# SECULARLY GROWING LOOP CORRECTIONS IN STRONG ELECTRIC FIELDS 

ETA, N.Astrakhantsev and F.K.Popov

(ITEP, MIPT, HSE, Moscow)
I. Pair creation by strong electric fields has been extensively studied by many authors. These studies were based either on tree-level calculations or were using Feynman diagrammatic technique. Common wisdom is that loop corrections should not bring anything substantially new to the Schwinger's pair creation picture. In fact, usually it is believed that loop contributions cannot bring anything else but the UV renormalization or corrections to the effective Lagrangian. The goal of my talk is to show that this is an incorrect intuition and the tree-level picture or the picture provided by the Feynman technique is incomplete.

In condensed matter theory it is known that IR loop corrections can become strong in non-stationary situations - loop corrections can be comparable to the treelevel contributions. In this note we observe similar effects in scalar electrodynamics on strong electric field backgrounds. These effects, as we will see, substantially change the picture of the particle production in strong electric fields.
II. We study massive scalar coupled to electromagnetic field in $(3+1)$ dimensions:

$$
S=\int d^{4} x\left[\frac{1}{2}\left|D_{\mu} \phi\right|^{2}-\frac{1}{2} m^{2}|\phi|^{2}-\frac{1}{4} F_{\mu \nu}^{2}-j_{\mu}^{c l} A^{\mu}\right]
$$

Here $D_{\mu}=\partial_{\mu}+i e A_{\mu}$.

We will study two particular types of background electric fields:
the constant field $A_{1}(t)=E t$, for which $j_{\mu}^{c l}=0$,
and the pulse $A_{1}(t)=E T \tanh \left(\frac{t}{T}\right)$, which transforms into $E t$, as $T \rightarrow \infty$.
III. The harmonic expansion of the scalar field is

$$
\phi(t, \vec{x})=\int \frac{d^{3} p}{(2 \pi)^{3}}\left[a_{\vec{p}} e^{i \vec{p} \vec{x}} f_{p}(t)+b_{\vec{p}}^{+} e^{-i \vec{p} \vec{x}} f_{-p}^{*}(t)\right],
$$

where:

$$
\begin{array}{r}
{\left[\partial_{t}^{2}+\omega_{p}^{2}(t)\right] f_{p}(t)=0,} \\
\omega_{p}(t)=\sqrt{m^{2}+[\vec{p}+e \vec{A}(t)]^{2}} \\
\vec{A}(t)=\left(A_{1}(t), 0,0\right) .
\end{array}
$$

We will use the WKB approximation:

$$
f_{p}(t)=\left\{\begin{array}{l}
\frac{A\left(p_{\perp}\right)}{\sqrt{2 \omega_{p}(t)}} e^{-i \int_{h c}^{t} \omega_{p}\left(t^{\prime}\right) d t^{\prime}}+\frac{B\left(p_{\perp}\right)}{\sqrt{2 \omega_{p}(t)}} e^{i \int_{h c}^{t} \omega_{p}\left(t^{\prime}\right) d t^{\prime}}, t<t_{h c} \\
\frac{C\left(p_{\perp}\right)}{\sqrt{2 \omega_{p}(t)}} e^{-i \int_{h c}^{t} \omega_{p c}\left(t^{\prime}\right) d t^{\prime}}+\frac{D\left(p_{\perp}\right)}{\sqrt{2 \omega_{p}(t)}} e^{i \int_{h c}^{t} \omega_{p}\left(t^{\prime}\right) d t^{\prime}}, t>t_{h c} .
\end{array}\right.
$$

In the vicinity of the point $t_{h c}, p_{1}+e A_{1}\left(t_{h c}\right)=0$, the WKB approximation breaks down. Here $\vec{p}_{\perp}=\left(p_{2}, p_{3}\right)$.

In the pulse background $\omega_{p}(t)$ is time independent when $t \ll T$ and $t \gg T$. In this case the in- and out-harmonics become just linear combinations of the ordinary plane waves.

