2 d^2\rho \equiv I,$$

where $E \equiv - (m_e e^4 / 2) \lambda^2$, $a_B \equiv 1/(m_e e^2)$, $\gamma = 0.5772\dots$.

Let us choose $z_0 = a_B$

For the Coulomb potential:

$$I \approx \ln \frac{z_0^2}{a_H^2} \rightarrow \ln \frac{B}{m_e e^3}$$

For the screened potential ($\mu = \sqrt{6m_e^2 + (2e^3 B / \pi)}$, $\mu \gg m_e$):

$$I \approx \ln \frac{z_0^2}{1/m_e^2} + \ln \frac{1}{\mu^2 a_H^2} \rightarrow \ln \frac{1}{e^4} + \ln \frac{1}{\mu^2 a_H^2} \approx \ln \frac{1}{e^6}$$

Qualitative consideration

Matching the solutions which is valid at $z \ll a_B$ and $z \gg 1/m_e$, we get:

$$2 \ln \frac{z_0}{a_B} + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2 \ln 2 = 2 \int_0^{z_0} dz \int \frac{\Phi(\rho, z)}{e} |R_{00}(\rho)|^2 d^2 \rho \equiv I,$$

where $E \equiv - (m_e e^4 / 2) \lambda^2$, $a_B \equiv 1/(m_e e^2)$, $\gamma = 0.5772\dots$.

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$$I \approx \ln \frac{z_0^2}{1/m_e^2} + \ln \frac{1}{\mu^2 a_H^2} \rightarrow \ln \frac{1}{e^4} + \ln \frac{1}{\mu^2 a_H^2} \approx \ln \frac{1}{e^6}$$

For $a_H < R$, $B > 1/(eR^2)$,

$$I \approx \ln \frac{1}{e^4} + \ln \frac{1}{\mu^2 R^2}$$

Qualitative consideration

Matching the solutions which is valid at $z \ll a_B$ and $z \gg 1/m_e$, we get:

$$2 \ln \frac{z_0}{a_B} + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2 \ln 2 = 2 \int_0^{z_0} dz \int \frac{\Phi(\rho, z)}{e} |R_{00}(\rho)|^2 d^2\rho \equiv I,$$

where $E \equiv - (m_e e^4 / 2) \lambda^2$, $a_B \equiv 1/(m_e e^2)$, $\gamma = 0.5772\dots$.

Let us choose $z_0 = a_B$

For the Coulomb potential:

$$I \approx \ln \frac{z_0^2}{a_H^2} \rightarrow \ln \frac{B}{m_e e^3}$$

For the screened potential ($\mu = \sqrt{6m_e^2 + (2e^3 B / \pi)}$, $\mu \gg m_e$):

$$I \approx \ln \frac{z_0^2}{1/m_e^2} + \ln \frac{1}{\mu^2 a_H^2} \rightarrow \ln \frac{1}{e^4} + \ln \frac{1}{\mu^2 a_H^2} \approx \ln \frac{1}{e^6}$$

For $a_H < R$, $B > 1/(eR^2)$,	For $\mu > 1/R$, $B > 1/(e^3 R^2)$,
$I \approx \ln \frac{1}{e^4} + \ln \frac{1}{\mu^2 R^2}$	$I \approx \ln \frac{1}{e^4}$

Potential of the pointlike potential

Potential along z -axis:

$$\Phi(0, z) = \frac{e}{|z|} \left(1 - e^{-|z|\sqrt{6m_e^2}} + e^{-\mu|z|} \right),$$

where $\mu \equiv \sqrt{6m_e^2 + (2e^3 B / \pi)}$.

