Link polynomials and AENV conjecture

Andrey Morozov

ITEP, MSU, Moscow

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AENV-conjecture

In their paper[1] Aganagic, Ekholm, Ng and Vafa proposed that the holomorphic curves, evaluated using the mirror symmetry of the topological string theories, should be equal to the classical limit of quantum $\mathcal{A}$-polynomials from the knot theory. This provided a task to find the unknown quantum $\mathcal{A}$-polynomials for the simplest links.

[1] M.Aganagic, T.Ekholm, L.Ng and C.Vafa, arXiv:1304.5778

Chern-Simons theory

- 3-dimensional topological gauge theory - Chern-Simons theory
- $S_{CS} = \frac{k}{4\pi}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$
- Wilson-loop averages:

$$H_Q^{\mathcal{K}}(A,q) = \left\langle \mathrm{Tr}_{Q_1} Pexp\left(\oint_{\mathcal{K}_1}\mathcal{A}\right)\ldots \mathrm{Tr}_{Q_n} Pexp\left(\oint_{\mathcal{K}_n}\mathcal{A}\right)\right\rangle_{CS(N,q)}$$

$Q_1\ldots Q_n$ are representations of the group $SU(N)$

$\mathcal{K} = \mathcal{K}_1\cup\ldots\cup\mathcal{K}_n$ is a collection of disjoint contours (i.e. a link)

$q = exp(\frac{2\pi i}{k+N}) \qquad A = q^N$

$\mathcal{A}$-polynomial for Unknot

Using the exact answers one can find recursive relations on the HOMFLY.
In the case of unknot $\mathcal{H}_r = S_r^* = \prod_{j=0}^{r-1} \frac{D_j}{\{q^{r-j}\}}$
where following notations are used $\{x\} = (x - x^{-1})$, $D_i = \{Aq^i\}$
Thus there exist recursive relation

$$\{q^r\}\mathcal{H}_r^{\mathsf{U}} = D_{r-1}\mathcal{H}_{r-1}^{\mathsf{U}}$$

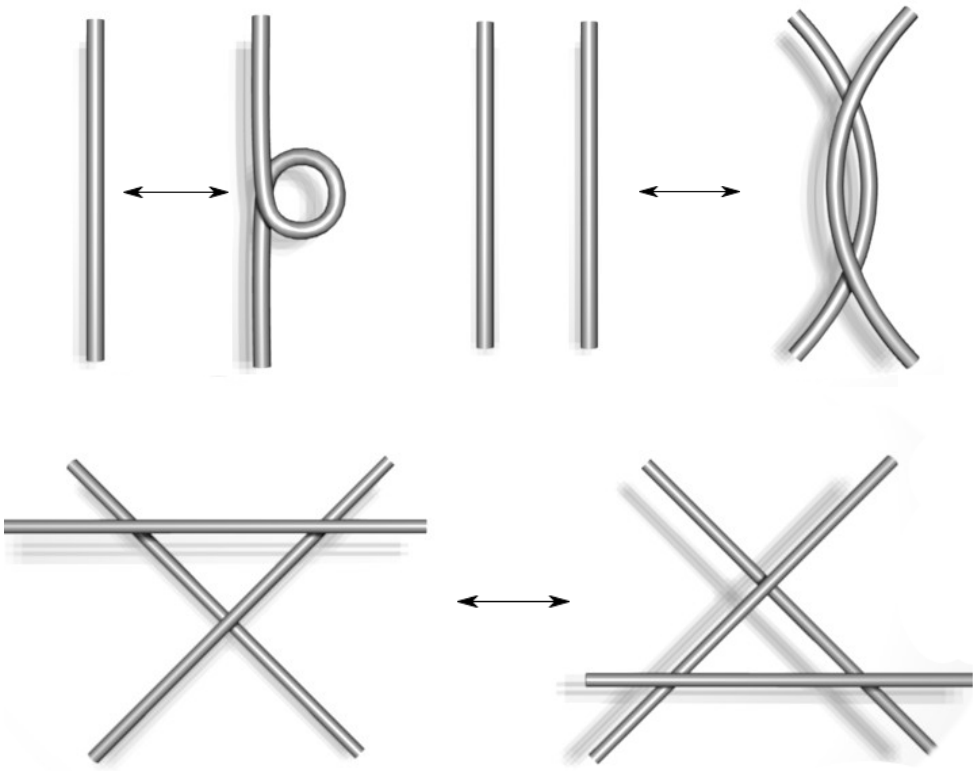
If two operators are introduced: $\hat{Q}\mathcal{H}_r = q^r\mathcal{H}_r, \quad \hat{P}\mathcal{H}_r = \mathcal{H}_{r-1}$
Then $\mathcal{A}$-polynomial is given by

$$\hat{\mathcal{A}}^{\mathsf{U}} = \hat{Q}^2(1 - A\hat{P}) - (1 - A^{-1}\hat{P}), \quad \hat{\mathcal{A}}^{\mathsf{U}}\mathcal{H}_r^{\mathsf{U}} = 0$$

The classical limit is given by $\hat{Q}^2 = \mu$, $\hat{P} = z$:
$A^{\mathsf{U}} = \mu(1 - Az) - (1 - A^{-1}z)$

Yang-Baxter Equation

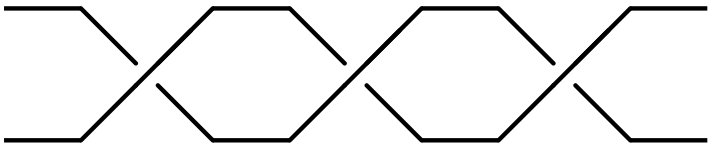
HOMFLY polynomials satisfy three Reidemeister moves



The third Reidemeister move is the Yang-Baxter equation which solutions are $\mathcal{R}$-matrices.

