Conformal algebra: R-matrix and star-triangle relations.

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Plan

1. **Introduction.**
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   - Conformal algebra $conf(\mathbb{R}^{p,q})$
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Star-triangle relation (STR)

To evaluate multi-loop Feynman integrals we have to consider integral

$$\int \frac{d^D z}{(x-z)^{2\alpha}\, z^{2\beta}\, (z-y)^{2\gamma}} \; .$$

where $x, y, z \in \mathbb{R}^D$, $x^{2\beta} = (x_\mu x^\mu)^\beta$.
Interesting special case $\alpha + \beta + \gamma = D$ firstly considered in CFT (see e.g. E.S.Fradkin, M.Ya.Palchik, Phys. Rep. 1978)

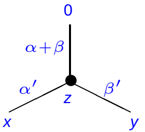
$$\int \frac{d^D z}{(x-z)^{2\alpha'}\, z^{2(\alpha+\beta)}\, (z-y)^{2\beta'}} = \frac{G(\alpha,\beta)}{(x)^{2\beta}\, (x-y)^{2(\frac{D}{2}-\alpha-\beta)}\, (y)^{2\alpha}} \; ,$$

where parameters $\boxed{\alpha' := \frac{D}{2} - \alpha}$, $\Rightarrow \alpha' + \alpha + \beta + \beta' = D$,

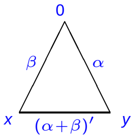
$$G(\alpha,\beta) = \frac{a(\alpha+\beta)}{a(\alpha)a(\beta)} \; , \quad a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2}\, 2^{2\beta}\, \Gamma(\beta)} \; .$$

Graphic representation of Star Triangle Relation (reconstruction of Feynman graphs):

$$x \overset{\alpha}{\text{———}} y \;=\; \frac{1}{(x-y)^{2\alpha}} \quad \Rightarrow$$



$$= G(\alpha,\beta)\cdot$$



Operator version of STR: (API, 2003)

$$\boxed{\hat{p}^{-2\alpha}\cdot\hat{q}^{-2(\alpha+\beta)}\cdot\hat{p}^{-2\beta} = \hat{q}^{-2\beta}\cdot\hat{p}^{-2(\alpha+\beta)}\cdot\hat{q}^{-2\alpha}} \quad !!!$$

where we have used Heisenberg algebra: $[\hat{q}_\mu,\ \hat{p}_\nu] = \delta_{\mu\nu}$.

Proof.

$$\langle x|\hat{p}^{-2\alpha}\cdot\hat{q}^{-2(\alpha+\beta)}\cdot\hat{p}^{-2\beta}|y\rangle = \langle x|\hat{q}^{-2\beta}\cdot\hat{p}^{-2(\alpha+\beta)}\cdot\hat{q}^{-2\alpha}|y\rangle$$

$$\langle x|\hat{p}^{-2\alpha}|y\rangle = a(\alpha)\,(x-y)^{-2\alpha'}\ .$$

Any STR is related to a solution $\mathcal{R}$ of the Yang-Baxter equation

$$\mathcal{R}_{12}(u)\,\mathcal{R}_{23}(u+v)\,\mathcal{R}_{12}(v) = \mathcal{R}_{23}(v)\,\mathcal{R}_{12}(u+v)\,\mathcal{R}_{23}(u)$$

u, v — are spectral parameters (YBE $\Rightarrow$ integrable models, e.g. Zamolodchikov's "Fishnet" diagram IM).

Our aim is to find $\mathcal{R}$ which corresponds to the operator STR:

$$\hat{p}^{\,2u} \cdot q^{2(u+v)} \cdot \hat{p}^{\,2v} = q^{2v} \cdot \hat{p}^{\,2(u+v)} \cdot q^{2u} \,. \qquad (1)$$