Potential in the plane $z = 0$ for $\mu \gg m_e$, $B \gg m_e^2/e^3$:

$$\Phi(\rho, 0) = \frac{e}{\rho} \left(e^{-\mu\rho} + \frac{\sqrt{6m_e^2}}{\mu} \right),$$

$\Phi(\rho, z)$ for $z \gg 1/m_e$:

$$\Phi(\rho, z) = \frac{e}{\sqrt{z^2 + \left(1 + \frac{e^3 B}{3\pi m_e^2} \right) \rho^2}}.$$

Potential at large distances becomes sensitive to the charge distribution inside proton for the fields

$$B \gtrsim 10^{10} B_0.$$

Potential of the nucleus with a finite radius

$$r = \sqrt{\rho^2 + z^2}, \quad \rho \lesssim a_H$$

$$\Phi(\rho, z) \approx \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + h(R)e^{-\mu r} \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + h(r)e^{-\mu R} \right), & r < R, \end{cases}$$

$$\Phi^{(1)}(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + e^{-\mu r} \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \right), & r < R. \end{cases}$$

$$\Phi^{(2)}(\rho, z) = \begin{cases} \frac{e}{r} \left(1 - e^{-r\sqrt{6m_e^2}} + e^{-\mu r} \cdot \frac{1}{2\mu R} (e^{\mu R} - e^{-\mu R}) \right), & r \geq R, \\ \frac{e}{R} \left(1 - e^{-R\sqrt{6m_e^2}} + e^{-\mu R} \cdot \frac{1}{2\mu r} (e^{\mu r} - e^{-\mu r}) \right), & r < R. \end{cases}$$

Analytical solution

Neglecting the constant of the order of one, we get:

$$\begin{aligned} I &\approx 2 \int_{\sqrt{R^2 + a_H^2}}^{z_0} \frac{dz}{z} \left[1 - e^{-z\sqrt{6m_e^2}} + h(R)e^{-\mu z} \right] \approx \\ &\approx 2 \left[\ln \frac{z_0}{\sqrt{R^2 + a_H^2}} - E_1 \left(\sqrt{R^2 + a_H^2} \sqrt{6m_e^2} \right) + h(R)E_1 \left(\mu \sqrt{R^2 + a_H^2} \right) \right], \end{aligned}$$

where

$$E_1(x) \equiv \int_x^\infty \frac{e^{-t}}{t} dt,$$

$$E_1(x)|_{x \ll 1} = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!},$$

$$E_1(x)|_{x \gg 1} = \frac{e^{-x}}{x} \left(1 - \frac{1}{x} + \frac{1 \cdot 2}{x^2} - \frac{1 \cdot 2 \cdot 3}{x^3} + \dots \right).$$

Equation for λ

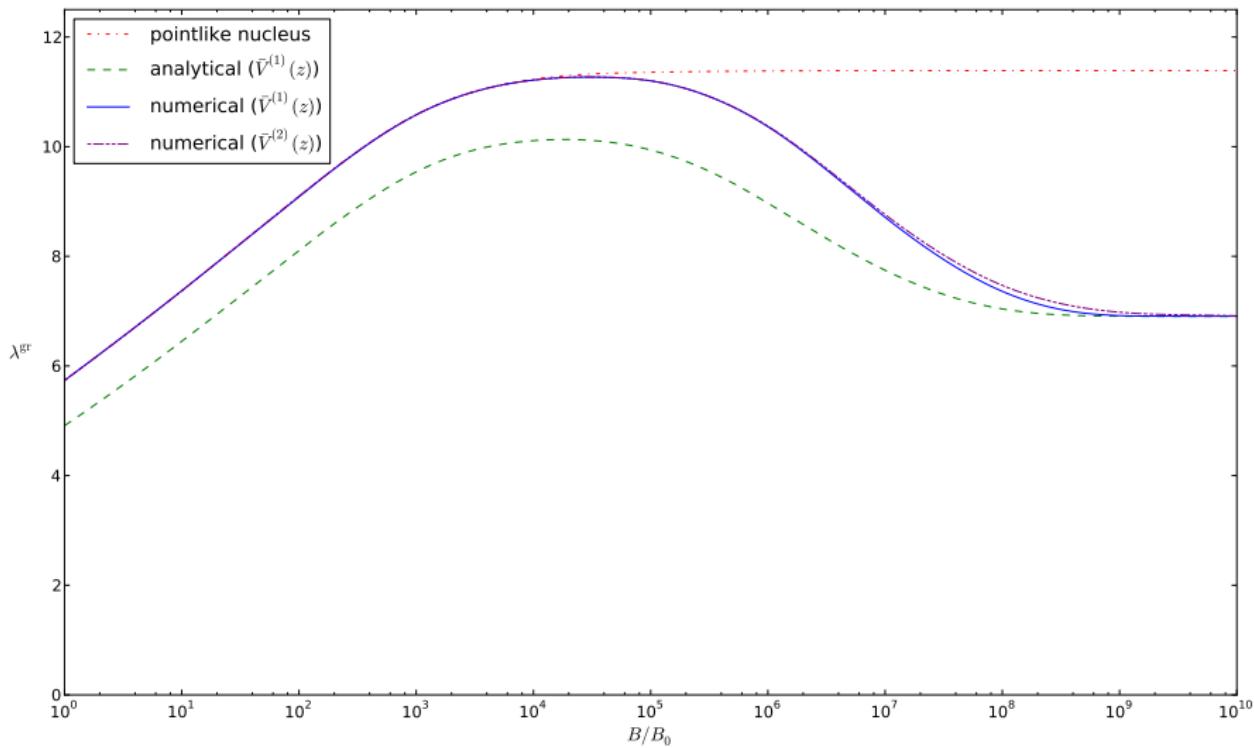
Equation for λ :

