

Semiclassical description of particle production in external fields.

The force of the “**Worldline Instanton**” approach

Petr Satunin



INR

Institute for Nuclear Research
of the Russian Academy of Sciences



06.06.2014, Suzdal

Outline

- Idea of the method
- The Schwinger effect — idea of *Affleck et.al '82* calculation
- Photon decay $\gamma \rightarrow e^+e^-$ in a weak magnetic field
 - Sensitivity to Lorentz Invariance Violation *P.S. '13*
- Neutrino $\nu \rightarrow W^+e^-$ in external magnetic field. The regime of exponential suppression
- Particle production in de Sitter spacetime *Guts '13*

An idea of "Worldline instanton" method

- Particle production in a certain external field ϕ_{ext}
$$\Gamma = \text{Im}F \left[\int_{p.b.c} Dx_\mu e^{-S[x_\mu, \phi_{ext}]} \right]$$
- Saddle point approximation. E.o.m.: $\frac{\delta S}{\delta x_\mu} \Big|_{x_\mu^{cl}} = 0 + \text{periodical b.c.}$
- x_μ^{cl} — closed trajectory. $\Gamma = \text{Im}F \left[A e^{-S[x_\mu^{cl}]} \right]$
- Fluctuations near classical solution $\delta x_\mu = x_\mu - x_\mu^{cl}$.
Integral over $\delta x_\mu \rightarrow$ prefactor
- 2nd variation of the action $\delta^2 S[\delta x_\mu]$ have odd number of negative modes $\rightarrow i$ in the prefactor \rightarrow particle production

Affleck, Alvarez, Manton, '82; Selivanov, Voloshin '85; Dunne, Shubert '05, '06; Monin '05; Monin, Voloshin '10 etc..

The Schwinger effect in WI formulation

Affleck, Alvarez, Manton, 1982

$$\langle 0|0 \rangle = Z = \int D\phi^* D\phi DA_\mu e^{-S_E[\phi^*, \phi, A_\mu^{ext}, A_\mu]} = e^{i\Gamma}.$$

$$P_{p,p} = 1 - |\langle 0|0 \rangle|^2 = 1 - e^{-2\text{Im}\Gamma} \approx 2\text{Im}\Gamma.$$

$$\begin{aligned} Z[A_\mu] &= \int D\phi^* D\phi e^{-\int d^4x(|D_\mu\phi|^2 + m^2|\phi|^2)} = \\ &= \det(-D_\mu^2 + m^2) = \exp \text{Tr} \ln (-D_\mu^2 + m^2) . \end{aligned}$$

$$Z[A_\mu] = Z_0 \exp \left[- \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{Tr} \left(e^{TD_\mu^2} \right) \right].$$

Operator $(-D_\mu^2)$ can be interpreted as QM Hamiltonian. Go to [Lagrange formalism](#)

$$Z[A_\mu] = Z_0 \exp \left[- \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \frac{1}{(2\pi)^2 N} \int_{p.b.c} Dx_\mu e^{-\int_0^T d\tau \left(\frac{\dot{x}_\mu^2}{4} + ie\dot{x}_\mu A_\mu \right)} \right].$$

The Schwinger effect in WI formulation

Affleck, Alvarez, Manton, 1982

$$P_{pp} \approx \int Dx_\mu \exp \left(-m \sqrt{\oint d\tau \dot{x}_\mu^2} + ie \oint A_\mu dx_\mu \right)$$

Saddle point equation:

$$\frac{m \ddot{x}_\mu}{\sqrt{\oint d\tau \dot{x}_\mu^2}} = -e F_{\mu\nu} \dot{x}_\nu$$

Solution — particle propagates on a circle:

