Non-Abelian strings and 2D-4D correspondence

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1 Introduction

Non-Abelian strings were suggested in $\mathcal{N}=2$ U(N) QCD

Hanany, Tong 2003

Auzzi, Bolognesi, Evslin, Konishi, Yung 2003

Shifman Yung 2004

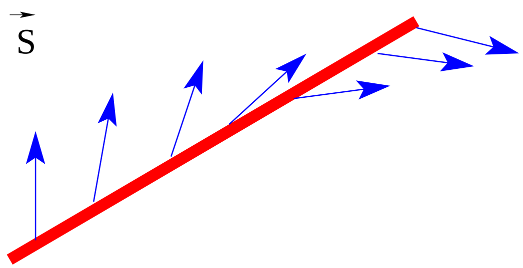
Hanany Tong 2004

Z_N Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/Z_2$

$$flux \sim \tau_3$$

Non-Abelian string : Orientational zero modes

Rotation of color flux inside SU(N).



For $U(N)$ gauge theory in the 4D bulk we have 2D $CP(N-1)$ model on the string

$\boxed{\text{2D-4D correspondence}}$

Most striking example: $\mathcal{N} = 2$ QCD

Coincidence of BPS spectra

4D U(N) $\mathcal{N} = 2$ QCD $\Longleftrightarrow$ 2D CP$(N-1)$ model
in the particular
vacuum with maximum
number of condensed
quarks, $r = N$

Observed by *Dorey 1998*

Explained by *Shifman Yung 2004* and *Hanany Tong 2004*
via non-Abelian strings

Can we generalize this 2D-4D correspondence to other r-vacua of
$\mathcal{N} = 2$ QCD?

2 r-Vacua

$\mathcal{N}=2$ QCD with gauge group $U(N)=SU(N)\times U(1)$ and
N_f flavors of fundamental matter – quarks

The field content:
U(1) gauge field A_μ
SU(N) gauge field A^a_μ, $a=1,...,N^2-1$
complex scalar fields a, and a^a
+ fermions

Complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) + fermions
$k=1,...,N$ is the color index, A is the flavor index, $A=1,...,N_f$

Mass term for the adjoint chiral field

$$\mathcal{W}_{\rm br}=\mu\,{\rm Tr}\,\Phi^2,$$

where

$$\Phi=\frac{1}{2}\,\mathcal{A}+T^a\,\mathcal{A}^a.$$

r Vacuum

First r (s)quarks condense,

F-terms in the potential

$$\left| \tilde{q}_A q^A + \sqrt{2}\, \frac{\partial \mathcal{W}_{\rm br}}{\partial \Phi} \right|^2 , \qquad \left| (\sqrt{2}\Phi + m_A) q^A \right|$$

Adjoint fields:

$$\langle \mathrm{diag}\Phi \rangle \approx -\frac{1}{\sqrt{2}} \left[m_1, ..., m_r, 0, ..., 0 \right] ,$$

For $r = N$ $U(N)$ gauge group is Higgsed

For $r < N$ classically unbroken gauge group

$$U(N-r) \qquad \rightarrow \qquad U(1)^{N-r} \qquad \rightarrow \qquad U(1)$$

adjoints $\qquad$ $(N-r-1)$ monopoles

Quark VEV's

$$\langle q^{kA} \rangle = \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \end{pmatrix},$$

$$k = 1, ..., r\,, \qquad A = 1, ..., N_f\,,$$

where for $r = N$

$$\xi_P \approx 2\,\mu m_P, \qquad P = 1, ..., N,$$

while for $r < N$

$$\xi_P \approx 2\,\mu m_P, \qquad \xi_N = 0, \qquad P = 1, ..., r.$$

Color-flavor locking

Both gauge $U(N)$ and flavor $SU(N)$ are broken, however diagonal $SU(N)_{C+F}$ is unbroken if quark masses are equal

$$\langle q \rangle \to U \langle q \rangle U^{-1}$$

3 Non-Abelian strings in $r = N$ vacuum

Example in $U(2) = U(1) \times SU(2)$

Abrikosov-Nielsen-Olesen string:

$$q|_{r\to\infty} \sim \sqrt{\xi}\, e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = 4\pi\xi$$