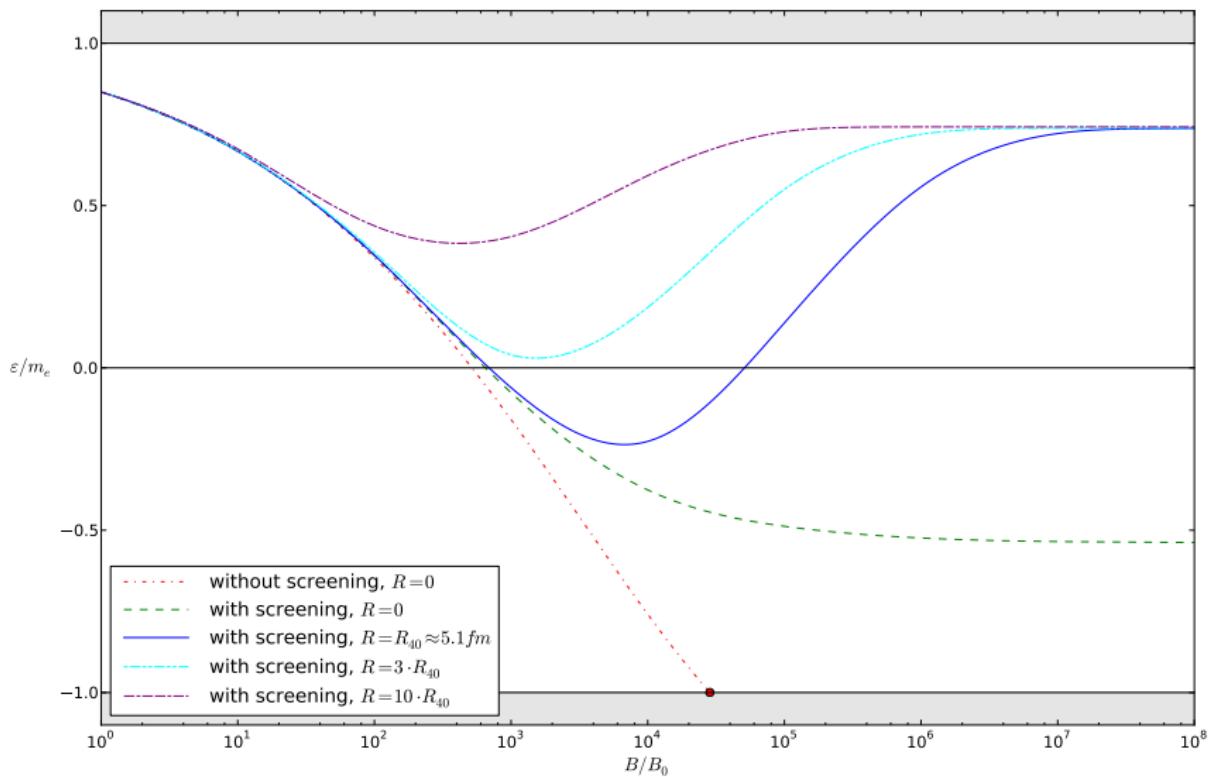
$$\frac{\lambda}{2} + \ln \lambda + \psi\left(1 - \frac{1}{\lambda}\right) + 2\gamma + \ln 2 = \ln \frac{a_B}{\sqrt{R^2 + a_H^2}} - E_1\left(\sqrt{R^2 + a_H^2}\sqrt{6m_e^2}\right) + h(R)E_1\left(\mu\sqrt{R^2 + a_H^2}\right).$$