In the constant field

$$
f_{p}(t)=f_{p_{\perp}}\left(p_{p h}\right), \quad \text { where } \quad p_{p h}=p_{1}+e E t .
$$

Also

$$
f_{p_{\perp}}^{\text {in* }}\left(p_{p h}\right)=f_{p_{\perp}}^{\text {out }}\left(-p_{p h}\right) .
$$

And in the constant electric field background one can construct a peculiar time-symmetric state:

$$
f_{p_{\perp}}^{s *}\left(p_{p h}\right)=f_{p_{\perp}}^{s}\left(-p_{p h}\right) .
$$

The free Hamiltonian cannot be diagonalized once and forever. However, in the pulse background inharmonics diagonalize it at the past infinity, while outharmonics do the same at the future infinity. At the same time, in the constant electric background none of the choices of the harmonic functions does the diagonalization of the free Hamiltonian at the past or future infinity.

In the Keldysh-Schwinger diagrammatic technique every particle is described by the matrix propagator, whose entries are the Keldysh propagator

$$
D^{K}=\frac{1}{2}\langle\{\phi(x), \bar{\phi}(y)\}\rangle,
$$

and the retarded and advanced propagators

$$
D^{A, R}=\mp \theta(\mp \Delta t)\langle[\phi(x), \bar{\phi}(y)]\rangle
$$

(and the same for the gauge fields, with $\phi \rightarrow a_{\mu}$ ).

After the spatial Fourier transformation the tree-level propagators look like:

$$
\begin{gathered}
D_{0}^{K}\left(p, t_{1}, t_{2}\right)=\frac{1}{2}\left[f_{p}\left(t_{1}\right) f_{p}^{*}\left(t_{2}\right)+f_{p}\left(t_{1}\right) f_{p}^{*}\left(t_{2}\right)\right] \\
D_{0}^{R, A}\left(p, t_{1}, t_{2}\right)=\mp \theta(\mp \Delta t)\left[f_{p}\left(t_{1}\right) f_{p}^{*}\left(t_{2}\right)-f_{p}^{*}\left(t_{1}\right) f_{p}\left(t_{2}\right)\right] \\
G_{0 \mu \nu}^{K}\left(p, t_{1}, t_{2}\right)=-g_{\mu \nu} \frac{\cos \left[|p|\left(t_{1}-t_{2}\right)\right]}{2|p|} \\
G_{0 \mu \nu}^{R, A}\left(p, t_{1}, t_{2}\right)=\mp i g_{\mu \nu} \theta(\mp \Delta t) \frac{\sin \left[|p|\left(t_{1}-t_{2}\right)\right]}{|p|}
\end{gathered}
$$

The retarded and advanced propagators allow to specify the spectrum of the quasi-particles, while the Keldysh propagators specify the state of the theory, i.e. defines which $\vec{p}$-levels are occupied.
E.g., if the quantum average was done with the use of an arbitrary state $|\psi\rangle$, the form of the Keldysh propagators would have been:

$$
\begin{gathered}
D^{K}\left(p, t_{1}, t_{2}\right)=\left[\langle\psi| a_{\vec{p}}^{+} a_{\vec{p}}|\psi\rangle+\frac{1}{2}\right] f_{p}\left(t_{1}\right) f_{p}^{*}\left(t_{2}\right)+(a \rightarrow \text { b, h.c. }), \\
G_{\mu \nu}^{K}\left(q, t_{1}, t_{2}\right)=\left[\langle\psi| \alpha_{\vec{q} \mu}^{+} \alpha_{\vec{q} \nu}|\psi\rangle-\frac{g_{\mu \nu}}{2}\right] \frac{e^{-i|q|\left(t_{1}-t_{2}\right)}}{2|q|}+\text { h.c. }
\end{gathered}
$$

IV. The current at the tree-level looks as:

$$
\left\langle: J_{x}:\right\rangle=2 e \int \frac{d^{3} p}{(2 \pi)^{3}}\left(p_{1}+e E t\right)\left[\left|f_{p}(t)\right|^{2}-\frac{1}{2 \omega_{p}(t)}\right] .
$$

In the constant electric field background we obtain that

$$
\left\langle: J_{x}:\right\rangle=0
$$

for the time-symmetric vacuum. The current vanishes as the consequence of the time translation and time reversal invariance of the theory in the constant electric field.