Braid representation

To describe the knot it is convenient to use its braid representation:



Then in the Reshetikhin-Turaev formalism[2] for each crossing the $\mathcal{R}$-matrix should be inserted and the answer for the HOMFLY polynomials then is given by the trace over the product of all the $\mathcal{R}$-matrices:

$$H_Q^{\mathcal{K}} = Tr_{\bigotimes\limits_{j=1}^{n} Q_j} \prod_i \mathcal{R}_i = Tr_{Q_1 \otimes Q_2} \prod_i \mathcal{R}_i$$

[2] N.Yu.Reshetikhin and V.G.Turaev, Comm. Math. Phys. **127** (1990) 1-26

Character Expansion

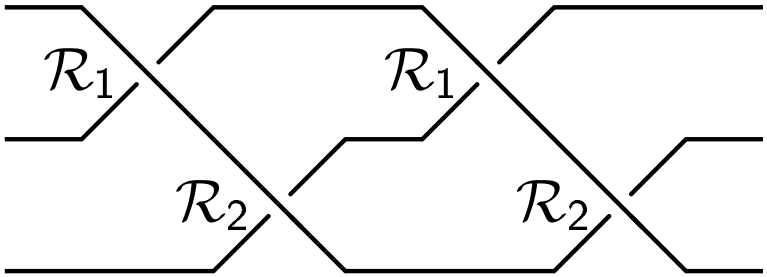
The eigenvectors of these $\mathcal{R}$-matrices with the different eigenvalues correspond to the different irreducible representations from the decomposition of the $Q^{\otimes m}$. Thus instead of using the $\mathcal{R}$-matrices acting on the vectors one could use one for the irreducible representations. Then the answer for the Knot polynomial will be an expansion into the Schur-functions (characters of different irreducible representations of the studied group).

$$H_Q^{\mathcal{K}}(A|q) = \sum_{T \vdash \bigotimes_{j=1}^{n} Q_j} h_Q^T S_T^*(A,q)$$

Coefficients h_Q^T are given by the trace of the product of the blocks in $\mathcal{R}$-matrices corresponding to the representation Q

$\mathcal{R}$-matrices

For the braid with m strands there are $m-1$ different $\mathcal{R}$ matrices corresponding to the crossings between different pairs of strands.



$\mathcal{R}_1 = \mathcal{R}_0 \otimes I \qquad \mathcal{R}_2 = I \otimes \mathcal{R}_0$

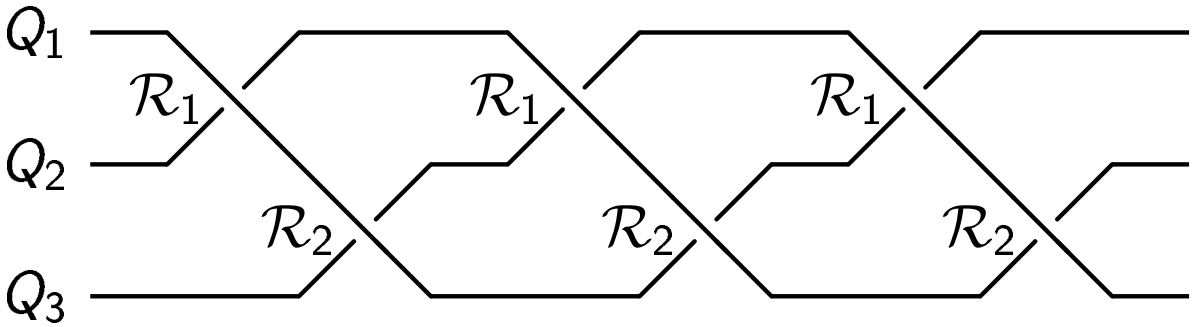
All of them have the same eigenvalues but they are not the same ones.

There exist some transition matrices which relate different $\mathcal{R}$-matrices.

$$\mathcal{R}_2 = U\mathcal{R}_1 U^+$$

Eigenvalue conjecture

In the case when $Q_1 = Q_2 = Q_3$ the eigenvalue conjecture implies that U-matrices have a certain form.



$$U = \begin{pmatrix} \frac{\sqrt{-\xi_1\xi_2}}{\xi_1-\xi_2} & \frac{\sqrt{\xi_1^2-\xi_1\xi_2+\xi_2^2}}{\xi_1-\xi_2} \\ -\frac{\sqrt{\xi_1^2-\xi_1\xi_2+\xi_2^2}}{\xi_1-\xi_2} & \frac{\sqrt{-\xi_1\xi_2}}{\xi_1-\xi_2} \end{pmatrix}$$

$\xi_i = \frac{\lambda_i}{\sqrt{\lambda_1\lambda_2}}$ are normalized eigenvalues of the corresponding $\mathcal{R}$-matrix

Similar formulae can be written for the blocks of other sizes.

