Stable Exact Cosmological Solutions in Induced Gravity Models

E.O. Pozdeeva

SINP MSU

based on E.O. Pozdeeva, S.Yu. Vernov, arXiv:1401.7550 [gr-qc]

QUARKS 2014

The Lyapunov stability

We study the stability of solutions that tend to fixed points for the induced gravity models in FRWL metric.

Lyapunov stability of fixed points for a general system of the first order autonomic equations

$$\dot{y}_k = F_k(y), \qquad k = 1, 2, \ldots, N. \tag{1}$$

We assume that $y(t)$ tends to a fixed point y_f. If all solutions of the dynamical system that start out near a fixed (equilibrium) point y_f,

$$F_k(y_f) = 0, \qquad k = 1, 2, \ldots, N \tag{2}$$

stay near y_f forever, then y_f is *a Lyapunov stable point*.
If all solutions that start out near the equilibrium point y_f converge to y_f, then the fixed point y_f is *an asymptotically stable* one.

A.M. Lyapunov, *Stability of motion*, Academic Press, New-York and London, 1966 (in English); A.M. Lyapunov, *General problem of stability of motion*, GITTL, Moscow–Leningrad, 1950 (in Russian)
L.S. Pontryagin, *Ordinary Differential Equations*, Adiwes International Series in Mathematics. Addison-Wesley Publ. Comp., London–Paris, 1962 (in English), "Nauka", Moscow, 1982 (in Russian)

The Lyapunov theorem states that to prove the stability of fixed point y_f of nonlinear system (1) it is sufficient to prove the stability of this fixed point for the corresponding linearized system (y is a column):

$$\dot{y} = Ay, \qquad A_{ik} = \frac{\partial F_i(y)}{\partial y_k}\Big|_{y=y_f}. \tag{3}$$

The stability of the linear system means that real parts of all roots λ_k of the characteristic equation

$$\det\left(A - \lambda I\right)\big|_{y=y_f} = 0 \tag{4}$$

are negative. Here I is the identity matrix.
If at least one root of (4) is positive, then the fixed point is unstable.

Cosmological models with non-minimally coupled scalar fields

The models with the Ricci scalar multiplied by a function of the scalar field can be described generally such as

$$S = \int d^4x \sqrt{-g}\left[U(\phi)R - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right], \qquad (5)$$

where $U(\phi)$ and $V(\phi)$ are differentiable functions of the scalar field ϕ, g is the determinant of the metric tensor $g_{\mu\nu}$, R the scalar curvature. We use the signature $(-,+,+,+)$.

Friedmann equations

Let us consider the spatially flat FLRW universe with the interval

$$ds^2 = \, - \, dt^2 + a^2(t) \left(dx_1^2 + dx_2^2 + dx_3^2\right) .$$

The Friedmann equations derived by variation of action (5) have the following form:

$$6UH^2 + 6\dot{U}H = \frac{1}{2}\dot{\phi}^2 + V, \tag{6}$$

$$2U\left(2\dot{H} + 3H^2\right) + 4\dot{U}H + 2\ddot{U} = \, - \, \frac{1}{2}\dot{\phi}^2 + V, \tag{7}$$

where the Hubble parameter is the logarithmic derivative of the scale factor: $H = \dot{a}/a$.
The variation of the initial action (5) with respect to ϕ gives

$$\ddot{\phi} + 3H\dot{\phi} + V' = 6\left(\dot{H} + 2H^2\right) U' \, , \tag{8}$$

where the prime indicates the derivative with respect to the scalar field ϕ. Combining early obtained Friedman eqs. (6) and (7) we obtain

$$4U\dot{H} - 2\dot{U}H + 2\ddot{U} + \dot{\phi}^2 = 0. \tag{9}$$

From Eqs. (6)–(9), one can get the following system of the first order equations:

$$\begin{aligned}
\dot{\phi} &= \psi, \\
\dot{\psi} &= -3H\psi - \frac{\left[(6U''+1)\psi^2 - 4V\right]U' + 2UV'}{2\left(3U'^2+U\right)}, \\
\dot{H} &= -\frac{2U''+1}{4\left(3U'^2+U\right)}\psi^2 + \frac{2U'}{3U'^2+U}H\psi - \frac{6U'^2}{3U'^2+U}H^2 + \frac{U'V'}{2\left(3U'^2+U\right)}.
\end{aligned} \tag{10}$$