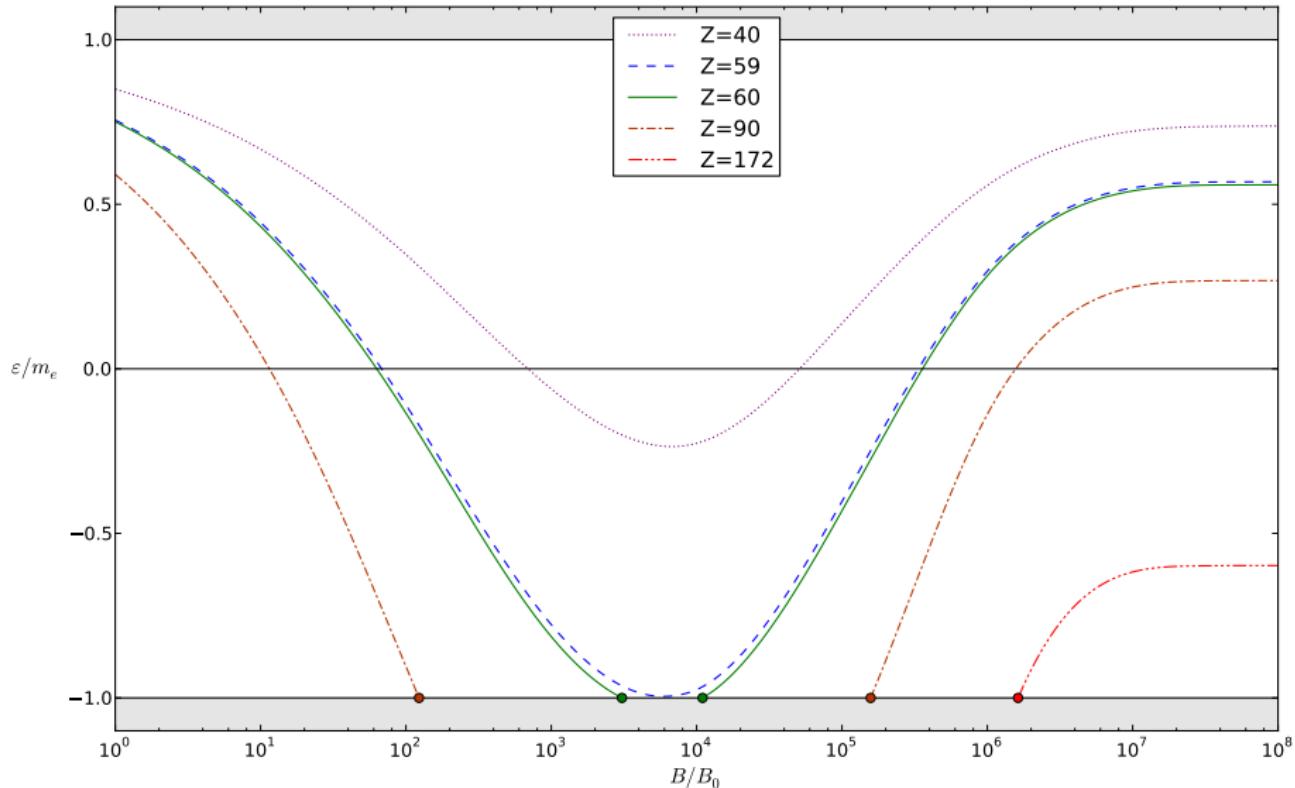
In the limit $B \gg 1/(e^3 R^2)$:

$$\lambda^{\lim} + 2 \ln \lambda^{\lim} + 2\psi\left(1 - \frac{1}{\lambda^{\lim}}\right) + 2\gamma + 2 \ln 2 = \ln\left(\frac{6}{e^4}\right).$$