Eigenvalue conjecture

Similar formulae can be written for the blocks of other sizes.

$$U = \begin{pmatrix} -\frac{\xi_1(\xi_2+\xi_3)}{\xi_{12}\xi_{13}} & -\frac{1}{\xi_{12}}\sqrt{\frac{(\xi_1^3+1)(\xi_2^3+1)}{\xi_1\xi_2\xi_{13}\xi_{23}}} & -\frac{1}{\xi_{13}}\sqrt{\frac{(\xi_1^3+1)(\xi_3^3+1)}{\xi_1\xi_3\xi_{12}\xi_{32}}} \\ \frac{1}{\xi_{12}}\sqrt{\frac{(\xi_1^3+1)(\xi_2^3+1)}{\xi_1\xi_2\xi_{13}\xi_{23}}} & \frac{\xi_2(\xi_1+\xi_3)}{\xi_{12}\xi_{23}} & \frac{1}{\xi_{23}}\sqrt{\frac{(\xi_2^3+1)(\xi_3^3+1)}{\xi_2\xi_3\xi_{12}\xi_{13}}} \\ -\frac{1}{\xi_{13}}\sqrt{\frac{(\xi_1^3+1)(\xi_3^3+1)}{\xi_1\xi_3\xi_{12}\xi_{32}}} & -\frac{1}{\xi_{23}}\sqrt{\frac{(\xi_2^3+1)(\xi_3^3+1)}{\xi_2\xi_3\xi_{12}\xi_{13}}} & -\frac{\xi_3(\xi_1+\xi_2)}{\xi_{13}\xi_{23}} \end{pmatrix}$$

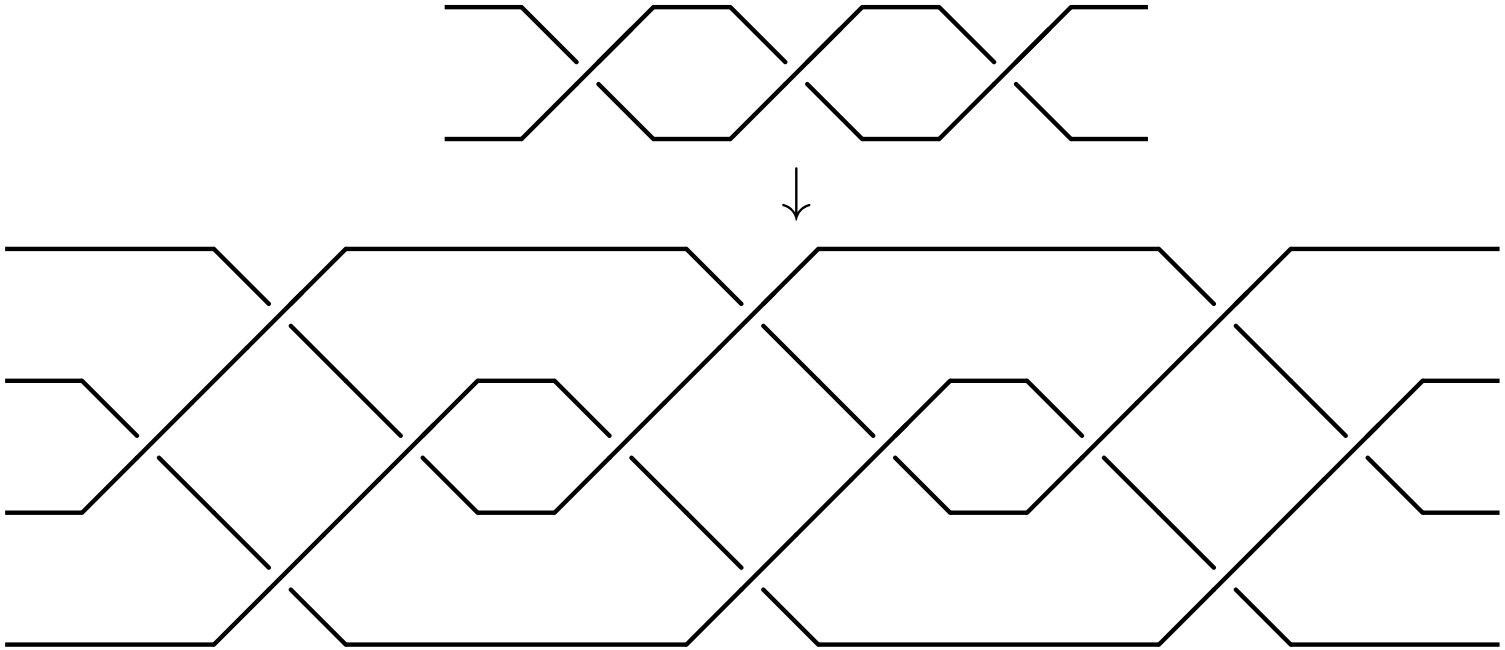
This method allows us to evaluate link HOMFLY polynomials for symmetrical representations $Q_1 = Q_2 = Q_3 \leq 4$

Cabling procedure

The main idea behind the cabling procedure is that

$$H^{\mathcal{K}}_{Q^{\otimes m}} = H^{\mathcal{K}^m}_{Q}$$

$\mathcal{K}^m$ is a knot $\mathcal{K}$ were the strand is replaced with the m parallel strands:



Projectors

The cabling of the knot gives the answer in the reducible representation $Q^{\otimes m}$ therefore to find the answer in some other irreducible or reducible representation T the operator called projector $P_T^{Q^{\otimes m}}$ should be constructed.

These projectors can be described as a combination of the added crossings between the m strands in the cable.
In other words the knot polynomials (Wilson-loop average) in the representation T can be represented as a linear combination of the knot polynomials in the fundamental representations but for the knots with $|T|$ times more strands.