Non-Abelian string:

$$q|_{r\to\infty} \sim \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = 2\pi\xi$$

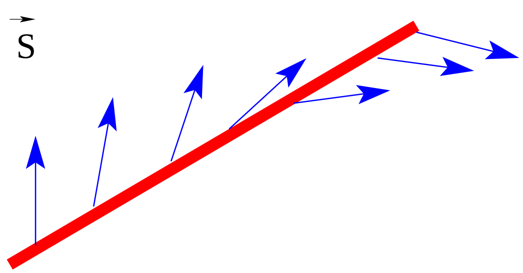
Here r and α are polar coordinates in the plane orthogonal to the string axis

String solution breaks $SU(2)_{C+F} \to$ 2 orentational zero modes.

$$\frac{SU(2)_{C+F}}{U(1)} = CP(1) = O(3)$$

We have two dimensional $O(3)$ sigma model living on the string world sheet.

$$S_{(1+1)} = \frac{\beta}{2} \int dt\, dz\, (\partial_k \vec{S})^2, \qquad \vec{S}^2 = 1$$



For $U(N)$ gauge group in the bulk we have 2D $CP(N-1)$ model on the string

$\mathrm{CP}(N-1)$ == U(1) gauge theory in the strong coupling limit

$$
\begin{aligned}
S_{\mathrm{CP}(N-1)} &= \int d^2x \left\{ \left|\nabla_\alpha n^P\right|^2 + \frac{1}{4e^2} F^2_{\alpha\beta} + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right. \\
&+ \left. |\sigma + m_P|^2 \, |n^P|^2 + \frac{e^2}{2} \left(|n^P|^2 - 2\beta \right)^2 \right\},
\end{aligned}
$$

where n^P are complex fields $P = 1, ..., N$,

Condition

$$|n^P|^2 = 2\beta \,,$$

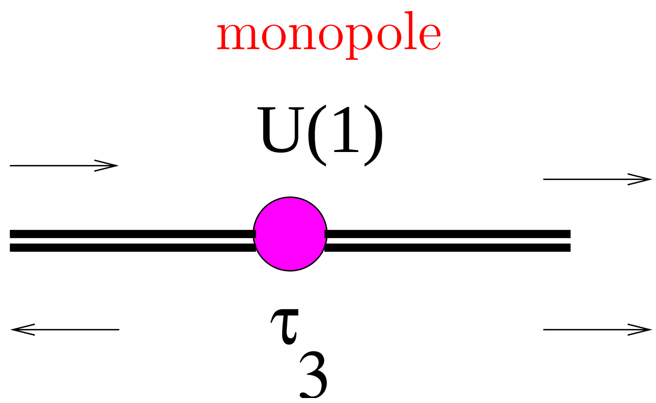
imposed in the limit $e^2 \to \infty$

4 Confined monopoles

Higgs phase for quarks $\Longrightarrow$ confinement of monopoles

Elementary monopoles – junctions of two different strings

Example in $U(2)$



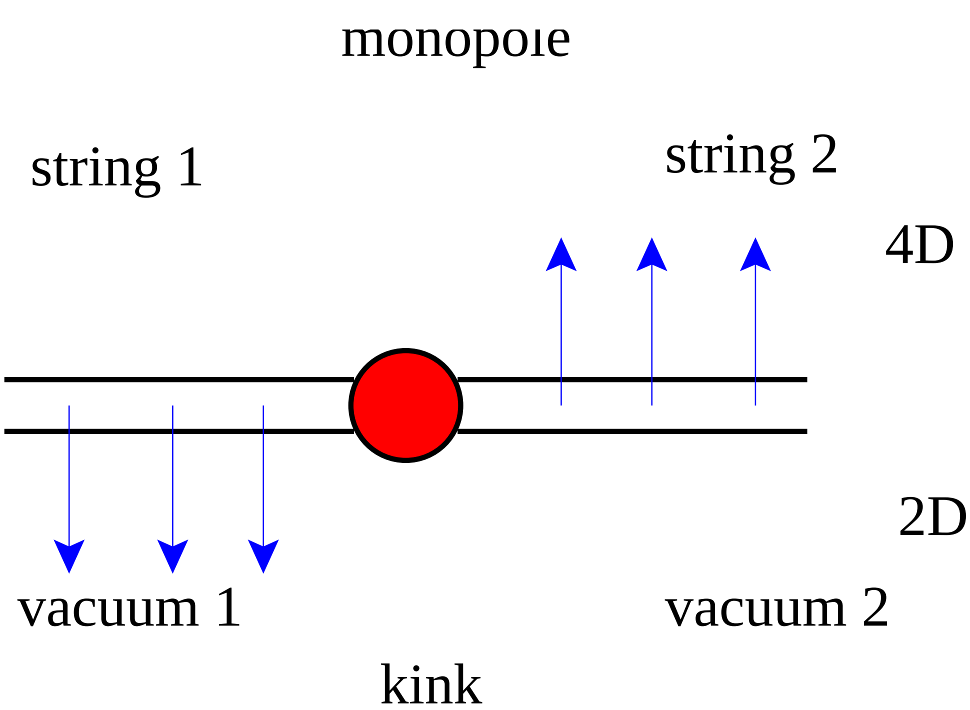
monopole flux $= 4\pi \times \mathrm{diag}\frac{1}{2}\{1, -1\}$