I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. Vernov, Class. Quant. Grav. **31** (2014) 065007 (arXiv:1206.2801)

De Sitter solutions

Substituting a constant values $H = H_f$, $\phi = \phi_f$ in Eqs. (6) and (8) we get:

$$6U(\phi_f)H_f^2 = V(\phi_f), \tag{11}$$

$$V'(\phi_f) = 12H_f^2U'(\phi_f). \tag{12}$$

From (11) we have

$$H_f^2 = \frac{V(\phi_f)}{6U(\phi_f)}, \tag{13}$$

therefore,

$$2\frac{U'(\phi_f)}{U(\phi_f)} = \frac{V'(\phi_f)}{V(\phi_f)}. \tag{14}$$

Stable and unstable de Sitter solutions

To consider the stability of the fixed point we use the Lyapunov theorem and consider the corresponding linearize system. Supposing that

$$\phi(t) = \phi_f + \varepsilon\phi_1(t), \qquad \psi(t) = \varepsilon\psi_1(t), \qquad H(t) = H_f + \varepsilon H_1(t),$$

$$\begin{aligned} U &= U_f + \varepsilon U_0'\phi_1(t), \quad U' = U_f' + \varepsilon U_f''\phi_1, \quad U'' = U_f'' + \varepsilon U_f'''\phi_1 \\ V &= V_f + \varepsilon V_f'\phi_1(t), \quad V' = V_f' + \varepsilon V_f''\phi_1, \quad V'' = V_f'' + \varepsilon V_f'''\phi_1 \end{aligned}$$

and substituting it to (10) we obtain the following linear system:

$$\begin{aligned} \dot{\phi}_1 &= \psi_1, \\ \dot{\psi}_1 &= -3H_f\psi_1 + \frac{V_f'U_f' + 2V_fU_f'' - U_fV_f''}{3(U_f')^2 + U_f}\phi_1, \\ \dot{H}_1 &= \frac{(U_f'V_f'' - V_f'U_f'')\phi_1 + 4H_fU_f'\psi_1 - 24H_f(U_f')^2H_1}{2(3(U_f')^2 + U_f)}. \end{aligned} \tag{15}$$

For the case of a generic $U(\phi)$ we get the following matrix A, determined system (15):

$$A = \left\| \begin{array}{ccc} 0 & 1 & 0 \\ \dfrac{V_f' U_f' + 2V_f U_f'' - U_f V_f''}{3(U_f')^2 + U_f} & -3H_f & 0 \\ \dfrac{U_f' V_f'' - V_f' U_f''}{2(3(U_f')^2 + U_f)} & -\dfrac{2H_f U_f'}{3(U_f')^2 + U_f} & -\dfrac{12H_f (U_f')^2}{3(U_f')^2 + U_f} \end{array} \right\|$$

and the corresponding characteristic equation

$$\det(A - \lambda I) = \left(-\frac{12H_f U_f'}{3(U_f')^2 + U_f} - \lambda \right) \left(\lambda(3H_f + \lambda) - \frac{V_f' U_f' + 2V_f U_f'' - U_f V_f''}{3(U_f')^2 + U_f} \right)$$

has the following roots:

$$\lambda_{1,2} = -\frac{3H_f}{2} \pm \sqrt{\frac{9H_f^2}{4} + \frac{V_f' U_f' + 2V_f U_f'' - U_f V_f''}{3(U_f')^2 + U_f}}, \qquad (16)$$

