Spectral duality and gauge theories

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in collaboration with
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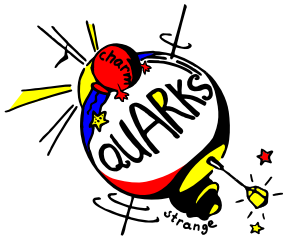
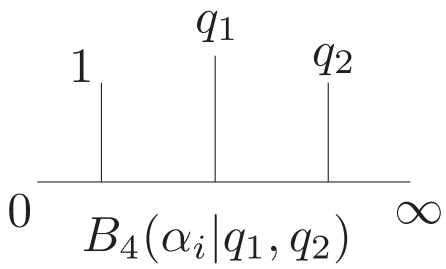


The plan

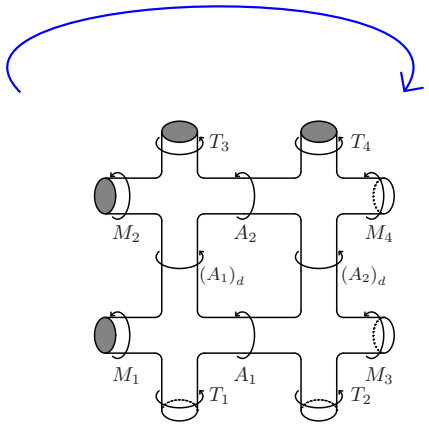
- Philosophy: branes, gauge theories and CFT
- Technicalities: XXZ/XXZ spectral duality
- Possible generalization: quantum p-q duality