$$x_\mu^{cl} = \frac{m}{eE} (\sin(2\pi\tau), 0, 0, \cos(2\pi\tau))$$

$$\Gamma = \frac{(eE)^2}{(2\pi)^3} e^{-\frac{\pi m^2}{eE}}$$

Photon decay in magnetic field

- The Optical Theorem: $\Gamma = \frac{1}{2\omega} \epsilon_\mu(k) \epsilon_\nu(k) \text{Im} \Pi_{\mu\nu}(k)$
- Scalar electrons: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi$
- $j_\mu = -ie(\phi^* D_\mu \phi - \phi D_\mu \phi^*)$,
- $\Pi_{\mu\nu}(k) = \int d^4y e^{iky} \langle j_\mu(y/2) j_\nu(-y/2) \rangle$.
- $\langle j_\mu(y/2) j_\nu(-y/2) \rangle = \frac{1}{Z[A_\mu]} \frac{\delta}{i\delta A_\mu(y/2)} \frac{\delta}{i\delta A_\nu(-y/2)} Z[A_\mu]$,
- $Z[A_\mu] = \int D\phi^* D\phi e^{-\int d^4x (|D_\mu \phi|^2 + m^2 |\phi|^2)} = \det(-D_\mu^2 + m^2) = \exp \text{Tr} \ln(-D_\mu^2 + m^2)$
- $\frac{\delta}{\delta A_\mu(y)} \rightarrow$ insertion $\oint d\tau \dot{x}_\mu(\tau) \delta(x(\tau) - y)$ to the path integral.

$$\begin{aligned} \langle j_\mu(y/2) j_\nu(-y/2) \rangle &\propto \int_0^\infty \frac{dT}{T^3} \frac{1}{N} \int_{p.b.c.} Dx_\mu \oint d\tau_1 \oint d\tau_2 \dot{x}_\mu(\tau_1) \dot{x}_\nu(\tau_2) \times \\ &\quad \times \delta(x(\tau_1) - y/2) \delta(x(\tau_2) + y/2) e^{-m^2 T - \int_0^1 d\tau \left(\frac{\dot{x}_\mu^2}{4T} + i\epsilon \dot{x}_\mu A_\mu \right)}. \end{aligned}$$

Photon decay in magnetic field

$$\text{Im } \Pi_{\mu\nu}(k) \propto \text{Im} \int_0^\infty \frac{dT}{T^3} \frac{1}{N} \int_{p.b.c.} Dx_\mu \oint d\tau_1 \oint d\tau_2 \dot{x}_\mu(\tau_1) \dot{x}_\nu(\tau_2) \delta(x(\tau_1) + x(\tau_2)) e^{-S_m[x_\mu; \tau_1, \tau_2]},$$

where $S_m[x_\mu; \tau_1, \tau_2] = m^2 T + \int_0^1 d\tau \left(\frac{\dot{x}_\mu^2}{4T} + ieA_\mu \dot{x}_\mu \right) - ik_\mu (x_\mu(\tau_1) - x_\mu(\tau_2))$. is the action of a charged particle in 4d space + 1d time with p.b.c.

Integrals over x_μ , T , ($\delta\tau \equiv \tau_1 - \tau_2$) at **saddle point**.

Saddle point equations:

- x_μ : $\frac{\ddot{x}_0}{2T} = \omega (\delta(\tau - \tau_1) - \delta(\tau - \tau_2)),$
 $\frac{\ddot{x}_i}{2T} - ieF_{ij}\dot{x}_j = -i\omega\delta_{i2} (\delta(\tau - \tau_1) - \delta(\tau - \tau_2)).$
- T : $m^2 - \frac{\int_0^1 d\tau \dot{x}_\mu^2}{4T^2} = 0.$
- $\delta\tau$: $k_\mu \dot{x}_\mu (\Delta\tau) = 0$

Photon decay in magnetic field

The solution of saddle point equations — 2 arcs of hyperbolas:

$$0 < \tau < \frac{1}{2} :$$

$$x_0^{cl} = \omega T \left(\tau - \frac{1}{4} \right), x_1^{cl} = 0,$$

$$x_2^{cl} = -i \text{Ash} \left(\eta \left(\tau - \frac{1}{4} \right) \right),$$

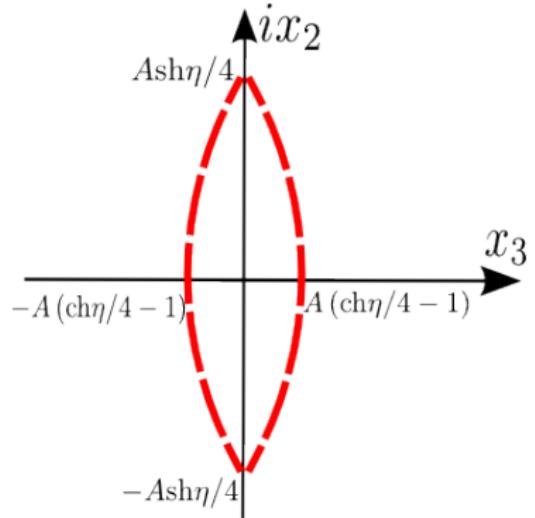
$$x_3^{cl} = -A \left[\text{ch} \left(\eta \left(\tau - \frac{1}{4} \right) \right) - \text{ch} \frac{\eta}{4} \right];$$

