On the particle excitations in the XXZ- spin chain.

A.Ovchinnikov[1]

[1]INR, Moscow

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Introduction

$$H_{XXZ} = \sum_{i=1}^{L} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right), \qquad \Delta = \cos(\eta).$$

- We study the excited states for the XXZ- spin chain corresponding to the complex roots of the Bethe Ansatz equations with the imaginary part equal to $\pi/2$.
- We propose the particle-hole symmetry which relates the eigenstates build up from the two different pseudovacuum states.
- We find the XXX- spin chain limit for the eigenstates with the complex roots.
- We comment on the low-energy excited states for the XXZ- spin chain.

Complex roots.

Bethe Ansatz solution:

$$\left(\frac{\sinh(t_\alpha - i\eta/2)}{\sinh(t_\alpha + i\eta/2)}\right)^L = \prod_{\gamma\neq\alpha} \frac{\sinh(t_\alpha - t_\gamma - i\eta)}{\sinh(t_\alpha - t_\gamma + i\eta)},$$

$$\phi(t) = \frac{1}{i}\ln\left(-\frac{\sinh(t - i\eta/2)}{\sinh(t + i\eta/2)}\right), \quad \phi_2(t) = \frac{1}{i}\ln\left(-\frac{\sinh(t - i\eta)}{\sinh(t + i\eta)}\right).$$

Quntum numbers n_α (α=1,...M):

$$L\phi(t_\alpha) = 2\pi n_\alpha + \sum_{\gamma\neq\alpha} \phi_2(t_\alpha - t_\gamma),$$

Known complex solutions (k-strings):

$$t_n = t_0 + i\eta n, \quad n = -(k-1)\dots(k-1).$$

New complex solutions:

$$t_\alpha = t'_\alpha + i\pi/2,$$

where t'_α- is Real.

Complex roots and the Particle-Hole symmetry.

From the XX- spin chain limit ($\Delta = 0$) by the continuity argument it follows that for the XXZ- spin chain:

1) The complex solution of the form $t_\alpha = t'_\alpha + i\pi/2$ are exist. In fact for the XX-chain we have the BA equations $L\phi(t_1) = 2\pi n_1$ which shows that for $n_1 > L/4$ we have $t_1 = t'_1 + i\pi/2$ ($M = L/2$).

THEOREM: For each comlex root t_α the complex conjugate t^*_α is also a root unless $t_\alpha = t'_\alpha + i\pi/2$.

2) Consider two eigenstates build up starting from two different pseudovacuum states ($|0\rangle = |\downarrow\rangle$ (t_α, n_α)) and $|\uparrow\rangle$ (t^*_α, n^*_α). Let n_{0i} are the holes in the usual BA state. Then we have for the "dual" states the relations

$$n^\star_i = L/2 - n_{0i}, \quad (n_{0i} > 0), \qquad n^\star_i = -L/2 - n_{0i}, \quad (n_{0i} < 0).$$

where $i = 1, \ldots M^*$, $M^* = L - M$

Evolution of the eigenstates with η at fixed quantum numbers.

Let us consider the evolution of the eigenstates with particles (complex roots) when we vary η from $\eta = \pi/2$ (XX) to $\eta = 0$ (XXX) at fixed quantum numbers n_α at arbitrary

$$M = \frac{L}{2} + k\,.$$

For example consider the state without the holes at the real axis. Define the following function:

$$n_L(t) = \frac{1}{2\pi}\left(L\phi(t) - \sum_\gamma \phi_2(t - t_\gamma)\right)\,.$$

Using this function since $n_{max} = [n_c]$ where $n_c = n_L(\infty)$, it easy to calculate the number of vacancies at the real axis R and the number of complex roots C:

$$R = \frac{L}{2} - k + 2\left[\frac{1}{2} + \frac{k\eta}{\pi}\right]\,,$$

$$C = M - R = 2k - 2\left[\frac{1}{2} + \frac{k\eta}{\pi}\right]\,.$$

We have the following eigenstate without the holes on the real axis:

$$|\phi(R, C)\rangle = |\phi\left(\frac{L}{2} - k + 2\left[\frac{1}{2} + \frac{k\eta}{\pi}\right],\;\; 2k - 2\left[\frac{1}{2} + \frac{k\eta}{\pi}\right]\right)\rangle.$$

Evolution of the states with particles.

When η decreases from $\pi/2$ to 0 the roots and the holes on the real axis at the points $\eta m = \pi/k(m+1/2)$ are pushed to the infinity and then jumps upwards to the line $Imt = \pi/2$ i.e. becomes complex roots.

The jumps happen when $n_C = n_L(\infty)$ as a function of η crosses the points n_α (the roots or the holes).

For example without the holes on the real axis we have:

$$|\phi(L/2,k)\rangle(\eta = \pi/2) \longrightarrow |\phi(L/2-k,2k)\rangle(\eta \to 0).$$

XXX- limit of states with particles.

In the XXX spin chain there is NO complex roots of this type! Then what is the limit of the states with particles at $\eta \to 0$? Since

$$B(t)|_{t-fixed} \to \eta f(t) S^{+} + O(\eta^{2}),$$

we find that

$$|\phi(L/2-k,2k)\rangle \to (S^{+})^{2k} |\phi(L/2-k)\rangle_{XXX}^{(2k)}.$$

For the XXX chain the number of the real roots is $L/2-k$, so that there is $L/2+k$ vacancies on the real axis. Then there is $2k$ holes, which is indicated in the last equation.

What are the positions of these holes?

The answer is the following. Let n_i to be the quantum numbers of the complex roots at $\eta \sim 0$. Then the positions of the holes n_{0i} are:

$$n_{0i} = L/2 - n_i \quad (n_i > 0), \qquad n_{0i} = -L/2 - n_i \quad (n_i < 0).$$

In fact, using the Particle-Hole symmetry we obtain:

$$|\phi(L/2-k,2k)\rangle_{XXZ} \to \left(|\phi(L/2-k)\rangle_{XXX}^{(2k)} \right)^{*}.$$

Comparing the two equations for the states we find n_{0i} since the states $|\phi(L/2-k)\rangle$ in the RHS of these equations belong to the same spin multiplet in XXX.

Low lying excitations.

Cosider the low-energy excitations.

1) $k = 1$ -Two particles. Dispersion relation for one particle:

$$\epsilon(p) = \frac{\pi}{2}\frac{\sin(\eta)}{\eta}\sin(p).$$

Totel energy and momentum

$$\Delta E = \epsilon(p_1) + \epsilon(p_2), \qquad \Delta P = p_1 + p_2.$$

2) $k = 0$ -One particle and one hole.

3) $k = -1$ -Two holes.

Luttinger liquid relation:

$$\Delta E = \frac{\pi v}{2L}\left(\xi(\Delta N)^2 + (1/\xi)(\Delta Q)^2\right).$$

$\Delta N_{1,2}$ can be positive or negative!

Conclusion

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