Branes, gauge theories and CFT

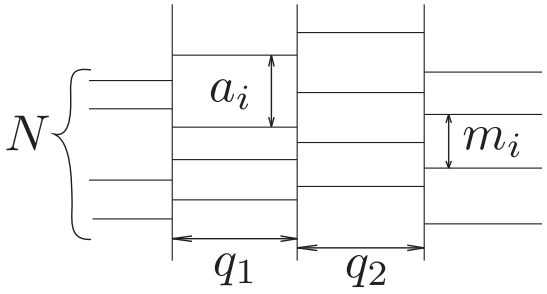


conformal block

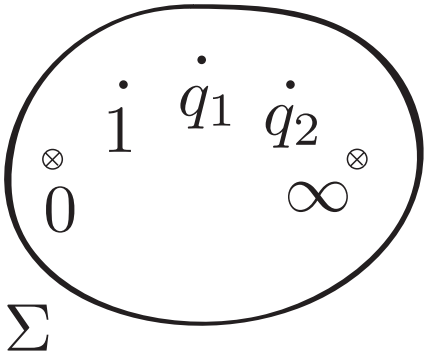
spectral duality



5d brane diagram



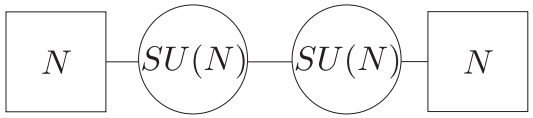
$SU(N)^2$ brane diagram



bare spectral curve

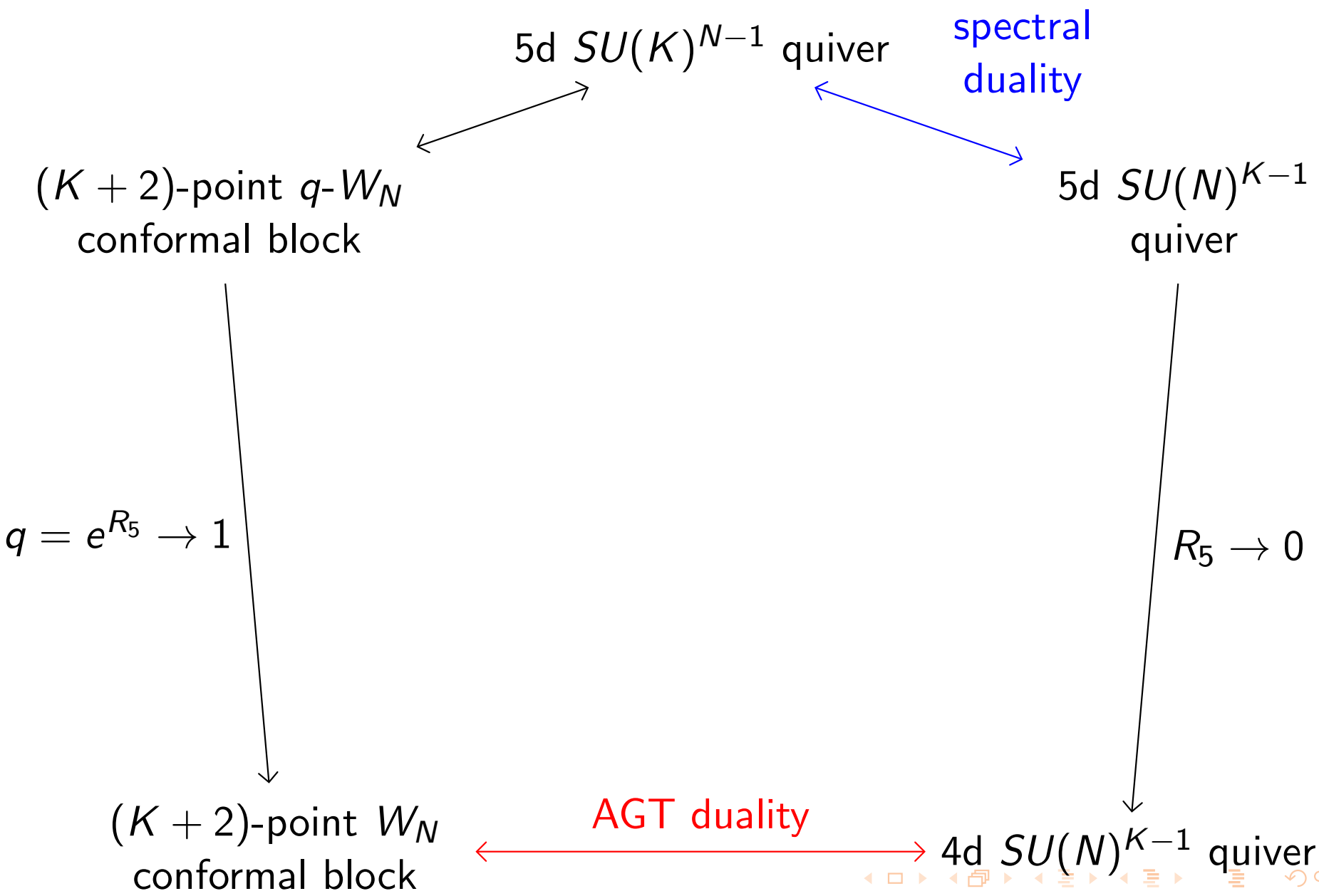