For the pulse background the result is that:

$$
\left\langle: J_{x}:\right\rangle \propto T e^{3} E^{2} e^{-\frac{\pi m^{2}}{e E}}
$$

The physical meaning of this answer is easy to understand. If we have a situation with the Schwinger's constant pair production per unit four-volume - $\Gamma \propto$ $(e E)^{2} e^{-\frac{\pi m^{2}}{e E}}$ - the density of the charge carriers grows linearly and, hence, the current should also grow during the whole period, $T$, when the background field is on.
V. In the limit

$$
\frac{t_{1}+t_{2}}{2}=t \rightarrow \infty \quad \text { and } \quad t_{1}-t_{2}=\mathrm{const}
$$

the leading one-loop correction to the photon's Keldysh propagator can be written in the following form:

$$
G_{\mu \nu}^{K}\left(q, t_{1}, t_{2}\right)=\left[n_{\mu \nu}(q, t)-\frac{g_{\mu \nu}}{2}\right] \frac{e^{-i|q|\left(t_{1}-t_{2}\right)}}{2|q|}+h . c .
$$

where:

$$
\begin{array}{r}
n_{\mu \nu}(q, t)=2 e^{2} \int_{t_{0}}^{t} d t^{\prime} \int_{-\infty}^{\infty} d \tau \frac{e^{-2 i|q| \tau}}{2|q|} \int \frac{d^{3} k}{(2 \pi)^{3}} \times \\
\times\left[f_{k}(\tau) D_{\mu} f_{k+q}(\tau)-D_{\mu} f_{k}(\tau) f_{k+q}(\tau)\right] \times \\
\times\left[f_{k}^{*}(-\tau) D_{\nu} f_{k+q}^{*}(-\tau)-D_{\nu} f_{k}^{*}(-\tau) f_{k+q}^{*}(-\tau)\right] .
\end{array}
$$

Here $D_{\mu} f_{p}(t) \equiv\left(\partial_{t}, i p_{1}+i e A_{1}(t), i p_{2}, i p_{3}\right) f_{p}(t)$ and $t_{0}$ is a moment of time, after which the interactions are adiabatically turned on.

In the pulse background we have the following large IR contribution: $n_{\mu \nu} \sim e^{2} T A_{\mu \nu}^{\prime}$, as $T \rightarrow \infty$, while $t_{0} \ll-T$ and $t \gg T$.

Due to the invariance of the harmonic functions $f_{k_{\perp}}\left(k_{1}+e E t\right)$ under simultaneous compensating shifts of $k_{1}$ and $t$ the two-point function has an IR divergence: $n_{\mu \nu} \sim e^{2}\left(t-t_{0}\right) A_{\mu \nu}$.

The regulator, $t_{0}$, of the IR divergence in question cannot be taken to the past infinity. When $E \rightarrow 0$ the harmonic functions are converted into plain waves, $f_{p}(t) \rightarrow e^{i \omega_{p} t}$. Then the factor, $A_{\mu \nu}$, of the divergence becomes proportional to $\int d^{3} k \delta\left(|q|+\omega_{k}+\omega_{k-q}\right)$. It is zero and one can take $t_{0} \rightarrow-\infty$. At the same time, if $\vec{E} \neq 0$ the sharp $\delta$-function gets eroded because there is no conservation of energy in time-dependent background fields.
The divergence in question has nothing to do with the vanishing photon's mass. In fact, let us add to the theory in question the Yukawa coupling to a massive real (neutral) scalar, $\lambda \varphi|\phi|^{2}$. Then the analogous expression for $n_{p}$ of $\langle\{\varphi, \varphi\}\rangle$ is even simpler: Under the $d \tau$ integral one just has the product of four harmonic functions without derivatives. Then $n_{p}$ does have the same type of divergence.