Eq. (1) can be written in two equivalent forms

$$\hat{p}_2^{2u} \cdot q_{12}^{2(u+v)} \cdot \hat{p}_2^{2v} = q_{12}^{2v} \cdot \hat{p}_2^{2(u+v)} \cdot q_{12}^{2v} \qquad (1 \leftrightarrow 2) \,,$$

where $[q_k^\mu, \hat{p}_j^\nu] = i\delta_{kj}\delta^{\mu\nu}$ and $q_{12}^\mu = q_1^\mu - q_2^\mu$. Then one can prove that $\mathcal{R}$-operator

$$\underline{\mathcal{R}_{12}(u-v) = q_{12}^{2(u_- - v_+)} \cdot \hat{p}_2^{2(u_+ - v_+)} \cdot \hat{p}_1^{2(u_- - v_-)} \cdot q_{12}^{2(u_+ - v_-)}} \in \mathrm{End}(V_{\Delta_1} \otimes V_{\Delta_2})$$

where V_Δ is the space of conformal fields with conf. dimension Δ and

$$u_+ = u + \frac{\Delta_1 - D}{2} \,, \;\; u_- = u - \frac{\Delta_1}{2} \,, \;\; v_+ = v + \frac{\Delta_2 - D}{2} \,, \;\; v_- = v - \frac{\Delta_2}{2} \,,$$

satisfies YB equation.

For $\Delta_1 = \Delta_2 = \Delta$ the operator $\mathcal{R}_{12}$ is

$$R_{ab}(\alpha;\xi) := (\hat{q}_{(ab)})^{2(\alpha+\xi)}(\hat{p}_{(a)})^{2\alpha}(\hat{p}_{(b)})^{2\alpha}(\hat{q}_{(ab)})^{2(\alpha-\xi)} = \\ = 1 + \alpha\, h_{(ab)}(\xi) + \alpha^2 \dots ,$$

where $\alpha = u - v$, $\xi = \frac{D}{2} - \Delta$ and Hamiltonian densities $h_{(ab)}(x)$ are

$$h_{(ab)}(\xi) = 2\,\ln(\hat{q}_{(ab)})^2 + (\hat{q}_{(ab)})^{2\xi}\,\ln(\hat{p}^{\,2}_{(a)}\,\hat{p}^{\,2}_{(b)})\,(\hat{q}_{(ab)})^{-2\xi} = \\ = \hat{p}^{-2\xi}_{(a)}\,\ln(\hat{q}_{(ab)})^2\,\hat{p}^{2\xi}_{(a)} + \hat{p}^{-2\xi}_{(b)}\,\ln(\hat{q}_{(ab)})^2\,\hat{p}^{2\xi}_{(b)} + \ln(\hat{p}^{\,2}_{(a)}\,\hat{p}^{\,2}_{(b)}) \, .$$

Using the standard procedure one can construct an integrable system with Hamiltonian $H(\xi) = \sum_{a=1}^{N-1} h_{(a,a+1)}(\xi)$. For $D = 1$ and $\xi = 1/2$ this Hamiltonian reproduces the Hamiltonian for the Lipatov's integrable model which is related to BFKL equation.

The $\mathcal{R}$-operator acts in the tensor product of two representation spaces of conformal algebra $\mathrm{conf}(\mathbb{R}^D) = so(D+1,1)$

$$\Phi_{\Delta_1}(x_1) \otimes \Phi_{\Delta_2}(x_2) \in V_{\Delta_1} \otimes V_{\Delta_2} \; ,$$

where $\Phi_\Delta(x)$ are spinless fields with conformal dimension Δ.
The meaning of $\mathcal{R}$: it intertwines two representations

$$\mathcal{R}_{12}(u-v) : \quad V_{\Delta_1} \otimes V_{\Delta_2} \to V_{\Delta_2} \otimes V_{\Delta_1} \; .$$

or

$$\mathcal{R}_{12}(u-v) \cdot A_{\Delta_1} \otimes B_{\Delta_2} = B'_{\Delta_2} \otimes A'_{\Delta_1} \cdot \mathcal{R}_{12}(u-v) \; .$$

where $A_\Delta \in \mathrm{End}(V_\Delta)$.