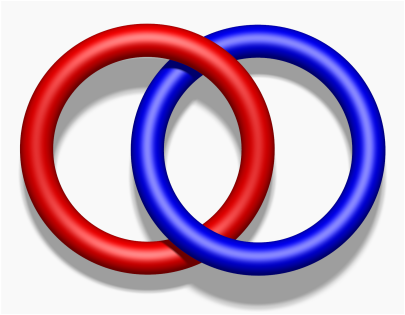
Cabling procedure

The answers for the fundamental R-matrices and projectors on any representations are known for any braid.
Due to the complicated calculations with the large matrices we could only provide the answers in the cases when $|Q_1| + |Q_2| + |Q_3| \leq 12$.

Important difference from the eigenvalue conjecture is that in this case we can study all the cases, even when representations are not the same.

Answers for the link polynomials

These methods allows one to construct from the known examples the whole answer for the symmetrical representations in the form of hypergeometric series:



Hopf link

$$H^{\mathrm{H}}_{r,s} = 1 + \sum_{k=1}^{\min(r,s)} (-1)^k A^{-k} q^{-k(r+s)+k(k+3)/2} \prod_{j=0}^{k-1} \frac{\{q^{r-j}\}\{q^{s-j}\}}{D_j}$$

Answers for the link polynomials

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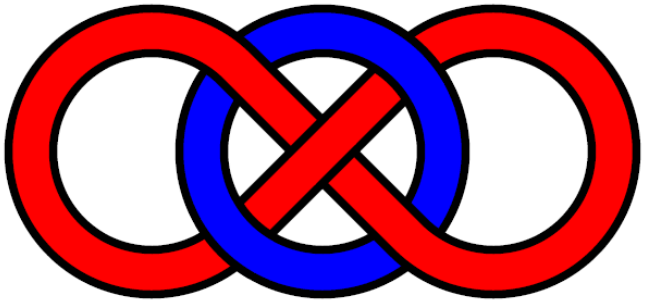

Whitehead link

$$H^{W}_{r,s}(A,q) = 1 + \sum_{k=1}^{\min(r,s)} \frac{1}{A^k q^{k(k-1)/2}} \frac{D_{-1}}{D_{k-1}} \prod_{j=0}^{k-1} \frac{D_{r+j} D_{s+j}}{D_{k+j}} \{q^{r-j}\}\{q^{s-j}\}$$

Answers for the link polynomials

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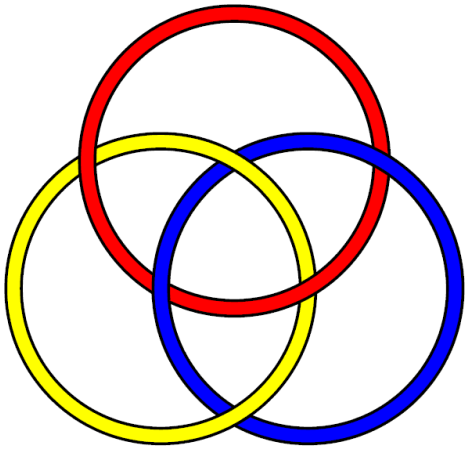


Borromean rings

$$H^{\mathrm{B}}_{r,s,t}(A,q) = 1 + D_{-1}\sum_{k=1}^{\min(r,s,t)}(-)^k\{q\}^k\,[k]!\,\frac{D_{k-2}!}{\left(D_{2k-1}!\right)^2}$$
$$\prod_{j=0}^{k-1} D_{r+j}D_{s+j}D_{t+j}\{q^{r-j}\}\{q^{s-j}\}\{q^{t-j}\}$$

Recursive relations for Hopf link

From the answer

$$H^{\mathsf{H}}_{r,s} = 1 + \sum_{k=1}^{\min(r,s)} (-1)^k A^{-k} q^{-k(r+s)+k(k+3)/2} \prod_{j=0}^{k-1} \frac{\{q^{r-j}\}\{q^{s-j}\}}{D_j}$$

several recursive relations can be constructed

$$\mathcal{H}^{\mathsf{H}}_{r,s} - \frac{q^{1-2r}}{A}\mathcal{H}^{\mathsf{H}}_{r,s-1} = S^*_r\left(S^*_s - \frac{q}{A}S^*_{s-1}\right)$$

$$H^{\mathsf{H}}_{r,s}(A\,|\,q) \;=\; 1 \;-\; \frac{\{q^r\}\{q^s\}}{Aq^{r+s-2}D_0}H^{\mathsf{H}}_{r-1,s-1}(Aq\,|\,q)$$

$$\{q^s\}\left(\mathcal{H}^{\mathsf{H}}_{r,s} - \frac{q^{1-2r}}{A}\mathcal{H}^{\mathsf{H}}_{r,s-1}\right) =$$

$$= qD_{s-2}\left(\mathcal{H}^{\mathsf{H}}_{r,s-1} - \frac{q^{1-2r}}{A}\mathcal{H}^{\mathsf{H}}_{r,s-2}\right)$$

$\mathcal{A}$-polynomial for Hopf link

The needed recursive relation for the $\mathcal{A}$-polynomial is

$$\mathcal{H}^{\mathsf{H}}_{r,s} - \frac{q^{1-2r}}{A}\mathcal{H}^{\mathsf{H}}_{r,s-1} = S^*_r\left(S^*_s - \frac{q}{A}S^*_{s-1}\right)$$