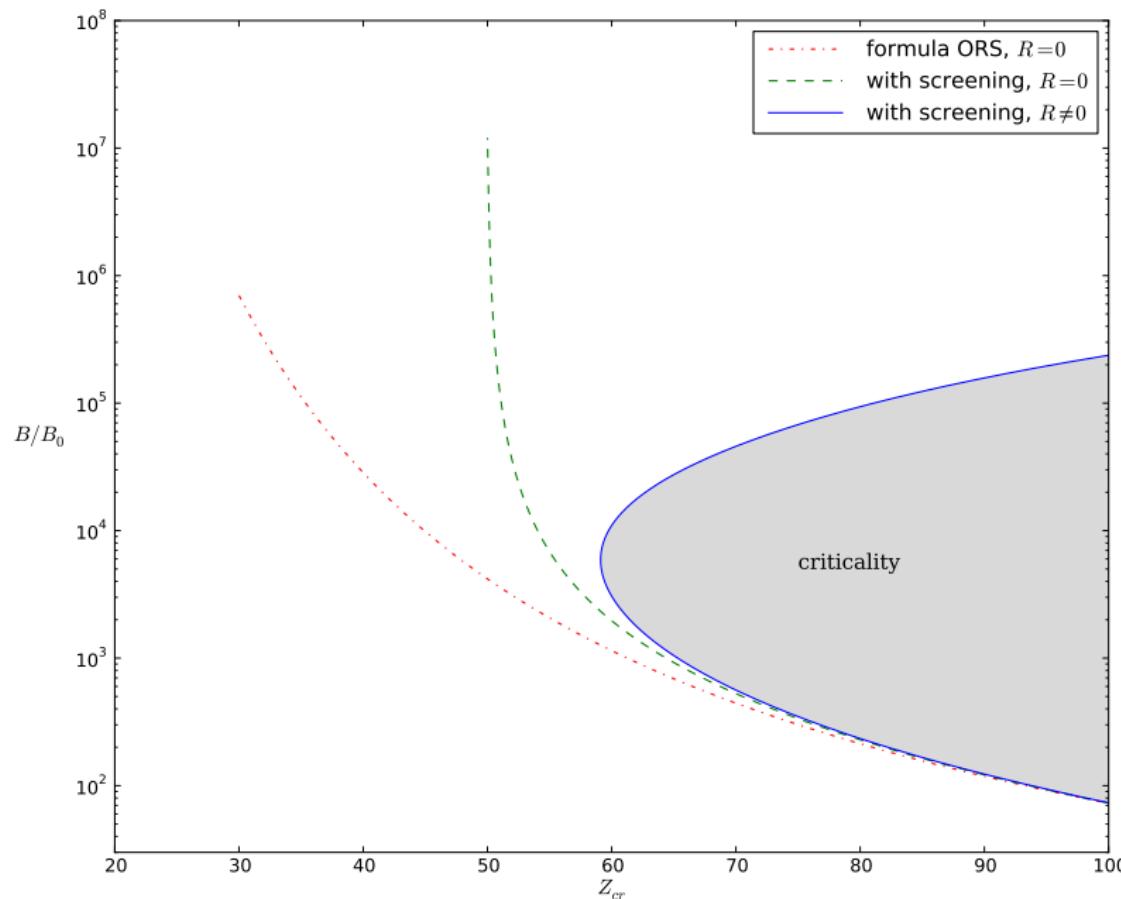
The dependence of λ^{gr} on magnetic field with the account of a finite nucleus size ($Z = 1$)







Critical nucleus charge with the account of finite nucleus radius



- The analytic formula for the ground energy level of the hydrogen atom in a superstrong magnetic field is derived with the account of a finite nucleus radius.
- The effect of the rising of the ground level was found for hydrogen-like ions.
- Finite nucleus radius leads to nontrivial dependence of the critical nucleus charge on magnetic field: ions are critical only in the finite range of magnetic fields.
- Only ions with $Z > 210$ remain critical regardless of the value of magnetic field.