To demonstrate this we construct L-operator (quantum analog of a Lax operator — another important object in quantum integr. models)

$$||(\mathrm{L}^{(\Delta)})^{\alpha}_{\beta}|| = \mathrm{L}^{(\Delta)} \quad : \quad V \otimes V_{\Delta} \to V \otimes V_{\Delta}$$

where V is the space of a matrix (e.g., spinor) representation T_s of $\mathrm{conf}(\mathbb{R}^D) \equiv \mathrm{conf}$. The L-operator satisfies *RLL* relations

$$\mathcal{R}_{23}(u-v)\,(\mathrm{L}_2^{(\Delta_1)})^{\alpha}_{\beta}(u)\,(\mathrm{L}_3^{(\Delta_2)})^{\beta}_{\gamma}(v) = (\mathrm{L}_2^{(\Delta_2)})^{\alpha}_{\beta}(v)\,(\mathrm{L}_3^{(\Delta_1)})^{\beta}_{\gamma}(u)\,\mathcal{R}_{23}(u-v)\;,$$

where

$$(\mathrm{L}_2^{(\Delta)})^{\alpha}_{\beta} = T_s(\mathcal{U}(\mathrm{conf}))^{\alpha}_{\beta} \otimes \rho_{\Delta}(\mathcal{U}(\mathrm{conf})) \otimes 1$$

$$(\mathrm{L}_3^{(\Delta)})^{\alpha}_{\beta} = T_s(\mathcal{U}(\mathrm{conf}))^{\alpha}_{\beta} \otimes 1 \otimes \rho_{\Delta}(\mathcal{U}(\mathrm{conf}))$$

$$\mathcal{R}_{23}(u-v) = 1 \otimes \rho_{\Delta_1}(\mathcal{U}(\mathrm{conf})) \otimes \rho_{\Delta_2}(\mathcal{U}(\mathrm{conf}))$$

and $\mathcal{U}(\mathrm{conf})$— enveloping algebra of conf. We will consider the general case of $\mathrm{conf}(\mathbb{R}^{p,q})\;,\quad p+q=D \Rightarrow n$.

$\mathbb{R}^{p,q}$ — pseudoeuclidean space with the metric

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q}) .$$

conf($\mathbb{R}^{p,q}$) — Lie algebra of the conformal group in $\mathbb{R}^{p,q}$ generated by $\{L_{\mu\nu}, P_\mu, K_\mu, D\}$ $(\mu,\nu = 0,1,\dots,p+q-1)$:

$$[L_{\mu\nu}\,,\,L_{\rho\sigma}] = i\,(g_{\nu\rho}\,L_{\mu\sigma} + g_{\mu\sigma}\,L_{\nu\rho} - g_{\mu\rho}\,L_{\nu\sigma} - g_{\nu\sigma}\,L_{\mu\rho})$$

$$[K_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,K_\nu - g_{\rho\nu}\,K_\mu)\ , \quad [P_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,P_\nu - g_{\rho\nu}\,P_\mu)\ ,$$

$$[D\,,\,P_\mu] = i\,P_\mu\ , \quad [D\,,\,K_\mu] = -i\,K_\mu\ ,$$

$$[K_\mu\,,\,P_\nu] = 2i\,(g_{\mu\nu}\,D - L_{\mu\nu})\ , \quad [P_\mu\,,\,P_\nu] = 0\ ,$$

$$[K_\mu\,,\,K_\nu] = 0\ , \quad [L_{\mu\nu}\,,\,D] = 0\ .$$

$L_{\mu\nu}$ – generators for the rotation group $SO(p,q)$ in $\mathbb{R}^{p,q}$,
P_ν – shift generators in $\mathbb{R}^{p,q}$,
D – dilatation operator,
K_ν – conformal boost generators.

We have the well known isomorphism:

$$\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$$

and on generators it looks like

$$L_{\mu\nu} = M_{\mu\nu} , \quad K_\mu = M_{n,\mu} - M_{n+1,\mu} ,$$
$$P_\mu = M_{n,\mu} + M_{n+1,\mu} , \quad D = -M_{n,n+1} , \quad (n = p + q) .$$

where M_{ab} $(a, b = 0, 1, \ldots, n+1)$ are generators of $so(p+1, q+1)$

$$[M_{ab}, M_{dc}] = i(g_{bd}M_{ac} + g_{ac}M_{bd} - g_{ad}M_{bc} - g_{bc}M_{ad}) ,$$
$$g_{ab} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, 1, -1) .$$