$$\lambda_3 = -\frac{12H_f U_f'}{3(U_f')^2 + U_f}. \qquad (17)$$

For the case of induced gravity $U(\phi) = \xi\phi^2$ the reconstruction procedure has been proposed in
A.Yu. Kamenshchik, A. Tronconi, G. Venturi, *Reconstruction of scalar potentials in induced gravity and cosmology*,
Phys. Lett. B **702** (2011) 191–196, arXiv:1104.2125.
They got a lot of potential for different types of the Hubble behaviors.
They start from the explicit function H(t),
solve only **LINEAR** differential equations and
construct the potentials reproducing interesting cosmological evolutions.
We use another reconstruction procedure and do not assume $H(t)$.
We assume the explicit form of $H = Y(\phi)$.
The two methods supplement each other and together allow one to construct different cosmological models with some required properties.

Let $H = Y(\phi)$ and the function $\mathcal{F}(\phi)$ is defined as follows

$$\dot{\phi} = \mathcal{F}(\phi). \tag{18}$$

Substituting $\dot{\phi}$ and $\ddot{\phi} = \mathcal{F}'\mathcal{F}$ into Eq. (9), one obtains the following equation:

$$4UY' + 2(\mathcal{F}' - Y)U' + (2U'' + 1)\,\mathcal{F} = 0. \tag{19}$$

The potential $V(\phi)$ can be obtained from (6):

$$V(\phi) = 6UY^2 + 6U'\mathcal{F}Y - \frac{1}{2}\mathcal{F}^2. \tag{20}$$

To find the function $\phi(t)$ and, hence, $H(t) = Y(\phi(t))$ we integrate Eq. (18).

A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.Yu. Vernov, *Reconstruction of Scalar Potentials in Modified Gravity Models*, Phys. Rev. D **87** (2013) 063503 (arXiv:1211.6272)

In terms of functions $\mathcal{F}$ and Y we get the following conditions for fixed point:

$$\mathcal{F}(\phi_f) = 0, \qquad V(\phi_f) = 6U(\phi_f)Y(\phi_f)^2, \qquad V'(\phi_f) = 12U'(\phi_f)Y(\phi_f)^2.$$

In this paper, we are interested in the induced gravity models with

$$U(\phi) = \frac{\xi}{2}\phi^2\,, \tag{21}$$

where ξ is the non-minimal coupling constant.
Equations (6)–(8) for such choice of the function $U(\phi)$ look as follows:

$$H^2 = \frac{V}{3\xi\phi^2} + \frac{1}{6\xi}\left(\frac{\dot\phi}{\phi}\right)^2 - 2H\frac{\dot\phi}{\phi}, \tag{22}$$

$$3H^2 + 2\dot H = \, -2\frac{\ddot\phi}{\phi} - 4H\frac{\dot\phi}{\phi} - \frac{4\xi+1}{2\xi}\left(\frac{\dot\phi}{\phi}\right)^2 + \frac{V}{\xi\phi^2}, \tag{23}$$

$$\ddot\phi + 3H\dot\phi + V' - 6\xi\phi\left(2H^2 + \dot H\right) = 0, \tag{24}$$

And the first order differential equations system (10) has the form:

$$
\begin{aligned}
\dot{\phi} &= \psi, \\
\dot{\psi} &= -3H\psi - \frac{\psi^2}{\phi} + \frac{1}{(1+6\xi)\phi}[4V(\phi) - \phi V'(\phi)], \\
\dot{H} &= \frac{4H\psi}{(1+6\xi)\phi} + \frac{V'(\phi)}{(1+6\xi)\phi} - \frac{12\xi}{1+6\xi}H^2 - \frac{1+2\xi}{2\xi(1+6\xi)}\left(\frac{\psi}{\phi}\right)^2.
\end{aligned}
\tag{25}
$$

We consider case $\xi \neq -1/6$.
For induced gravity models we get from equation (19) that

$$
\mathcal{F}(\phi) = \phi^{-1-1/(2\xi)}\left[B_1 - \int \phi^{(2\xi+1)/(2\xi)}\left(Y'\phi - Y\right)d\phi\right], \tag{26}
$$

where B_1 is an arbitrary constant.