$$\frac{1}{2} < \tau < 1 :$$

$$x_0^{cl} = -\omega T \left(\tau - \frac{3}{4} \right), x_1^{cl} = 0,$$

$$x_2^{cl} = i \text{Ash} \left(\eta \left(\tau - \frac{3}{4} \right) \right),$$

$$x_3^{cl} = A \left[\text{ch} \left(\eta \left(\tau - \frac{3}{4} \right) \right) - \text{ch} \frac{\eta}{4} \right].$$



$$\text{Here } A = \frac{\omega}{2eH \text{ch} \frac{\eta}{4}}, \quad \eta = 2TeH,$$

$$T = \frac{4m}{\omega eH}$$

$$S[x_{cl}] = \frac{8m^3}{3\omega eH} \gg 1, \quad \Gamma \propto e^{-S[x_{cl}]}$$

Photon decay in magnetic field. Prefactor.

- $\Gamma \propto \text{Im} \Pi_{\mu\nu}$.
- Integrate fluctuations over classical trajectory $\delta x_\mu \equiv x_\mu - x_\mu^{cl}$ etc.
- Single negative mode: $\delta x_\mu^{-1} \propto x_\mu^{cl}$.
- $\Gamma \propto \alpha_{em} \frac{eH}{m} e^{-S[x_{cl}]} = \alpha_{em} \frac{eH}{m} e^{-\frac{8m^3}{3\omega_e H}}$
- Coincides with

Klepikov '54

Photon decay in magnetic field

Advantages

- Clear geometrical interpretation
- Simplicity of the calculations
- Simple generalization to exotic models

Disadvantages

- **Only** the case of exponential suppression
- A bit more complicated calculations for fermions

Photon decay in magnetic field

Sensitivity to Lorentz Invariance violation

P.S., arXiv:1301.5707 [hep-th], PhysRevD.87.105015

QED with Lorentz Invariance Violation

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\varkappa \bar{\psi} \gamma^i D_i \psi + \frac{ig}{M^2} D_j \bar{\psi} \gamma^i D_i D_j \psi + \frac{\xi}{4M^2} F_{kj} \partial_i^2 F^{kj}$$

$$\omega_{LV} = -\varkappa k + \left(\frac{\xi}{2} - g\right) \frac{k^3}{M^2}$$

$$\Gamma \propto \exp \left[-\frac{8m^3}{3\omega e H} \left(1 - \frac{\omega \cdot \omega_{LV}}{2m^2} \right)^{3/2} \right]$$

- Large negative ω_{LV} → strong suppression of the process
- Positive ω_{LV} → Vacuum photon decay Rubtsov, P.S., Sibiryakov, 12
- We can constrain LIV from possible detection of photons in cosmic rays (energies approx. 10^{20} eV)

"Neutrino" decay $\nu \rightarrow W^+ e^-$

Decay of a massless particle to two with different masses

Toy model with scalar particles

ϕ — scalar electron, χ — scalar W, ξ — scalar neutrino

$$\mathcal{L} = D_\mu \phi^* D_\mu \phi - m_e^2 \phi^* \phi + D_\mu \chi^* D_\mu \chi - m_W^2 \chi^* \chi + \frac{1}{2} (\partial_\mu \xi)^2 + ig \xi \chi^* \phi + h.c.$$

$$\Sigma^\xi(y, z) = g^2 \langle \chi^*(y) \phi(y) \phi^*(z) \chi^*(z) \rangle$$

$$\langle \phi^*(z) \phi(y) \rangle = \langle z \left| \frac{1}{D_\mu^2 + m_e^2} \right| y \rangle = \int_0^\infty dT e^{-m_e^2 T} \langle z | e^{-D_\mu^2 T} | y \rangle.$$

$$\Sigma^\xi \propto \int_0^\infty \frac{dT_1}{T_1^2} \int_0^\infty \frac{dT_1}{T_1^2} \int_y^z Dx \int_z^y Dx e^{-S[x_\mu]}$$

$$S[x_\mu] = m_e^2 T_1 + \int_0^{T_1} \frac{\dot{x}_\mu^2}{4} d\tau - ie \int_0^{T_1} A_\mu \dot{x}_\mu d\tau + m_W^2 T_2 + \int_0^{T_2} \frac{\dot{x}_\mu^2}{4} d\tau - ie \int_0^{T_2} A_\mu \dot{x}_\mu d\tau.$$