Then the quadratic Casimir operator for $\mathrm{conf}(\mathbb{R}^{p,q})$ is

$$C_2 = \frac{1}{2} M_{ab} M^{ab} = \frac{1}{2} (L_{\mu\nu} L^{\mu\nu} + P_\mu K^\mu + K_\mu P^\mu) - D^2 .$$

The first main result is that the explicit form of $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$-type L-operator which we search in the form:

$$\mathrm{L}(u_+, u_-) = u\,\mathrm{I} + \frac{1}{2} T_s(M^{ab}) \otimes \rho(M_{ab}) \ .$$

(looks like split Casimir operator for $o(p+1, q+1)$) where M_{ab} are generators of $so(p+1, q+1)$ and

$$\rho(M_{ab}) = y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a} \ .$$

Now we specify the spinor matrix representation T_s of the algebra $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$.

Spinor reps T_s of $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$

Let $n = p + q = 2\nu (= D)$ be even integer and γ_μ $(\mu = 0, ..., n-1)$ be $2^{\frac{n}{2}}$-dimensional gamma-matrices in $\mathbb{R}^{p,q}$:

$$\gamma_\mu \, \gamma_\nu + \gamma_\nu \, \gamma_\mu = 2 \, g_{\mu\nu} \, I \; ,$$

$$\gamma_{n+1} \equiv \alpha \, \gamma_0 \cdot \gamma_1 \cdots \gamma_{n-1} \; , \quad \alpha^2 = (-1)^{q+n(n-1)/2} = (-1)^{q-\nu} \; ,$$

where α is such that $\gamma_{n+1}^2 = I$. Using gamma-matrices γ_μ in $\mathbb{R}^{p,q}$ one can construct representation T_s of $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$

$$\boxed{\begin{array}{c} T_s(L_{\mu\nu}) = \frac{i}{4} \, [\gamma_\mu, \, \gamma_\nu] \equiv \ell_{\mu\nu} \; , \quad T_s(K_\mu) = \gamma_\mu \, \frac{(1-\gamma_{n+1})}{2} \equiv k_\mu \; , \\ T_s(P_\mu) = \gamma_\mu \, \frac{(1+\gamma_{n+1})}{2} \equiv p_\mu \; , \quad T_s(D) = -\frac{i}{2} \, \gamma_{n+1} \equiv d \; . \end{array}}$$

We choose the representation for γ_μ in $\mathbb{R}^{p,q}$ as:

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_\mu \\ \overline{\boldsymbol{\sigma}}_\mu & \mathbf{0} \end{pmatrix}, \quad \gamma_{n+1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

where $\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu + \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,, \quad \overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu + \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,.$

Thus, the representation T_s of $conf(\mathbb{R}^{p,q})$ is

$$\ell_{\mu\nu} = \begin{pmatrix} \frac{i}{4}(\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu - \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu) & \mathbf{0} \\ \mathbf{0} & \frac{i}{4}(\overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu - \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{\mu\nu} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix},$$

$$p^\mu = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \overline{\boldsymbol{\sigma}}^\mu & \mathbf{0} \end{pmatrix}, \quad k^\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^\mu \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad d = -\frac{i}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}.$$

Recall that

$$\boldsymbol{\sigma}_{\mu\nu} = ||(\boldsymbol{\sigma}_{\mu\nu})_\alpha{}^\beta||\,, \quad \overline{\boldsymbol{\sigma}}_{\mu\nu} = ||(\overline{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}||\,,$$

are inequivalent spinor representations of $so(p,q) = spin(p,q)$.

Any element of $conf(\mathbb{R}^{p,q})$ in the representation T_s is

$$A = i(\omega^{\mu\nu}\,\ell_{\mu\nu} + a^\mu\, p_\mu + b^\mu\, k_\mu + \beta\, d) =$$

$$= \begin{pmatrix} \frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\boldsymbol{\sigma}_{\mu\nu} & ib^\mu\boldsymbol{\sigma}_\mu \\ ia^\mu\overline{\boldsymbol{\sigma}}_\mu & -\frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} .$$

It can be considered as the matrix of parameters $\omega^{\mu\nu}$, a^μ, b^μ, $\beta \in \mathbb{R}$.