Let $Y(\phi)$ is a generic quadratic polynomial

$$H = Y(\phi) = C_0 + C_1\phi + C_2\phi^2, \qquad (27)$$

where C_0, C_1, and C_2 are arbitrary constants, but $C_2 \neq 0$. From (26) we obtain

$$\mathcal{F}(\phi) = \frac{2\left((8\xi+1)C_0 - (4\xi+1)C_2\phi^2\right)\xi\phi}{(4\xi+1)(8\xi+1)} + B\phi^{-(1+2\xi)/(2\xi)}, \qquad (28)$$

where B is an arbitrary constant. Note that the function $\mathcal{F}(\phi)$ does not depend on C_1. We assume that $\xi \neq -1/4$ and $\xi \neq -1/8$.
When $B = 0$, the function $\mathcal{F}(\phi)$ is a cubic polynomial and the general solution for Eq. (18) can be written in terms of elementary functions.

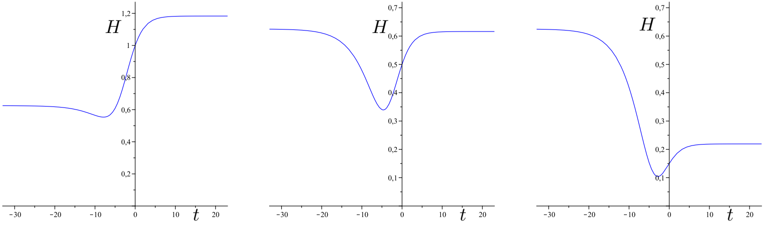

Figure: Non-monotonic functions $H(t)$, given by (27) for $\phi = \phi_+$. The values of parameter are $\xi = 1$, $C_2 = 7/2$, and $C_0 = 5/8$, and $t_0 = 2\ln(8/9)$. The parameter $C_1 = -1$ (left), $C_1 = -2$ (middle), $C_1 = -2.7$ (right).

For $C_0 \neq 0$ we obtain

$$\phi_{\pm}(t) = \pm\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(8\xi+1)C_0 e^{-\omega(t-t_0)} + (4\xi+1)C_2}}, \qquad (29)$$

where $\omega = 4\xi C_0/(4\xi+1)$, t_0 is an arbitrary integration constant. At $C_0 = 0$ the general solution for Eq. (18) is

$$\tilde{\phi}_{\pm}(t) = \pm\frac{\sqrt{8\xi+1}}{\sqrt{4\xi C_2(t-t_0)}}. \qquad (30)$$

Solutions that correspond to $\mathcal{F}(\phi) = 0$ are

$$\phi_{f_0} = 0, \qquad \phi_{f_\pm} = \pm\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(4\xi+1)C_2}}, \qquad (31)$$

Note that ϕ_{f_0} is a singular point for system (25).

For de Sitter solutions with $\phi_{f_\pm}$ the value of the de Sitter Hubble parameter is

$$H_f = C_0 + C_1\phi_f + C_2\phi_f^2.$$

The corresponding potential is the sixth degree polynomial:

$$\begin{aligned} V(\phi) = {} & \frac{(16\xi+3)(6\xi+1)\xi}{(8\xi+1)^2}C_2^2\phi^6 + \frac{6(6\xi+1)\xi}{8\xi+1}C_1C_2\phi^5 + \\ & + \left[3\xi C_1^2 + \frac{2(6\xi+1)(20\xi+3)\xi}{(8\xi+1)(4\xi+1)}C_0C_2\right]\phi^4 + \\ & + \frac{6(6\xi+1)\xi}{4\xi+1}C_0C_1\phi^3 + \frac{(16\xi+3)(6\xi+1)\xi}{(4\xi+1)^2}C_0^2\phi^2\,. \end{aligned} \tag{32}$$