$\Gamma : Pol_{N,2}(x,z) = 0$

Seiberg-Witten curve

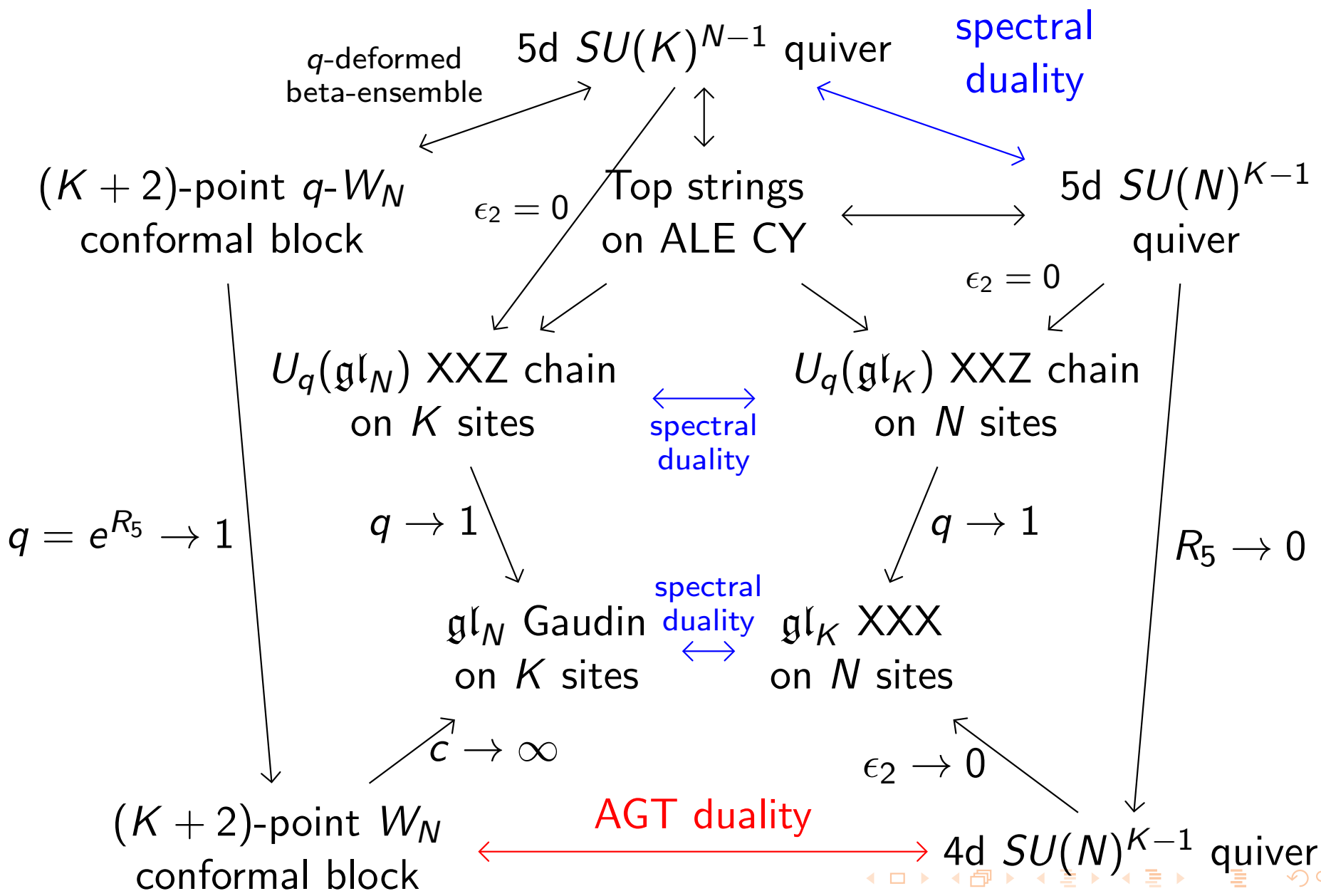


$SU(N)^2$ quiver gauge theory

The pentagon of dualities



The double pentagon of dualities



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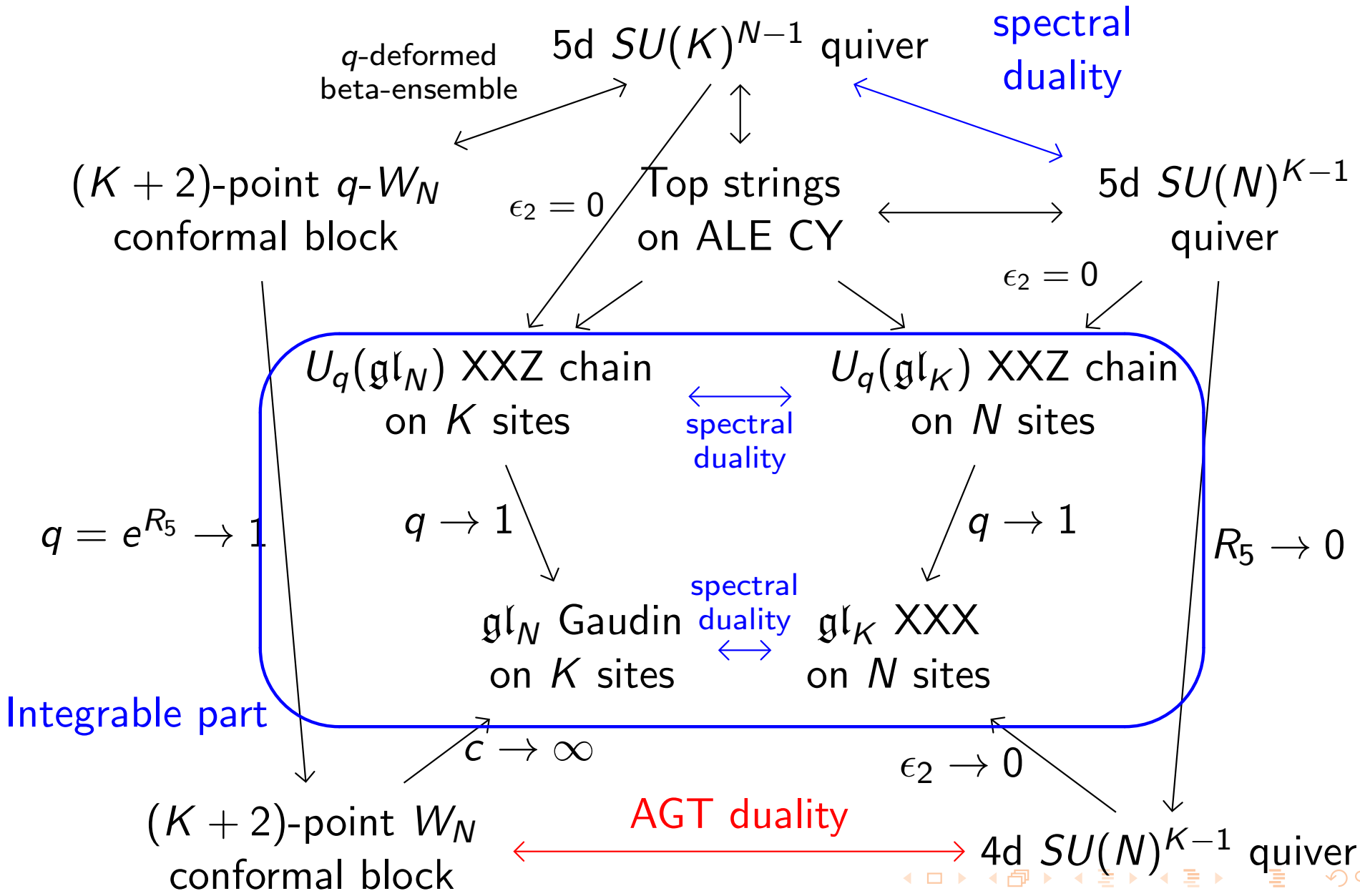