Further we will consider L-operator

$$\mathrm{L}^{(\Delta)}(u) = u\,\mathrm{I} + \frac{1}{2} T_s(M^{ab}) \otimes \rho_\Delta(M_{ab})$$

Diff. representation of $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$

The standard differential representation ρ_Δ of $conf(\mathbb{R}^{p,q})$ can be obtained by the method of induced representations (G. Mack and A. Salam (1969))

$$\rho_\Delta(P_\mu) = -i\partial_{x_\mu} \equiv \hat{p}_\mu \ , \quad \rho_\Delta(D) = x^\mu \hat{p}_\mu - i\Delta \ ,$$

$$\rho_\Delta(L_{\mu\nu}) = \hat{\ell}_{\mu\nu} + S_{\mu\nu} \ , \quad \rho_\Delta(K_\mu) = 2x^\nu\,(\hat{\ell}_{\nu\mu} + S_{\nu\mu}) + (x^\nu x_\nu)\hat{p}_\mu - 2i\Delta x_\mu \ ,$$

$$\hat{\ell}_{\mu\nu} \equiv (x_\nu \hat{p}_\mu - x_\mu \hat{p}_\nu) \ ,$$

where $x_\mu \equiv \hat{q}_\mu$ are coordinates in $\mathbb{R}^{p,q}$, $\Delta \in \mathbb{R}$ – conformal parameter, $S_{\mu\nu} = -S_{\nu\mu}$ are spin generators with commutation relations as for $\hat{\ell}_{\mu\nu}$ and $[S_{\mu\nu}, x_\rho] = 0 = [S_{\mu\nu}, \hat{p}_\rho]$. For the quadratic Casimir operator we have:

$$\rho_\Delta(C_2) = \frac{1}{2}\left(S_{\mu\nu}\,S^{\mu\nu} - \hat{\ell}_{\mu\nu}\,\hat{\ell}^{\mu\nu}\right) + \Delta(\Delta - n) \ .$$

The representations ρ_Δ and $\rho_{n-\Delta}$ are contragradient to each other and in particular we have $\rho_\Delta(C_2) = \rho_{n-\Delta}(C_2)$.

In the representation ρ elements of $conf(\mathbb{R}^{p,q})$ act on the fields $\Phi(\mathbf{x})$:

$$\rho(\omega^{\mu\nu} L_{\mu\nu} + a^\mu P_\mu + b^\mu K_\mu + \beta D)\, \Phi(\mathbf{x}) =$$

$$= \mathrm{Tr}_{T_s} \left[\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \left(T_s(M^{ab}) \cdot \rho(M_{ab}) \right) \right] \Phi(\mathbf{x}) \; .$$

where $\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$ is the matrix of parameters, and the matrix of generators is

$$\frac{1}{2} T_s(M^{ab}) \cdot \rho_\Delta(M_{ab}) = (T_s \otimes \rho) \left(\frac{1}{2} M^{ab} \otimes M_{ab} \right) =$$

$$= \begin{pmatrix} \frac{\Delta - n}{2} \cdot \mathbf{1} + \mathbf{S} - \mathbf{p} \cdot \mathbf{x} \, , & \mathbf{p} \\ \mathbf{x} \cdot \mathbf{S} - \overline{\mathbf{S}} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} + (\Delta - \frac{n}{2}) \cdot \mathbf{x} \, , & -\frac{\Delta}{2} \cdot \mathbf{1} + \overline{\mathbf{S}} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix} \, ,$$

Here we introduced

$$\mathbf{p} = \tfrac{1}{2} \boldsymbol{\sigma}^\mu \hat{p}_\mu = -\tfrac{i}{2} \boldsymbol{\sigma}^\mu \partial_{x_\mu} \, , \quad \mathbf{x} = -i \, \overline{\boldsymbol{\sigma}}^\mu x_\mu \, ,$$

$$\overline{\mathbf{S}} = \tfrac{1}{2} \overline{\boldsymbol{\sigma}}^{\mu\nu} S_{\mu\nu} \, , \quad \mathbf{S} = \tfrac{1}{2} \boldsymbol{\sigma}^{\mu\nu} S_{\mu\nu} \; .$$