Neutrino decay $\nu \rightarrow W^+ e^-$

Two particles with different masses \rightarrow two arcs with different curvatures

$$x_0^L = A_0 \left(\tau - \frac{1}{4} \right),$$

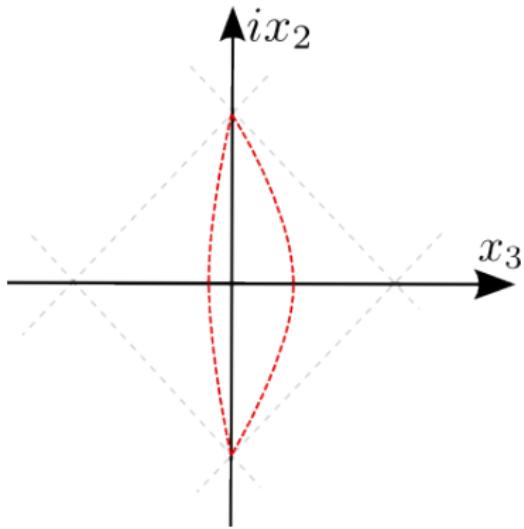
$$x_2^L = i A_L \operatorname{sh} \left(4\theta_1 \left(\tau - \frac{1}{4} \right) \right),$$

$$x_3^L = A_L \left[\operatorname{ch} \left(4\theta_1 \left(\tau - \frac{1}{4} \right) \right) - \operatorname{ch} \theta_1 \right].$$

$$x_0^R = -A_0 \left(\tau - \frac{3}{4} \right),$$

$$x_2^R = i A_R \operatorname{sh} \left(4\theta_2 \left(\tau - \frac{3}{4} \right) \right),$$

$$x_3^R = A_R \left[\operatorname{ch} \left(4\theta_2 \left(\tau - \frac{3}{4} \right) \right) - \operatorname{ch} \theta_2 \right].$$



$$\frac{\sin^2 \theta_2}{\sin^2(\theta_1 + \theta_2)} = \frac{\theta_2^2}{(\theta_1 + \theta_2)^2} - \frac{m_e^2}{\omega^2},$$

$$\frac{\sin^2 \theta_1}{\sin^2(\theta_1 + \theta_2)} = \frac{\theta_1^2}{(\theta_1 + \theta_2)^2} - \frac{m_W^2}{\omega^2}.$$

$$\theta_i = T_i eH, \quad A_0 = \frac{4\omega}{eH} \cdot \frac{\theta_1 \theta_2}{\theta_1 + \theta_2},$$

$$A_L = \frac{\omega}{eH} \cdot \frac{\operatorname{sh}\theta_2}{\operatorname{sh}(\theta_1 + \theta_2)}, \quad A_R = -\frac{\omega}{eH} \cdot \frac{\operatorname{sh}\theta_1}{\operatorname{sh}(\theta_1 + \theta_2)}.$$

Neutrino decay $\nu \rightarrow W^+ e^-$

$$\frac{\sinh^2 \theta_2}{\sinh^2(\theta_1 + \theta_2)} = \frac{\theta_2^2}{(\theta_1 + \theta_2)^2} - \frac{m_e^2}{\omega^2},$$

$$\frac{\sinh^2 \theta_1}{\sinh^2(\theta_1 + \theta_2)} = \frac{\theta_1^2}{(\theta_1 + \theta_2)^2} - \frac{m_W^2}{\omega^2}.$$

We work **in the limit** $\theta_2 \ll \theta_1 \ll 1$.

$$\theta_1 = \frac{\sqrt{3}}{2} \frac{m_W^2}{m_e \omega}, \quad \theta_2 = \sqrt{3} \frac{m_e}{\omega}$$