To consider the stability of solutions of induced gravity models we denote $y(t) = (\phi(t), \psi(t), H(t))$. Let $y_f = (\phi_f, \psi_f, H_f)$ is a fixed point. We get $\psi_f = 0$. Also from Eqs. (22) and (24) we obtain:

$$V(\phi_f) = 3\xi\phi_f^2 H_f^2, \qquad V'(\phi_f) = 12\xi\phi_f H_f^2, \tag{33}$$

consequently,

$$V(\phi_f) = \frac{1}{4}\phi_f V'(\phi_f). \tag{34}$$

Let

$$\phi(t) = \phi_f + \varepsilon \phi_1(t), \qquad \psi(t) = \varepsilon \psi_1(t), \qquad H(t) = H_f + \varepsilon H_1(t). \quad (35)$$

To first order in ε we obtain the following system of linear equations

$$\begin{aligned} \dot{\phi}_1 &= \psi_1, \\ \dot{\psi}_1 &= \frac{1}{1+6\xi}\left[3\frac{V'(\phi_f)}{\phi_f} - V''(\phi_f)\right]\phi_1 - 3H_f\psi_1, \\ \dot{H}_1 &= \frac{V''(\phi_f)\phi_f - V'(\phi_f)}{(1+6\xi)\phi_f^2}\phi_1 + \frac{4H_f}{(1+6\xi)\phi_f}\psi_1 - \frac{24\xi H_f}{1+6\xi}H_1. \end{aligned} \quad (36)$$

With the result that we get the following matrix

$$A = \begin{Vmatrix} 0 & 1 & 0 \\ \frac{36\xi H_f^2 - V''(\phi_f)}{1+6\xi} & -3H_f & 0 \\ \frac{V''(\phi_f)\phi_f - V'(\phi_f)}{(1+6\xi)\phi_f^2} & \frac{4H_f}{(1+6\xi)\phi_f} & -\frac{24\xi H_f}{1+6\xi} \end{Vmatrix} \tag{37}$$

and roots of Eq. (4) are as follows:

$$\begin{aligned} \lambda_1 &= -\frac{3}{2}H_f + \frac{\sqrt{9(22\xi+1)H_f^2 - 4V''(\phi_f)}}{2\sqrt{1+6\xi}}, \\ \lambda_2 &= -\frac{3}{2}H_f - \frac{\sqrt{9(22\xi+1)H_f^2 - 4V''(\phi_f)}}{2\sqrt{1+6\xi}}, \\ \lambda_3 &= -\frac{24\xi H_f}{1+6\xi}. \end{aligned} \tag{38}$$

So, we obtain the conditions on H_f and $V''(\phi_f)$ that are sufficient for the stability of de Sitter solutions in induced gravity models. If we assume $\xi > 0$, then we get that a fixed point can be stable for $H_f > 0$ only.

Stability conditions

Let us consider the stability conditions for solutions, describing by formulae (27) and (29). Using (27), we get the following conditions on parameters of the potentials:

$$\lambda_3 \Leftrightarrow H_f > 0 \Leftrightarrow \begin{cases} C_1 > -\frac{2(6\xi+1)\sqrt{C_0 C_2}}{\sqrt{(4\xi+1)(8\xi+1)}}, & \text{for } \phi_f = \phi_{f_+} \equiv \frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(4\xi+1)C_2}}, \\ \\ C_1 < \frac{2(6\xi+1)\sqrt{C_0 C_2}}{\sqrt{(4\xi+1)(8\xi+1)}}, & \text{for } \phi_f = \phi_{f_-} \equiv -\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(4\xi+1)C_2}}. \end{cases}$$

We consider the case $C_0 > 0, \ C_2 > 0$.