In 2D $CP(N-1)$ model on the string we have

N vacua $= N$ Z_N strings

and kinks interpolating between these vacua

Kinks = confined monopoles



$$M^{kink} = M^{\text{monopole}}$$

5 2D-4D correspondence in $r = N$ vacuum

Exact superpotential in CP$(N-1)$ model

$$\mathcal{W}_{\rm CP}(\sigma) = \frac{1}{4\pi}\left\{\sum_{P=1}^{N}\,(\sigma + m_P)\,\ln\frac{\sigma + m_P}{e\Lambda}\right.$$

$$\left. - \sum_{K=N+1}^{N_F}\,(\sigma + m_K)\,\ln\frac{\sigma + m_K}{e\Lambda}\right\},$$

Vacuum equation (chiral ring equation)

$$\prod_{P=1}^{N}(\sigma + m_P) = \Lambda^{(N-\tilde{N})}\prod_{K=N+1}^{N_f}(\sigma + m_K)$$

N roots $\Longrightarrow$ N vacua σ_P, $P = 1, ..., N$

Kink masses

$$M^{\rm BPS}_{PP'} = 2\left|\mathcal{W}_{\rm CP}(\sigma_{P'}) - \mathcal{W}_{\rm CP}(\sigma_P)\right| \,, \qquad P, P' = 1, ..., N$$

Compare with monopole masses

$$M^{\rm monopole}_{PP'} = \left|\frac{\sqrt{2}}{2\pi i}\oint_{\beta_{PP'}} d\lambda_{SW}\right|, \qquad P, P' = 1, ...N$$

$$M^{\rm monopole}_{PP'} = M^{\rm kink}_{PP'}, \qquad P, P' = 1, ..., N$$

6 Quantum deformation

Rewrite identically exact superpotential

$$\mathcal{W}^{\rm cl}(\sigma) = \frac{1}{4\pi}\left\{2\,{\rm Tr}\left[(\sigma-\sqrt{2}\,\Phi^{\rm cl})\,\ln\frac{\sigma-\sqrt{2}\,\Phi^{\rm cl}}{e\Lambda}\right]\right.$$

$$\left. - \quad \sum_{A=1}^{N_f}(\sigma+m_A)\,\ln\frac{\sigma+m_A}{e\Lambda}\right\},$$

where for $r=N$ vaquum

$$\left\langle {\rm diag}\Phi^{\rm cl}\right\rangle = -\frac{1}{\sqrt{2}}\left[m_1,...,m_N\right],$$

Now we propose that quantum superpotential is

$$\mathcal{W}(\sigma) = \frac{1}{4\pi}\left\{2\left\langle{\rm Tr}\left[(\sigma-\sqrt{2}\,\Phi)\,\ln\frac{\sigma-\sqrt{2}\,\Phi}{e\Lambda}\right]\right\rangle\right.$$

$$\left. - \quad \sum_{A=1}^{N_f}(\sigma+m_A)\,\ln\frac{\sigma+m_A}{e\Lambda}\right\},$$

where quantum average is taken over the bulk theory.

Calculate quantum average over the bulk theory.