The integrable part of the story

$$
\begin{tikzcd}
N\text{-site } U_q(\mathfrak{gl}_K) \text{ XXZ} \arrow[r, "\text{spectral duality}"] \arrow[d, "\substack{q=e^{\gamma}\to 1\\ v=q^{2x}\to 1+2\gamma x\\ q^{2v\partial_v}\to e^{\partial_x}}"'] & K\text{-site } U_q(\mathfrak{gl}_N) \text{ XXZ} \arrow[d, "\substack{q\to 1\\ q^{2v\partial_v}\to e^{\partial_x}\text{, finite!}\\ v=q^{2x}\to 1+2\gamma x}"] \\
N\text{-site } \mathfrak{gl}_K \text{ XXX} \arrow[r, "\text{spectral duality}"] \arrow[d] & K\text{-site } \mathfrak{gl}_N \text{ trig Gaudin} \arrow[d] \\
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[*Mukhin, Tarasov, Varchaneko*, `math/0610799`]

[*Mukhin, Tarasov, Varchaneko*, `math/0112005`, `math/0605172`]

The plan

- Philosophy: branes, gauge theories and CFT
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The $U_q(\mathfrak{gl}_K)$ XXZ spin chain

The chain is built out of Lax operators located at the sites:

$$\mathbf{T}(v) = Q\mathbf{L}^N(v)\cdots\mathbf{L}^1(v)\,. \qquad \text{Twist } Q = \mathrm{diag}(Q_1,\ldots,Q_K).$$

$U_q(\mathfrak{gl}_K)$ indices are **blue**, site indices are **green**.
Lax operators satisfy the "*RLL*" relations

$$R^{(12)}_{\text{trig}}(u/v)\mathbf{L}(u)\otimes\mathbf{L}(v) = (\mathbf{1}\otimes\mathbf{L}(v))(\mathbf{L}(u)\otimes\mathbf{1})R^{(12)}_{\text{trig}}(u/v)\,.$$

More explicitly

$$\mathbf{L}^i(v) = \sum_{a,b}\mathbf{L}^i_{ab}\otimes E_{ab} =$$

$$= \sum_{a,b}\left(\delta_{ab}q^{\mathbf{H}^i_a} + (q-q^{-1})\frac{\delta_{a>b}v + \delta_{b\geq a}v_i}{v-v_i}\mathbf{S}_{ba}\right)\otimes E_{ab}\,,$$

The operators $\mathbf{S}_{ab}$ and $q^{\mathbf{H}^i_a}$ satisfy the $U_q(\mathfrak{gl}_K)$ algebra.

Quantum commuting operators in $U_q(\mathfrak{gl}_K)$ XXZ spin chain

Commuting subalgebra

$$[\mathcal{H}_i, \mathcal{H}_j] = 0\,.$$

For $K = 2$ [*Sklyanin, Reshetikhin, etc.*]

$$\mathcal{H}_1(v) = \operatorname{tr} \mathbf{T}(v)\,,$$
$$[\mathcal{H}_1(v), \mathcal{H}_1(u)] = 0\,.$$

One more combination (quantum determinant) gives the Casimir operator

$$\mathcal{H}_2(v) = \operatorname{qdet}\mathbf{T}(v) = \mathbf{T}_{11}(v)\mathbf{T}_{22}(q^2 v) - q^{-1}\mathbf{T}_{21}(v)\mathbf{T}_{12}(q^2 v)\,,$$
$$[\mathcal{H}_2(v), \text{anything}] = 0\,.$$

For $K > 2$ this is not sufficient, more operators needed.

Quantum commuting operators in $U_q(\mathfrak{gl}_K)$ XXZ spin chain

For $K > 2$: Universal difference operator

$$\hat{D}(v) = \sum_{m=0}^{K} (-1)^m \mathcal{H}_m(v) q^{2v\partial_v} = \sum_{m=0}^{K} \sum_{A=\{a_1<\ldots<a_m\}} \det_{\mathrm{col}\, q} \left(\mathbf{T}_{AA}(v) q^{2v\partial_v} \right),$$

where $\mathbf{T}_{AA}$ is $m \times m$ submatrix of $\mathbf{T}$ and the *column q-determinant* is

$$\det_{\mathrm{col}\, q} \mathbf{M}_{AA} = \sum_{\sigma \in \mathfrak{S}_m} (-q)^{\mathrm{inv}(\sigma)} \mathbf{M}_{a_{\sigma(1)} a_1} \cdots \mathbf{M}_{a_{\sigma(m)} a_m},$$

and $\mathrm{inv}(\sigma) = \{\text{number of pairs } (a, b) \text{ such that } \sigma(b) > \sigma(a)\}$.

$$[\mathcal{H}_m(v), \mathcal{H}_n(u)] = 0$$

[*Mukhin, Tarasov, Varchenko*, `math/0605015`]
[*Chervov, Falqui, Rubtsov, Silantyev*, `1210.3529`]

Building spins in (anti)symmetric reps from q-Bose (q-Fermi)