This relation can be described by quantum $\mathcal{A}$-polynomial,

$$\mathcal{A}^{\mathsf{H}} = \hat{Q}^2_r - \frac{q}{A}\hat{P}_s$$

which classical limit is

$$A^{\mathsf{H}} = \mu_r - \frac{1}{A}z_s$$

$\mathcal{A}$-polynomial for Whitehead link

The recursive relation is of the 4-th order with A-polynomial

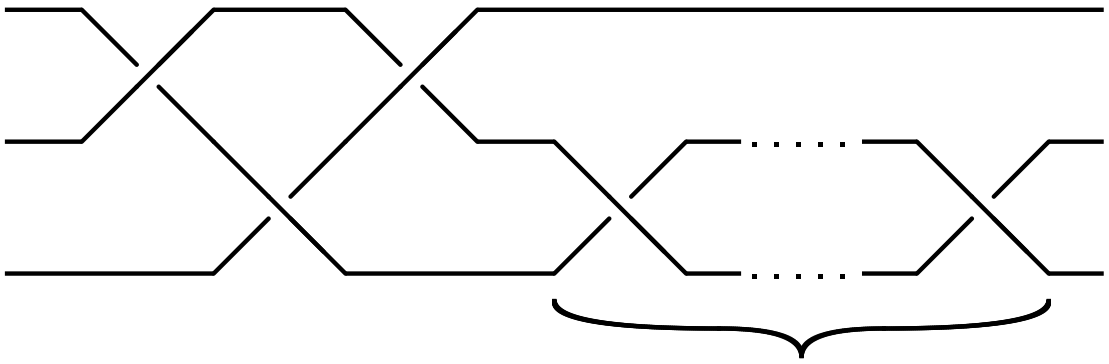
$$A^{\mathrm{W}} = \mu_s^2\mu_r A^4 + \Big(-\mu_r^2\mu_s^2 A^4 + (\mu_r^2 - \mu_r\mu_s + 2\mu_r - \mu_s)\mu_s A^2 + (\mu_s - \mu_r)\Big) A z_s +$$

$$+\Big((\mu_r\mu_s + \mu_s - 2\mu_r)\mu_r\mu_s A^4 + (\mu_r^2 - 4\mu_r\mu_s + \mu_s^2 + \mu_r - 2\mu_s)A^2 + 1\Big) z_s^2 +$$

$$+\Big((\mu_r\mu_s - \mu_s^2 + 2\mu_s - \mu_r)\mu_r A^2 + (\mu_s - \mu_r - 1)\Big) A z_s^3 + A^2\mu_r z_s^4 = 0$$

Evolution method

Knowing answers for several simplest links one can find the answers for the whole series of links



$k = 2m$ crossings

The answer for the whole series has a form

$$H^k_{[r],[1]} = aq^{kr} + bq^{-k}$$

Evolution method

We know the answers for several links from these series:

Hopf: $$H^{k=0}_{[r],[1]} = 1 - \frac{1}{Aq^{r-1}}\frac{\{q^r\}\{q\}}{D_0}$$

Whitehead: $$H^{k=2}_{[r],[1]} = 1 + \frac{\{q^r\}\{q\}D_r D_{-1}}{AD_0}$$

Thus the general answer is provided by

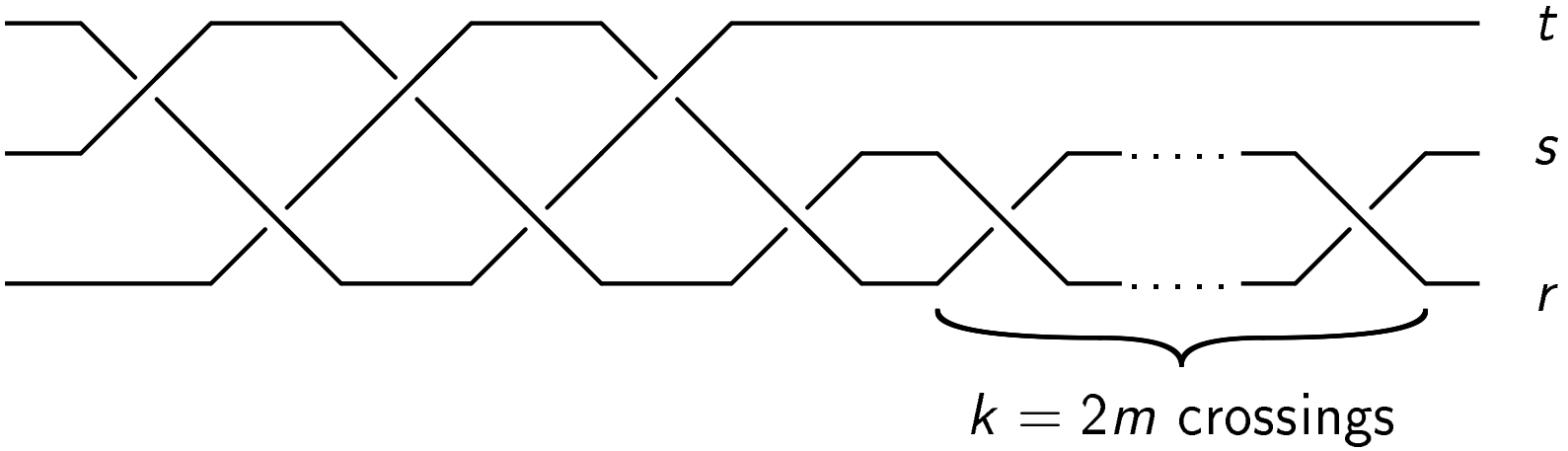
$$a = \frac{\{Aq^r\}\{q\}}{A^2 q^{2r-2}\{q^{r+1}\}\{A\}}(q^{2r-2}A^2 - q^{2r} + 1)$$

$$b = \frac{\{A/q\}\{q^r\}}{A^2 q^{2r}\{q^{r+1}\}\{A\}}(q^{2r}A^2 + q^2 - 1)$$

The same procedure can be applied also to other representations.