If a solution tends to ϕ_{f_+}, then

$$\begin{aligned}
\lambda_{1+} &= -\frac{3}{2}H_{f_+} + \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 + \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right), \\
\lambda_{2+} &= -\frac{3}{2}H_{f_+} - \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 + \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right).
\end{aligned} \tag{39}$$

If a solution tends to ϕ_{f_+}, then

$$\begin{aligned}
\lambda_{1+} &= -\frac{3}{2}H_{f_+} + \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 + \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right), \\
\lambda_{2+} &= -\frac{3}{2}H_{f_+} - \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 + \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right).
\end{aligned} \tag{39}$$

Analogically, if a solution tends to ϕ_{f-}, then

$$\begin{aligned}
\lambda_{1-} &= -\frac{3}{2}H_{f_-} + \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 - \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right), \\
\lambda_{2-} &= -\frac{3}{2}H_{f_-} - \frac{3\sqrt{C_0(8\xi+1)}}{2\sqrt{C_2(4\xi+1)}}\left(C_1 - \frac{2(14\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}\right).
\end{aligned} \tag{40}$$

We obtain that all λ_i are real.

We get conditions under that λ_i are negative. Using that $H = Y(\phi) = C_0 + C_1\phi + C_2\phi^2$ in considering model we obtain

$$\lambda_{1+} = -\frac{4\xi}{4\xi+1}C_0\,. \tag{41}$$

We consider the case $C_0 > 0$ and $\xi > 0$, hence, $\lambda_{1+} < 0$. So, the stability of the fixed point ϕ_{f_+} depends on sign of

$$\lambda_{2+} = -3\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(4\xi+1)C_2}}C_1 - \frac{32\xi+6}{4\xi+1}C_0\,. \tag{42}$$

We see that

$$\lambda_{2_+} < 0 \qquad \Leftrightarrow \qquad C_1 > -\frac{2(16\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}. \tag{43}$$

To explore the stability of the stable point ϕ_{f_-}, we consider $\lambda_{i_\pm}$ as functions of the parameter C_1 and take notice that $\lambda_{1_-}(C_1) = \lambda_{2_+}(-C_1)$ and $\lambda_{2_-}(C_1) = \lambda_{1_+}(-C_1)$. Consequently, we get

$$\lambda_{1-} = 3\frac{\sqrt{(8\xi+1)C_0}}{\sqrt{(4\xi+1)C_2}}C_1 - \frac{32\xi+6}{4\xi+1}C_0\,, \qquad \lambda_{2-} = -\frac{4\xi}{4\xi+1}C_0\,. \quad (44)$$

$$\lambda_{1_-} < 0 \qquad \Leftrightarrow \qquad C_1 < \frac{2(16\xi+3)\sqrt{C_0 C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}. \quad (45)$$

Now we are ready to analyse the stability of the fixed points. Let us start with ϕ_{f_+}. We consider only the case $C_0 > 0$, $C_2 > 0$ and $\xi > 0$. We see that $\lambda_{1_+} < 0$ and $\lambda_{3_+} < 0$ is a more strong restriction on C_1 than $\lambda_{2_+} < 0$. Therefore, the fixed point ϕ_{f_+} is stable at $\lambda_{2_+} < 0$. The analogous reasoning gives that ϕ_{f_-} is stable at $\lambda_{1_-} < 0$.

Now let us analyse the stability of solutions with nonmonotonic Hubble parameter. For such a solution that tends to a stable point ϕ_{f_+} condition (43) should be satisfied. So, we get a such stable solution at

$$
\begin{aligned}
-\frac{2(16\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}} < C_1 < 0, \quad \phi(t)=\phi_+(t),\\
0 < C_1 < \frac{2(16\xi+3)\sqrt{C_0C_2}}{3\sqrt{(8\xi+1)(4\xi+1)}}, \quad \phi(t)=\phi_-(t).
\end{aligned}
\tag{46}
$$

For example, we get that at $\xi=1$, $C_2=7/2$ and $C_0=5/8$, solutions are stable if $C_1 > -19\sqrt{7}/18 \approx -2.7927$. Therefore, solutions, presented in Fig. 1, are stable.

Conclusion

We have analysed the stability of kink-type solutions for the induced gravity models in the FLRW metric. Using the Lyapunov theorem we have found sufficient conditions of stability. The obtained results allow us to prove that the exact solutions, with non-monotonic behaviors of the Hubble parameter, are stable if condition (46) is satisfied.

Thank you for attention