Gaiotto, Gukov, Seiberg 2013: method of resolvents

$$T(\sigma) = \left\langle \mathrm{Tr}\, \frac{1}{\sigma - \sqrt{2}\,\Phi} \right\rangle$$

Cachazo, Seiberg, Witten 2003: exact solution for chiral rings in $\mathcal{N} = 1$ QCD

$$T(\sigma)_{r=N} = \sum_{P=1}^{N} \frac{1}{\sigma + m_P}$$

For $r = N$ vacuum this gives classical expression

7 World sheet theory for non-Abeliabn string in $r = N-1$ vacuum

Use Cachazo-Seiberg-Witten solution for resolvent

$$\partial_\sigma \mathcal{W}(\sigma) = \frac{1}{4\pi}\left\{\sum_{A=1}^{N-1} \ln\frac{(\sigma+m_A)}{\Lambda} - \sum_{A=N}^{N_f} \ln\frac{(\sigma+m_A)}{\Lambda}\right.$$
$$+ \quad \left.(2N-N_f)\ln\frac{t}{\Lambda} - \sum_{A=1}^{N-1}\ln\frac{t_A}{\Lambda} + \sum_{A=N}^{N_f}\ln\frac{t_A}{\Lambda}\right\},$$

where

$$t = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 - \frac{4S}{\mu}}\right), \qquad S = \frac{1}{32\pi^2}\,\langle \mathrm{Tr}\, W_\alpha W^\alpha \rangle$$

and

$$t_A = \frac{1}{2}\left(\sqrt{\sigma^2 - \frac{4S}{\mu}} + \frac{\sigma + \frac{4S}{\mu m_A}}{\sqrt{1 - \frac{4S}{\mu m_A^2}}}\right)$$

Bulk instantons produce gaugino condensate

It penetrates into 2D theory on the string and opens a cut in σ plane

8 2D-4D correspondence in $r = N - 1$ vacuum

Kink mass

$$M_{PP'}^{\rm kink} = 2\,|\mathcal{W}(\sigma_{P'}) - \mathcal{W}(\sigma_P)|$$

$$M_{PP'}^{\rm kink} = \left| \frac{1}{\pi} \int_{\sigma_P}^{\sigma_{P'}} d\sigma \left\{ \frac{2N - N_f}{2} \frac{\sigma}{\sqrt{\sigma^2 - \frac{4S}{\mu}}} \right.\right.$$
$$\left.\left. - \quad \frac{1}{2} \sum_{A=1}^{N-1} \frac{\sigma\sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}}\,(\sigma + m_A)} + \frac{1}{2} \sum_{A=N}^{N_f} \frac{\sigma\sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}}\,(\sigma + m_A)} \right\} \right|$$

Compare with monopole masses

$$M_{PP'}^{\rm monopole} = \left| \frac{\sqrt{2}}{2\pi i} \oint_{\beta_{PP'}} d\lambda_{SW} \right|, \qquad P, P' = 1, \ldots.N$$

$$M_{PP'}^{\rm monopole} = M_{PP'}^{\rm kink}, \qquad P, P' = 1, ..., N$$

9 Example in U(2)

U(2) gauge theory with $N_f = 2$

Exact formula for the kink mass in $r = 1$ vacuum $(m_1 = m_2 = m)$

$$M^{\rm kink} = \left| \frac{1}{2\pi} \left\{ m \, \ln \frac{m + \sqrt{m^2 - 4\Lambda^2}}{m - \sqrt{m^2 - 4\Lambda^2}} + 2\sqrt{m^2 - 4\Lambda^2} \right\} \right|$$

Compare with the kink mass in $r = 2$ vacuum given by CP(1) model

$$M^{\rm kink}_{r=N} = \left| \frac{1}{2\pi} \left\{ \Delta m \, \ln \frac{\Delta m + \sqrt{\Delta m^2 + 4\Lambda^2}}{\Delta m - \sqrt{\Delta m^2 + 4\Lambda^2}} - 2\sqrt{\Delta m^2 + 4\Lambda^2} \right\} \right|,$$