Evolution method

The same procedure can be applied to the series starting fron the Borromean rings link



The important difference is that the links of these series are symmetric only between representations r and s.

Conclusion

- The HOMFLY polynomials in any symmetric representations for the Hopf, Whitehead and Borromean rings links were constructed
- The recursive relations on these polynomials were constructed and quantum $\mathcal{A}$-polynomials were found
- Using the Evolution method general answers for series of links starting from Hopf link and Borromean rings link have been constructed

THANK YOU FOR YOUR ATTENTION!

Answers for the projectors

The general answer for the projectors is known in the matrix form.

The answer which describes the connections between the colored and the fundamental knot polynomials is known only in the several simplest examples.

$$P_2 = -\frac{R_1 + q^{-1}}{q + q^{-1}} \qquad P_{11} = \frac{R_1 - q}{q + q^{-1}}$$

There is also a suggestion of the recursive procedure which in principle should be able to construct any projector.

Diagonal $\mathcal{R}$-matrices

For the crossing between representations T_1 and T_2 the eigenvalues are described by the irreducible representations in the decomposition $Q_i \vdash T_1 \otimes T_2$.

$$|\lambda_i| = q^{\varkappa_{Q_i}}, \quad \varkappa_{Q_i} = \frac{1}{2} \sum_{\{i,j\} \in Q_i} (i-j)$$

If there are more than two strands then the following decomposition should be studied:

$$T_1 \otimes T_2 \otimes .. \otimes T_m = (\sum_i Q_i) \otimes .. \otimes T_m = \sum_j \bar{Q}_j$$

The eigenvalues for the $\bar{Q}_j$ are the same ones as for the corresponding Q_i

Non-diagonal $\mathcal{R}$-matrices

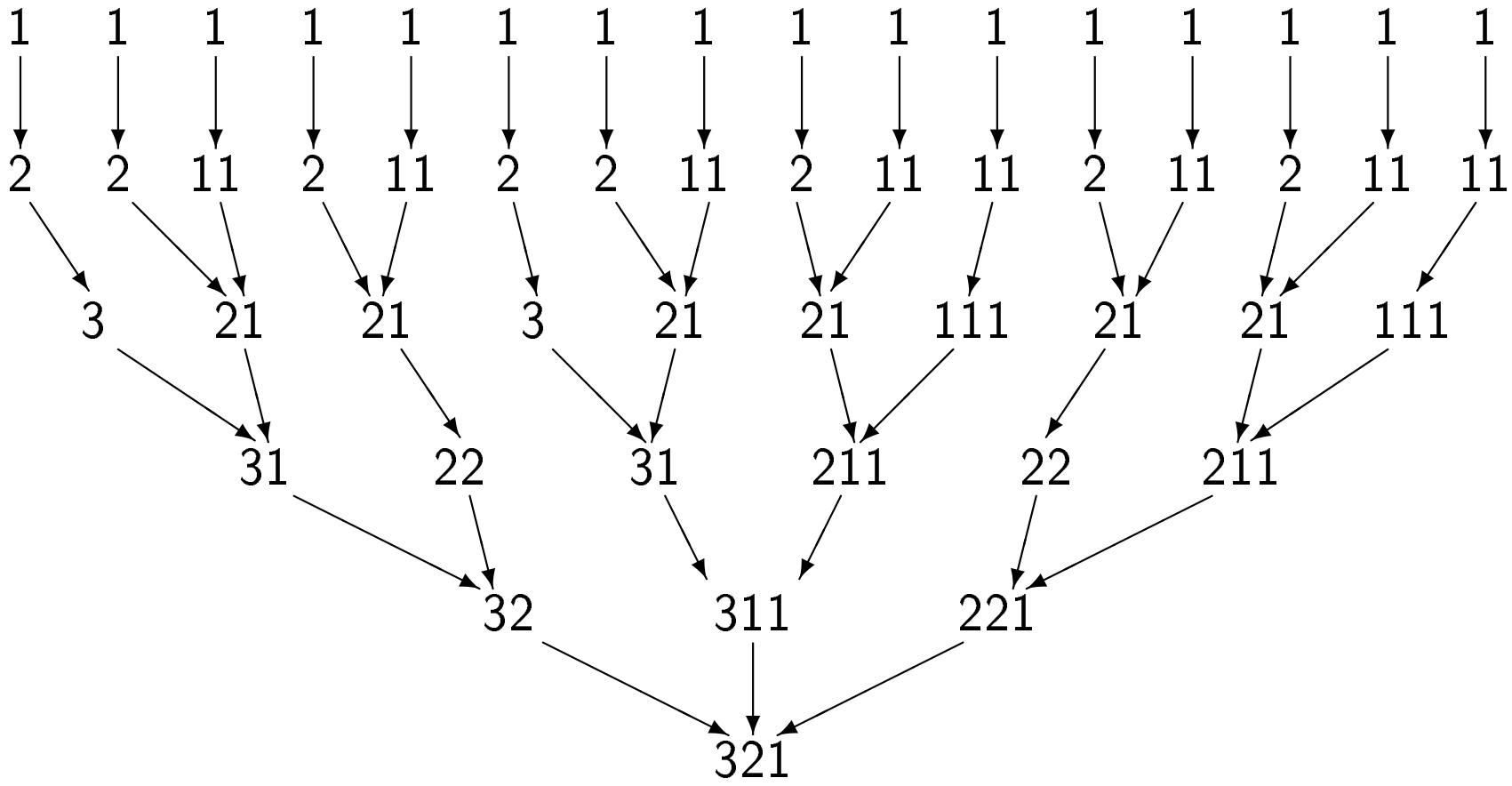
The general formula is known only for the fundamental representations. The matrix is block-diagonal with blocks 1×1 equal to q or $-q^{-1}$ and 2×2 equal to the

$$b_j = \begin{pmatrix} -\frac{1}{q^j [j]_q} & \frac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} \\ \frac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} & \frac{q^j}{[j]_q} \end{pmatrix}$$

Where $[j]_q$ is the quantum j: $[j]_q \equiv \frac{q^j - q^{-j}}{q - q^{-1}}$.