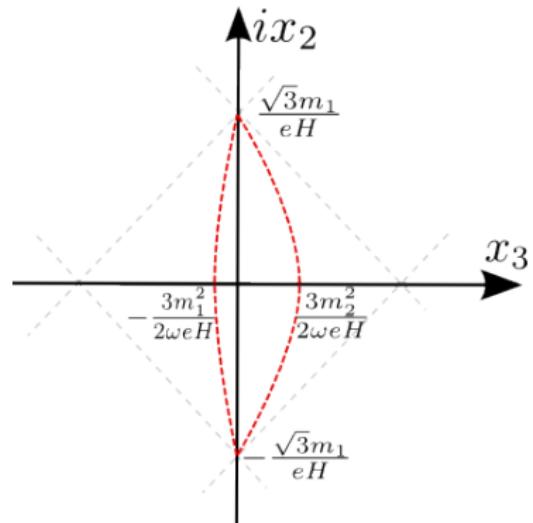
$$\Gamma \propto e^{-\frac{\sqrt{3} m_e m_W^2}{\omega e H}}.$$

Limits of applicability:

- $\frac{\sqrt{3} m_e m_W^2}{\omega e H} \gg 1$
- $\omega \gg \frac{m_W^2}{m_e}$

only subcritical magnetic field

$$H \ll H_{cr}$$



Coincides with

Borisov et.al. '85 Erdas, Lissia '03

Neutrino decay $\nu \rightarrow W^+ e^-$

Another limit $\theta_2 \ll 1, \theta_1 \gg 1$:

$$\theta_1 = \frac{m_W^2}{2\omega m_e}, \quad \theta_2 = \frac{m_W^2}{2\omega^2}$$

$$\Gamma \propto e^{-\frac{m_W^4}{4\omega^2 e H}}.$$

coincides with

Mikheev, Kuznetsov, Serghienko '12

Limits of applicability:

- $\frac{m_W^4}{4\omega^2 e H} \gg 1$
- $\omega \ll \frac{m_W^2}{m_e}$
- $H \gg H_{cr}$

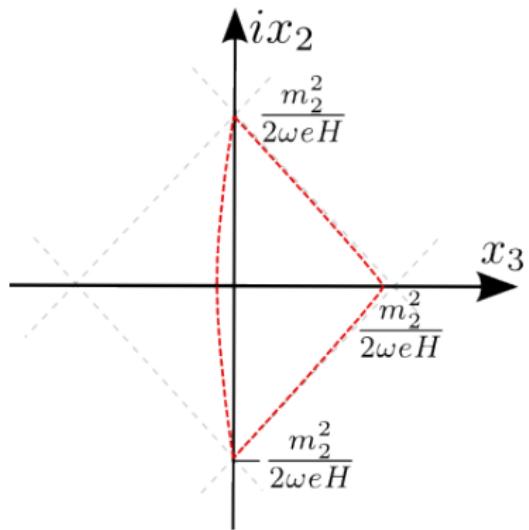


Figure: Electron trajectory \rightarrow lightcone

Prefactor does not coincide due to g has a mass dimension. Vector neutrino + scalars \rightarrow dimensionless $g \rightarrow$ OK

Pair production in de Sitter spacetime

Sergey Guts, arXiv:1312.2429 [hep-ph]

Euclidean de Sitter \sim sphere

$$\begin{cases} ds^2 = dX_{d+1}^2 - \sum_i dX_i^2 \\ X_{d+1}^2 - \sum_i X_i^2 = -R^2 \end{cases} \Rightarrow \begin{cases} ds^2 = -dX_{d+1}^2 - \sum_i dX_i^2 \\ X_{d+1}^2 + \sum_i X_i^2 = +R^2 \end{cases} \quad i = \overline{1, d}$$

Scalar field in dS

$$\Gamma_{\text{Eucl}} = \text{Im} \int_0^\infty \frac{dT}{T} e^{-\tilde{m}^2 T} \int_{x(0)=x(1)} \mathcal{D}x(\tau) \exp \left[- \int_0^1 d\tau \frac{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}{4T} \right].$$

Pair production in de Sitter spacetime

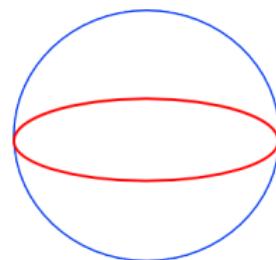
Sergey Guts, arXiv:1312.2429 [hep-ph]

2D de Sitter spacetime

Classical solution

$$\varphi = 2\pi\tau, \theta = \pi/2$$

"Equatorial instanton"



Negative mode — translations at the meridional direction

$$\Gamma_{\text{Eucl}} = \frac{4\tilde{m}V}{\pi r} e^{-2\pi\tilde{m}r}.$$

Some open questions...

Thank you for your attention!