where $\Delta m = m_1 - m_2$

10 Conclusions

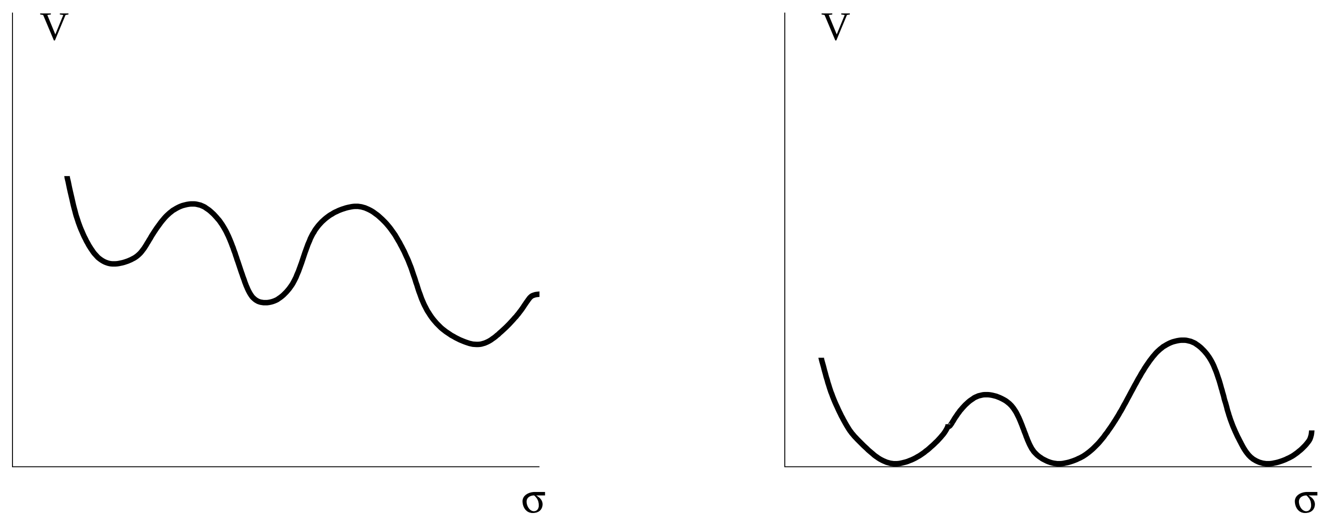
- Bulk instantons penetrates into the theory on the non-Abelian string
- 2D-4D correspondence (coincidence of BPS spectra of 2D kinks and 4D monopoles) is still valid in $r = N - 1$ vacuum

11 Classical theory on the string in $r = N-1$ vacuum

$$\begin{aligned}
S^{\rm cl}_{(2,2)} &= \int d^2x \left\{ \left|\nabla_\alpha n^P\right|^2 + \left|\nabla_\alpha n^N\right|^2 + \left|\tilde{\nabla}_\alpha \rho\right|^2 + \left|\nabla_\alpha z\right|^2 + \frac{1}{4e^2} F^2_{\alpha\beta} + \frac{1}{e^2}\left|\partial_\alpha \sigma\right|^2 \right. \\
&+ \left|\sigma + m_P\right|^2 \left|n^P\right|^2 + \left|\sigma + m_N\right|^2 |\rho|^2 + |\sigma|^2 \left|n^N\right|^2 + |\sigma|^2 |z|^2 \\
&+ \left. \frac{e^2}{2}\left(|n^P|^2 + |n^N|^2 + |z|^2 - |\rho|^2 - 2\beta\right)^2 \right\}, \\
& P = 1, ..., N-1\,.
\end{aligned}$$

The physical meaning of the n^N and z fields is related to "unwinding" of the $(N-1)$-th string into the N-th string which is, in fact, absent. The coefficient b of this WCP model is equal to the sum of the charges of all charged fields,