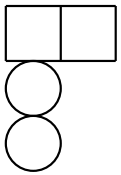
Then the paths on the tree of the diagrams should be considered

Tree of the diagrams

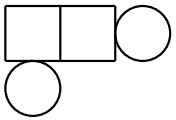


Non-diagonal $\mathcal{R}$-matrices

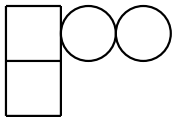
$\mathcal{R}_{k-1}$ is described by the level k in the tree. The paths can go as singlets or doublets. In doublets two paths differ only on the level k



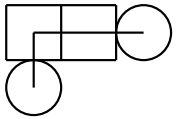
$2 \to 211$
$-q^{-1}$ in R



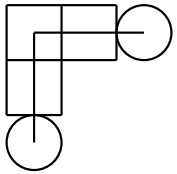
$2 \to 31$
b_3 block in R



$11 \to 31$
q in R



$2 \to 31$
b_3 block in R



$21 \to 311$
b_4 block in R

If the length of a hook, connecting the two added cells is equal to j then block b_{j-1} should be used

Non-diagonal $\mathcal{R}$-matrices

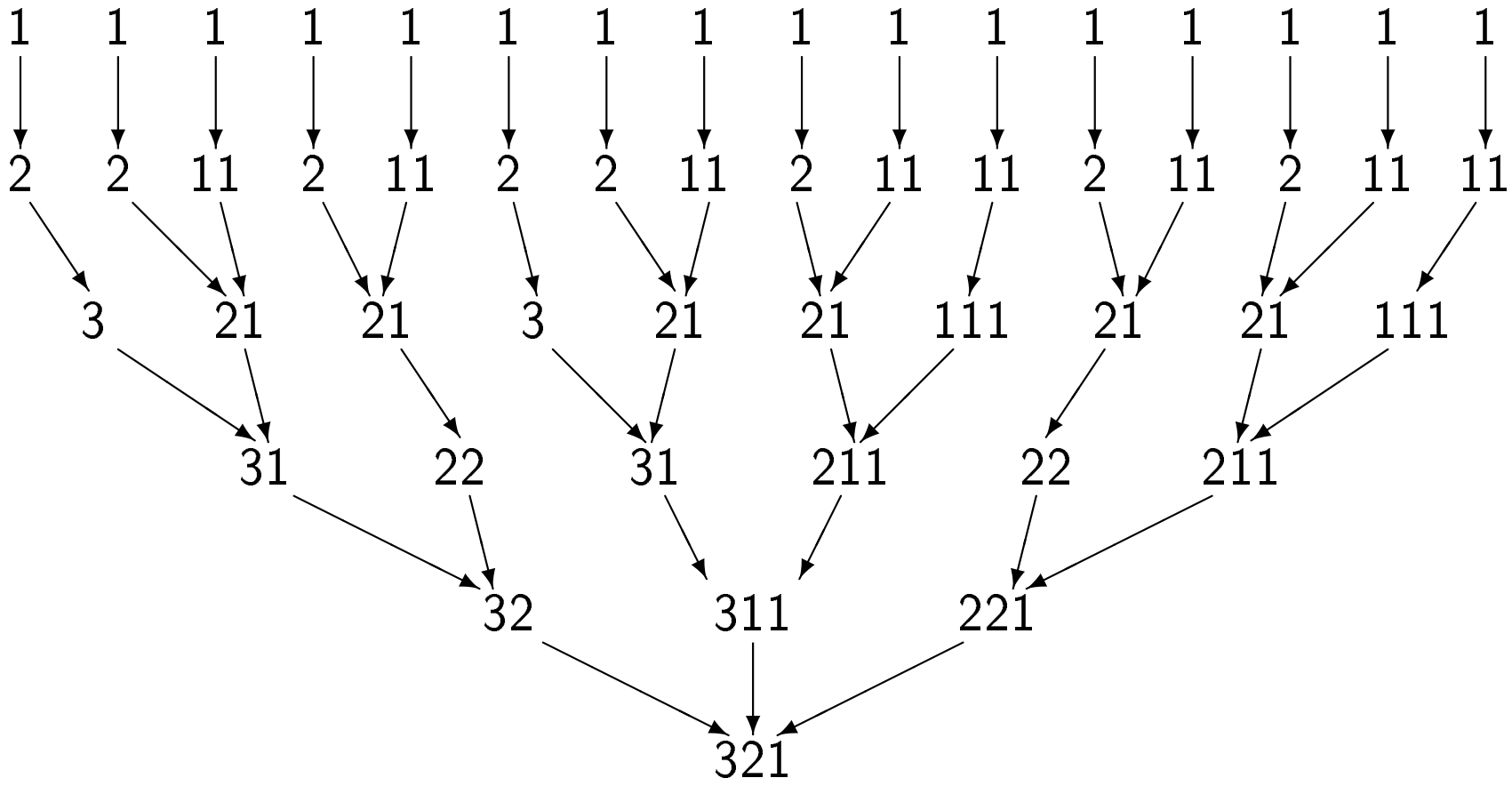
$$R_{3|311} = \begin{pmatrix} -\frac{1}{[3]_q q^3} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & & & & \\ \frac{\sqrt{[2]_q[4]_q}}{[2]_q} & \frac{q^3}{[3]_q} & & & & \\ & & q & & & \\ & & & -q^{-1} & & \\ & & & & -\frac{1}{[3]_q q^3} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} \\ & & & & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & \frac{q^3}{[3]_q} \end{pmatrix}$$