This divergence is due to photon production. As we will see now, its coefficient is just a piece of the collision integral, which is due to the particle creation by the background field. We have here the simultaneous creation of one photon, $e^{i q t}$, and two oppositely charged scalars, $f_{k}$ and $f_{k+q}$.

In the limit

$$
\frac{t_{1}+t_{2}}{2}=t \rightarrow \infty \quad \text { and } \quad t_{1}-t_{2}=\text { const }
$$

there are no large IR corrections to $D^{K}, D^{R, A}$ and $G^{R, A}$. Also there are no large IR corrections to the vertexes.

Even if $e^{2}$ is very small, after a long enough time period loop correction, $e^{2}\left(t-t_{0}\right)$, becomes comparable to the tree-level contribution. I.e. the loop correction, $n_{\mu \nu}$, is essentially a classical quantity. That is not a very unusual phenomenon in non-stationary quantum field theory. These observations put forward the question of the summation of all unsuppressed loop corrections in the limit $t-t_{0} \rightarrow \infty$.
VI. To understand the physics in the strong electric fields one has to sum up leading IR contributions from all loops. We would like to perform the summation of those terms which are powers of $e^{2}\left(t-t_{0}\right)$ and to drop terms, which are suppressed by higher powers of $e$. In order to do that, we have to solve the system of DysonSchwinger equations for the exact propagators, $D^{K, R, A}$ and $G^{K, R, A}$, and for the vertexes in the IR limit $\left(t-t_{0} \rightarrow\right.$ $\infty)$.
Taking into account that all vertexes, retarded, advanced propagators and also the Keldysh propagator for the scalars receive subleading corrections, we can put them to their tree-level values in the system of Dyson-Schwinger equations, if we are interested only in the leading corrections. Then this system reduces to the single equation for the exact Keldysh propagator of the gauge field.

Keeping trace of only leading corrections as $\left(t-t_{0}\right) \rightarrow$ $\infty$, one can convert the integral DS equation into the integrodifferential form, i.e., into the form of the Boltzman's kinetic equation:

$$
\begin{aligned}
\frac{\partial n_{\mu \nu}(q, t)}{\partial t}= & -\Gamma_{1 \mu}^{\rho}(q)\left[-g_{\rho \nu}+n_{\rho \nu}(q, t)\right]+\Gamma_{2 \mu}^{\rho}(q) n_{\rho \nu}(q, t) \\
& \text { where } \quad \Gamma_{1 \mu \nu}(q)=e^{2} \int \frac{d^{3} k}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d \tau \frac{e^{-2 i|q| \tau}}{|q|} \times \\
& \times\left[f_{k}(\tau) D_{\mu} f_{k-q}(\tau)-D_{\mu} f_{k}(\tau) f_{k-q}(\tau)\right] \times \\
\times & {\left[f_{k}^{*}(-\tau) D_{\nu} f_{k-q}^{*}(-\tau)-D_{\nu} f_{k}^{*}(-\tau) f_{k-q}^{*}(-\tau)\right] } \\
& \text { and } \quad \Gamma_{2 \mu \rho}(q)=e^{2} \int \frac{d^{3} k}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d \tau \frac{e^{-2 i|q| \tau}}{|q|} \times \\
& \times\left[f_{k}^{*}(\tau) D_{\mu} f_{k-q}^{*}(\tau)-D_{\mu} f_{k}^{*}(\tau) f_{k-q}^{*}(\tau)\right] \times \\
\times & {\left[f_{k}(-\tau) D_{\rho} f_{k-q}(-\tau)-D_{\rho} f_{k}(-\tau) f_{k-q}(-\tau)\right] }
\end{aligned}
$$

One can check that $n_{\mu \nu}$ and $\Gamma_{1,2 \mu \nu}(q)$ are transversal $n_{\mu \nu} q^{\nu}=\Gamma_{1,2 \mu \nu}(q) q^{\nu}=0$.

The physical meaning of the RHS of this equation is very simple. The first term describes the photon production by the background field. The second term describes the decay of the produced photons into charged pairs.

The situation is similar to the one in de Sitter space (arXiv:1309.2557).