$$b = (N-1) - 1 + 1 + 1 = N,$$



a b

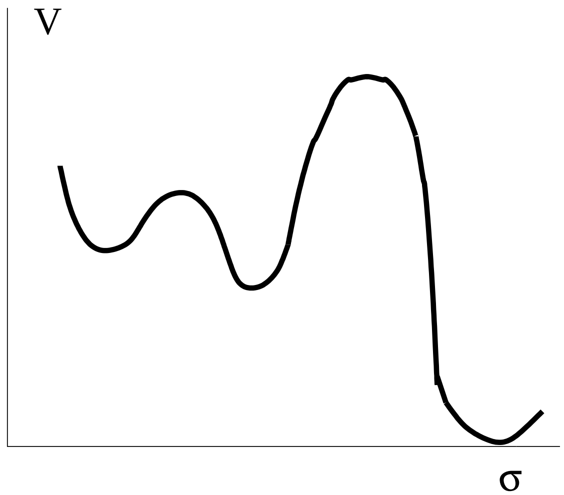
a. Schematic picture of the scalar potential in the theory. The complex variable σ is schematically represented by the horizontal axis. Minima of the potential correspond to elementary non-Abelian strings.

$$V(\sigma_P) = T_P, \qquad P = 1, ..., N$$

b. The same potential in the limit $\mu = 0$

$$V_{\rm def}(\sigma) = 4\pi\,|\mu\sigma|$$

$r = N - 1$ Vacuum



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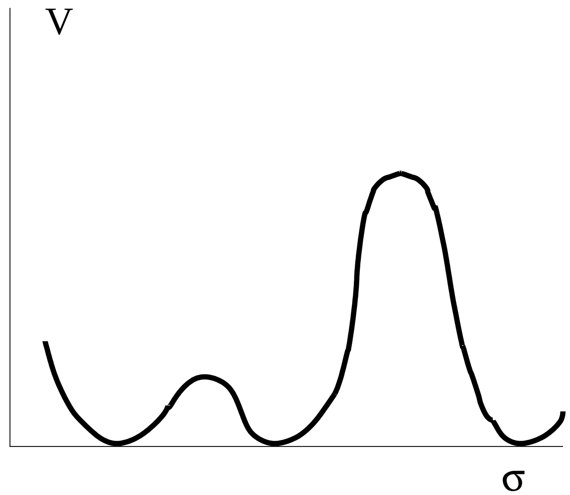

b

$$V_{\text{def}}(\sigma) = 4\pi \left| \mu \sqrt{\sigma^2 - \frac{4S}{\mu}} \right|$$

$r = N$ Vacuum, $\nu = \tilde{N}$

$$\xi_P = -2\sqrt{2}\,\mu\, e_P, \qquad P = 1, ..., N,$$

where e_P are the double roots of the Seiberg–Witten curve,

$$y^2 = \prod_{P=1}^{N} (x - \phi_P)^2 - 4\left(\frac{\Lambda}{\sqrt{2}}\right)^{N-\tilde{N}} \prod_{A=1}^{N_f}\left(x + \frac{m_A}{\sqrt{2}}\right) = \prod_{P=1}^{N}(x - e_P)^2$$

At small masses the double roots of the Seiberg–Witten curve are

$$\sqrt{2}e_I = -m_{I+N}, \qquad \sqrt{2}e_J = \Lambda_{\mathcal{N}=2}\,\exp\left(\frac{2\pi i}{N-\tilde{N}}J\right)$$

for $\tilde{N} < N - 1$, where

$$I = 1, ..., \tilde{N} \;\text{ and }\; J = \tilde{N}+1, ..., N\,.$$

The $\tilde{N}$ first roots are determined by the masses of the last $\tilde{N}$ quarks — a reflection of the fact that the non-Abelian sector of the dual theory is not asymptotically free and is at weak coupling in the domain.

$r < N$ Vacuum, $\nu = N_f - r$

$$\xi_P = -2\sqrt{2}\mu\sqrt{e_P^2 - e_N^2}, \qquad P = 1, ..., r$$

Seiberg-Witten curve

$$y^2 = \prod_{P=1}^{N-1} (x - e_P)^2 \, (x - e_N^+)(x - e_N^-), \qquad e_N^+ + e_N^- = 0.$$

Cachazo, Seiberg, Witten, 2003:

$$e_N^2 = \frac{2S}{\mu}, \qquad S = \frac{1}{32\pi^2} \left\langle \mathrm{Tr}\, W_\alpha W^\alpha \right\rangle .$$

First ν roots at small masses

$$\sqrt{2}e_I = -m_{I+r}, \qquad I = 1, ..., \nu\,. \qquad e_J \sim \Lambda_{\mathcal{N}=2}, \qquad J = \nu + 1, ..., r$$