Prototypical example: $\mathfrak{sl}_2$

- Bose (Fermi) generators $[\mathbf{a}_a, \mathbf{a}_b]_\pm = [\mathbf{a}_a^\dagger, \mathbf{a}_b^\dagger]_\pm = 0$, $[\mathbf{a}_a, \mathbf{a}_b^\dagger]_\pm = 1$,
- $\mathbf{s}_{ab} = \mathbf{a}_a^\dagger \mathbf{a}_b - \frac{1}{2}\delta_{ab}\mathbf{a}_c^\dagger \mathbf{a}_c \Rightarrow [\mathbf{s}_{ab}, \mathbf{s}_{cd}] = \delta_{bc}\mathbf{s}_{ad} - \delta_{ad}\mathbf{s}_{cb}$
- Vacuum is trivial rep $|0\rangle$.
- Symmetric reps are $\mathbf{a}_a^\dagger \cdots \mathbf{a}_b^\dagger |0\rangle$
- Quadratic Casimir is $\frac{1}{2}\mathbf{s}_{ab}\mathbf{s}_{ba} = \frac{\mathbf{N}}{2}\left(\frac{\mathbf{N}}{2} + 1\right)$, where $\mathbf{N} = \mathbf{a}_a^\dagger \mathbf{a}_a$.

q-deformation

- q-Bose (q-Fermi) creation and annihilation operators $\mathbf{A}^i_a$ and $\mathbf{B}^i_a$ act on $|0\rangle$:

$$\mathbf{A}^i_a\mathbf{A}^j_b = q^{\delta_{ij}\mathrm{sgn}(a-b)}\mathbf{A}^j_b\mathbf{A}^i_a\,, \qquad \mathbf{B}^i_a\mathbf{B}^j_b = q^{\delta_{ij}\mathrm{sgn}(a-b)}\mathbf{B}^j_b\mathbf{B}^i_a\,,$$

$$\mathbf{A}^i_a\mathbf{B}^j_b = q^{\delta_{ij}(\delta_{ab}-\mathrm{sgn}(a-b))}\mathbf{B}^j_b\mathbf{A}^i_a - \delta_{ab}\delta_{ij}q^{\mathbf{H}^i_a+1}\,,$$

$$[\mathbf{H}^i_a, \mathbf{A}^j_b] = \delta_{ab}\delta_{ij}\mathbf{A}^j_b\,, \qquad [\mathbf{H}^i_a, \mathbf{B}^j_b] = -\delta_{ab}\delta_{ij}\mathbf{B}^j_b$$

- $\mathbf{A}^i_a\mathbf{B}^i_a = [\mathbf{H}^i_a]_q$ count the number of different creation operators, they are Cartan elements of $U_q(\mathfrak{gl}_K)^{\otimes N}$. $c_i = \sum_{a=1}^K \mathbf{A}^i_a\mathbf{B}^i_a$ are Casimirs.
- The Lax operator is built from $\mathbf{A}$'s and $\mathbf{B}$'s:

$$\mathbf{S}^i_{ab} = \mathbf{A}^i_a\mathbf{B}^i_b\,, \qquad \mathbf{L}^i_{ab}(v) = \delta_{ab}\frac{q^{\mathbf{H}^i_a}}{q-q^{-1}} + \delta_{b>a}\mathbf{A}^i_b\mathbf{B}^i_a + \frac{v_i\mathbf{A}^i_b\mathbf{B}^i_a}{v-v_i}$$

The spectral duality

Consider the N-site $U_q(\mathfrak{gl}_K)$ and K-site $U_q(\mathfrak{gl}_N)$ XXZ spin chains. Denote their universal difference operators by $\hat{D}_{N,K}$ and $\hat{D}_{K,N}$

Main conjecture

$$\prod_{i=1}^{N}(v - v_i)\hat{D}_{N,K}(v) = \prod_{a=1}^{K}\left(1 - w_a q^{2v\partial_v}\right)\hat{D}_{K,N}(q^{-2v\partial_v}),$$

where v_i are the inhomogeneities in $\mathbf{L}^i$ and $w_a = Q_a \prod_{i=1}^{N} q^{\mathbf{H}^i_a}$.

Duality acts as the **Fourier transform!**

The spectral duality

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What evidence do we have?

- Isomorphism of q-Bose algebras.
- The classical limit.
- Degeneration $q \to 1$, XXX/trigonometric Gaudin duality.

The spectral duality

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What evidence do we have?

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XXX limit

q-Bose turns into Bose:

$$\widetilde{\mathbf{A}}^i_a, \widetilde{\mathbf{B}}^j_b \xrightarrow{q\to 1} \mathbf{a}^i_a, \mathbf{b}^j_b\,,$$

$$[\mathbf{a}^i_a, \mathbf{a}^j_b] = [\mathbf{b}^i_a, \mathbf{b}^j_b] = 0\,, \qquad [\mathbf{a}^i_a, \mathbf{a}^j_b] = -\delta_{ij}\delta_{ab}\,.$$

$$\hat{D}_{N,K}(v) \xrightarrow{q\to 1} \det_{\text{col}}\left(\mathbf{1} - \mathbf{T}(x)e^{\partial_x}\right),$$

where still $\mathbf{T}(x) = Q\mathbf{L}^N(v)\cdots\mathbf{L}^1(v)$, but $\mathbf{L}^i_{ab}(x) = \delta_{ab} + \frac{1}{x-x_i}\mathbf{b}^i_a\mathbf{a}^i_b$ and

$$\det_{\text{col}}\mathbf{M} = \sum_{\sigma\in\mathfrak{S}_K}(-1)^{\text{inv}(\sigma)}\mathbf{M}_{\sigma(1),1}\cdots\mathbf{M}_{\sigma(K),K}\,.$$