Path Projectors

Projectors can be constructed n the same manner as the $\mathcal{R}$-matrices. For the P_Q in the tree only the paths which pass through the Q should remain.

$$P_{2\otimes 1^3|311} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Representation tree



Characteristic equations

The definition using the paths is very constructive but does not provide the connections between the colored HOMFLY and the fundamental ones. To find these connections the definition which uses the R-matrices should be constructed.

This can be done using the operator properties of the R-matrices:

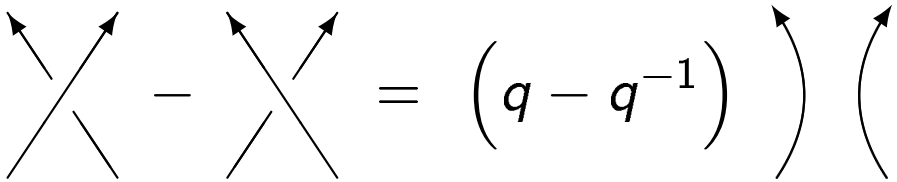
The projector on eigenvalue λ_i of the operator with the characteristic equation $\prod_{i=1}^{n}(\hat{R}-\lambda_i)=0$ is

$$\hat{P}_{\lambda_j}=\prod_{i\neq j}\frac{\hat{R}-\lambda_i}{\lambda_j-\lambda_i}$$

Characteristic equations for the R-matrices

For the fundamental R-matrices the characteristic equation is known from the mathematics and is described by the skein relations:

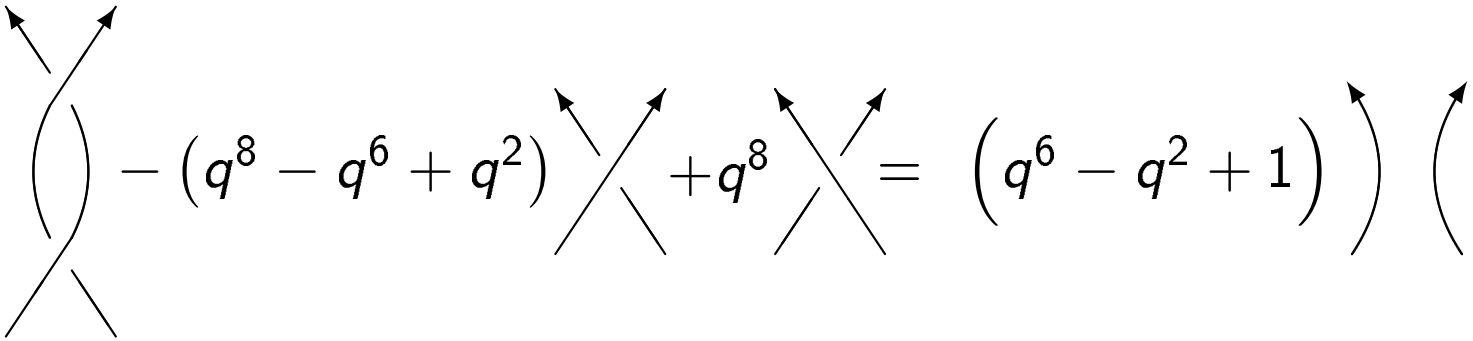
$$R - R^{-1} - \left(q - q^{-1}\right) = (R - q)\left(R + \frac{1}{q}\right) = 0$$



Colored skein relations

From the characteristic equation the analogue of the skein relations for the colored $\mathcal{R}$-matrices can be constructed though it is not very useful:

$$\left(\mathcal{R}_{2\otimes 2} - q^6\right)\left(\mathcal{R}_{2\otimes 2} + q^2\right)\left(\mathcal{R}_{2\otimes 2} - 1\right) = 0$$



The projectors on [2] and [11]

From the characteristic equation the projectors can be constructed

$$(R-q)\left(R+\frac{1}{q}\right)=0$$

$$\Downarrow$$

$$P_2=-\frac{R_1+q^{-1}}{q+q^{-1}} \qquad P_{11}=\frac{R_1-q}{q+q^{-1}}$$

The projectors on [21]

There are also additional characteristic equations for the combinations of the R-matrices:

$$(R_1 - R_2)\Big((R_1 - R_2)^2 - (q^2 + 1 + q^{-2})\Big) = 0$$

Together with the characteristic equations on each of the R-matrices this gives

$$P_{21} = \frac{(R_1 - R_2)^2}{q^2 + 1 + q^{-2}}$$

Recursive formula for the projector

$$B_{|Q|} = \left(\prod_{i=1}^{|Q|} R_i\right)\left(\prod_{j=1}^{|Q|} R_{|Q|+1-j}\right)$$

The projector on the representation Q made from T with the addition of the cell $\{k, l\}$ is equal to

$$P_{Q=T\cup(k,l)} = \sum_i \prod_{(i,j)\neq(k,l)} \frac{B_{|Q|} - q^{2j-2i}}{q^{2l-2k} - q^{2j-2i}} P_T$$

The product is over all possible additions of cells $\{i, j\}$.