Normal ordering. $:F(\mathbf{b},\mathbf{a},x,\partial_x): = \left\{\begin{matrix}\text{all } \mathbf{b}\text{'s, } x\text{'s to the left,}\\ \text{all } \mathbf{a}\text{'s, } \partial_x\text{'s to the right}\end{matrix}\right\}$

Normal ordering theorem I.

$$\det_{\text{col}}\left(\mathbf{1} - \mathbf{T}(x)e^{\partial_x}\right) = :\det_{\text{col}}\left(\mathbf{1} - \mathbf{T}(x)e^{\partial_x}\right):$$

Normal ordering in XXX. Examples

Examples.

- $K = 1$: trivial,

$$\left[1 - Q_1\left(1 + \frac{1}{x - x_1}\mathbf{b}_1^1\mathbf{a}_1^1\right)\cdots\left(1 + \frac{1}{x - x_N}\mathbf{b}_1^N\mathbf{a}_1^N\right)\right]e^{\partial_x} = :\text{same}:$$

- $K = 2,\ N = 1$:

$$\det_{\text{col}}{}^{2\times 2}\left[\mathbf{1} - Q\left(\mathbf{1} + \frac{1}{x - x_1}\mathbf{b}^1(\mathbf{a}^1)^{\mathsf{T}}\right)e^{\partial_x}\right] =$$

$$= \left(1 - Q_1 e^{\partial_x} - \frac{1}{x - x_1}Q_1\mathbf{b}_1^1\mathbf{a}_1^1 e^{\partial_x}\right)\left(1 - Q_2 e^{\partial_x} - \frac{1}{x - x_1}Q_2\mathbf{b}_2^1\mathbf{a}_2^1 e^{\partial_x}\right) -$$

$$- \frac{1}{x - x_1}Q_2\mathbf{b}_2^1\mathbf{a}_1^1 e^{\partial_x}\frac{1}{x - x_1}Q_1\mathbf{b}_1^1\mathbf{a}_2^1 e^{\partial_x} = \left(1 - Q_1 e^{\partial_x}\right)\left(1 - Q_2 e^{\partial_x}\right) -$$

$$- \frac{1}{x - x_1}Q_1\mathbf{b}_1^1\mathbf{a}_1^1 e^{\partial_x}\left(1 - Q_2 e^{\partial_x}\right) - \frac{1}{x - x_1}Q_2\mathbf{b}_2^1\mathbf{a}_2^1 e^{\partial_x}\left(1 - Q_1 e^{\partial_x}\right) -$$

$$\cancel{- \frac{1}{x - x_1}\frac{1}{x - x_1 + 1}Q_1 Q_2\mathbf{b}_2^1[\mathbf{a}_1^1, \mathbf{b}_1^1]\mathbf{a}_2^1 e^{2\partial_x}} \cancel{- \left[e^{\partial_x}, \frac{1}{x - x_1}\right]Q_2\mathbf{b}_2^1\mathbf{a}_2^1 e^{\partial_x}} = :\text{same}:$$

Degeneration $q \to 1$

$$
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Trigonometric Gaudin limit

q-Bose turns into Bose:

$$\widetilde{\mathbf{A}}^i_a, \widetilde{\mathbf{B}}^j_b \xrightarrow{q\to 1} \mathbf{a}^i_a, \mathbf{b}^j_b\,,$$

$$[\mathbf{a}^i_a, \mathbf{a}^j_b] = [\mathbf{b}^i_a, \mathbf{b}^j_b] = 0\,, \qquad [\mathbf{a}^i_a, \mathbf{a}^j_b] = -\delta_{ij}\delta_{ab}\,.$$

$$\hat{D}_{K,N}(q^{-2v\partial_v}) \xrightarrow{q=e^\gamma\to 1} (-2\gamma)^N \widetilde{\det_{\text{col}}}\Big(x - \mathbf{L}_{\text{tG}}(e^{-\partial_x})\Big),$$

where $\left[\mathbf{L}_{\text{tG}}(e^{-\partial_x})\right]_{ij} = \delta_{ij}x_i + \delta_{i>j}\sum_{a=1}^K \mathbf{a}^j_a\mathbf{b}^i_a + \sum_{a=1}^K \frac{Q_a}{e^{-\partial_x}-Q_a}\mathbf{a}^j_a\mathbf{b}^i_a$ and $\widetilde{\det_{\text{col}}}$ is **very** peculiar deformation of determinant

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$$\widetilde{\det_{\text{col}}}\mathbf{M} = \sum_{m=0}^{N-2}(-1)^m \sum_{J=\{i_1<\ldots<i_m\}} \sum_{\sigma\in\mathfrak{S}_N} (-1)^{\text{inv}(\sigma)} \prod_{\alpha=1}^m \left(\sum_{j=1}^{i_\alpha-1}\delta_{\sigma(j)>i_\alpha}\right)\cdot$$
$$\cdot\,\mathbf{M}_{\sigma(1),1}\cdots\mathbf{M}_{\sigma(i_1-1),i_1-1}\delta_{\sigma(i_1)i_1}\mathbf{M}_{\sigma(i_1+1),i_1+1}\cdots$$
$$\cdots\mathbf{M}_{\sigma(i_2-1),i_2-1}\delta_{\sigma(i_2)i_2}\mathbf{M}_{\sigma(i_2+1),i_2+1}\cdots\mathbf{M}_{\sigma(K),K}$$

Normal ordering in Gaudin.

Normal ordering theorem II.

$$\prod_{a=1}^{K}\left(1-Q_a e^{\partial_x}\right)\widetilde{\det_{\text{col}}}\left(x-\mathbf{L}_{\text{tG}}(e^{-\partial_x})\right)=$$

$$=:\prod_{a=1}^{K}\left(1-Q_a e^{\partial_x}\right)\det_{\text{col}}\left(x-\mathbf{L}_{\text{tG}}(e^{-\partial_x})\right):$$

(No tilde on the right)

Example. $K=N=1$:

$$\left(1-Q_1 e^{\partial_x}\right)\left(x-x_1-\frac{Q_1}{e^{-\partial_x}-Q_1}\mathbf{a}_1^1\mathbf{b}_1^1\right)=$$

$$=\left(x-x_1-\cancel{\frac{Q_1}{e^{-\partial_x}-Q_1}}-\frac{Q_1}{e^{-\partial_x}-Q_1}\mathbf{b}_1^1\mathbf{a}_1^1-\cancel{\frac{Q_1}{e^{-\partial_x}-Q_1}\left[\mathbf{a}_1^1,\mathbf{b}_1^1\right]}\right)\left(1-Q_1 e^{\partial_x}\right)$$

Degeneration. Resume.

XXX/trig Gaudin spectral duality

$$\prod_{i=1}^{N}(x-x_i)\det_{\text{col}}\left(\mathbf{1}-\mathbf{T}(x)e^{\partial_x}\right)=:\prod_{i=1}^{N}(x-x_i)\det_{\text{col}}\left(\mathbf{1}-\mathbf{T}(x)e^{\partial_x}\right):=$$

$$=:\prod_{a=1}^{K}\left(1-Q_a e^{\partial_x}\right)\det_{\text{col}}\left(x-\mathbf{L}_{\text{tG}}(e^{-\partial_x})\right):=$$

$$=\prod_{a=1}^{K}\left(1-Q_a e^{\partial_x}\right)\widetilde{\det_{\text{col}}}\left(x-\mathbf{L}_{\text{tG}}(e^{-\partial_x})\right)$$

The classical results hold under the normal ordering!

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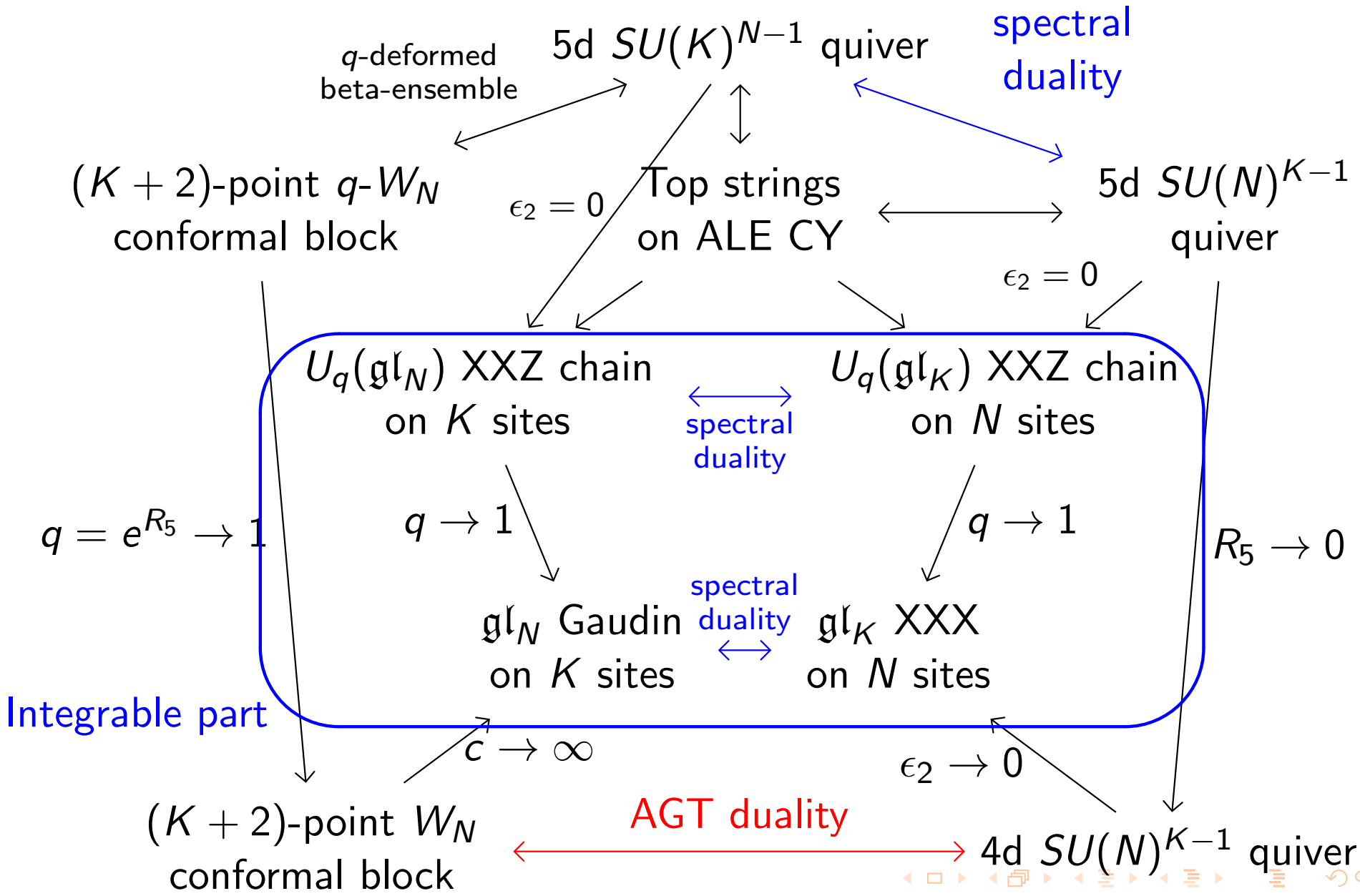
The double pentagon of dualities



Further generalization. "Double" quantization.

XXZ chain arises in the limit $\epsilon_2 \to 0$ (Nekrasov-Shatashvili).

What happens for $\epsilon_2 \neq 0$?

Bethe ansatz equations for *quantum* XXZ chains $\overset{QC}{\longleftrightarrow}$ Level set of Hamiltonians of the *classical* trigonometric Ruijsenaars-Schneider models

[*Gorsky, Zabrodin, Zotov,* `arXiv:1310.6958`]

$\epsilon_2 \neq 0$ quantizes the *quantum/classical (QC) dual* RS system. Spectral duality corresponds to the quantum *p*-*q* duality in the dual system:

$$\mathbf{H}^{\mathrm{RS}}(x) M_Y(x_i) = \epsilon_Y M_Y(x_i)$$
$$M_Y(x_i = q^{R_i+N-i}) = M_R(x_i = q^{Y_i+N-i})$$
$$\mathbf{H}^{\mathrm{RS}}(Y) M_Y(x_i = q^{R_i+N-i}) = \epsilon_R M_Y(x_i = q^{R_i+N-i})$$

Energies (Y) are exchanged with coordinates (x_i). On the classical level — canonical transformation $(x, p) \to (\epsilon, \phi)$.

Degeneration $q \to 1$

$$
\begin{tikzcd}
\text{XXZ} \arrow[r, "\text{spectral duality}"] \arrow[d, "\substack{q=e^{\gamma}\to 1 \\ v=q^{2x}\to 1+2\gamma x \\ q^{2v\partial_v}\to e^{\partial_x}}"'] & \text{XXZ} \arrow[d, "\substack{q\to 1 \\ q^{2v\partial_v}\to e^{\partial_x}\text{, finite!} \\ v=q^{2x}\to 1+2\gamma x}"] \\
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Spectral duality and p-q duality

$$
\begin{tikzcd}
\text{XXZ} \arrow[r, "\text{QC duality}"] \arrow[d, "q\to 1"'] & \text{trig. RS} \arrow[r, "\text{p-q duality}"] \arrow[d, "q\to 1"] & \text{trig. RS} \arrow[d, "q\to 1"] & \text{XXZ} \arrow[l, "\text{QC duality}"'] \arrow[d, "q\to 1"] \\
\text{XXX} \arrow[r, "\text{QC duality}"] \arrow[d] & \text{rat. RS} \arrow[r, "\text{p-q duality}"] \arrow[d] & \text{trig. Calogero} \arrow[d] & \text{trig. Gaudin} \arrow[l, "\text{QC duality}"'] \arrow[d] \\
\text{rat. Gaudin} \arrow[r, "\text{QC duality}"] & \text{rat. Calogero} \arrow[r, "\text{p-q duality}"] & \text{rat. Calogero} & \text{rat. Gaudin} \arrow[l, "\text{QC duality}"']
\end{tikzcd}
$$

Conclusions and prospects

- Spectral duality is key to many results in gauge/string theory.
- We demonstrate spectral duality between $U_q(\mathfrak{gl}_K)$ N-site and $U_q(\mathfrak{gl}_N)$ K-site XXZ spin chains.
- Beyond the $\epsilon_2 \to 0$ limit the spectral duality becomes p-q duality in the *QC dual* integrable system.
- It would be interesting to investigate the role pf p-q duality in CFT.

Thank you for your attention!