The action of spin generators $S_{\mu\nu}$ on spin-tensor fields of the type $(\ell, \dot{\ell})$ is

$$[S_{\mu\nu}\Phi]^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} = (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_1}^{\ \ \alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha\alpha_2\cdots\alpha_{2\ell}} + \cdots + (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_{2\ell}}^{\ \ \alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell-1}\alpha} +$$
$$+ (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_1}_{\ \ \dot\alpha}\,\Phi^{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} + \cdots + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_{2\ell}}_{\ \ \dot\alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\ell-1}\dot\alpha}_{\alpha_1\cdots\alpha_{2\ell}}\ .$$

For symmetric representations it is convenient to work with the generating functions

$$\Phi(x,\lambda,\tilde\lambda) = \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}}(x)\,\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2\ell}}\,\tilde\lambda_{\dot\alpha_1}\cdots\tilde\lambda_{\dot\alpha_{2\dot\ell}}\ ,$$

where λ and $\tilde\lambda$ are auxiliary spinors and the action of $S_{\mu\nu}$ is given by differential operators over spinors $S_{\mu\nu} = \lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\overline{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda}$:

$$[S_{\mu\nu}\Phi]\,(x,\lambda,\tilde\lambda) = \left[\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda}\right]\Phi(x,\lambda,\tilde\lambda)\ ,$$

where $\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda = \lambda_\alpha\,(\boldsymbol{\sigma}_{\mu\nu})^\alpha_{\ \ \beta}\,\partial_{\lambda_\beta},\ \tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda} = \tilde\lambda^{\dot\alpha}\,(\bar{\boldsymbol{\sigma}}_{\mu\nu})_{\dot\alpha}^{\ \ \dot\beta}\,\partial_{\tilde\lambda^{\dot\beta}}$.

For 4-dimensional case $\mathbb{R}^{p,q} = \mathbb{R}^{1,3}$ we have 2-component Weyl spinors $\lambda, \tilde{\lambda}$ and tensor fields $\Phi^{\dot{\alpha}_1 \cdots \dot{\alpha}_{2\dot{\ell}}}_{\alpha_1 \cdots \alpha_{2\ell}}(x)$ are automatically symmetric under permutations of dotted and undotted indices separately. Then for $n = 4$ we have

$$\boldsymbol{\sigma}_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) , \qquad \overline{\boldsymbol{\sigma}}_\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3) ,$$

where $\sigma_0 = \mathrm{I}_2$ and $\sigma_1, \sigma_2, \sigma_3$ are standard Pauli matrices. Consequently we obtain for the self-dual components of $S_{\mu\nu}$

$$\mathbf{S} = \frac{1}{2}\, \boldsymbol{\sigma}^{\mu\nu}\, S_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\, \lambda_1 \partial_{\lambda_1} - \frac{1}{2}\, \lambda_2 \partial_{\lambda_2} & \lambda_2 \partial_{\lambda_1} \\ \lambda_1 \partial_{\lambda_2} & -\frac{1}{2}\, \lambda_1 \partial_{\lambda_1} + \frac{1}{2}\, \lambda_2 \partial_{\lambda_2} \end{pmatrix}$$

and for anti-self-dual components of $S_{\mu\nu}$

$$\overline{\mathbf{S}} = \frac{1}{2}\, \overline{\boldsymbol{\sigma}}^{\mu\nu}\, S_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\, \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{1}}} - \frac{1}{2}\, \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{2}}} & \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{1}}} \\ \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{2}}} & -\frac{1}{2}\, \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{1}}} + \frac{1}{2}\, \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{2}}} \end{pmatrix}$$

Consider $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1,q+1)$-type operator:

$$\mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u) \equiv \mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u_+,u_-) = u\,\mathrm{I} + \tfrac{1}{2}T_s(M^{ab}) \otimes \rho_{\Delta,\ell,\dot{\ell}}(M_{ab}) =$$

$$= \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S} - \mathbf{p}\cdot\mathbf{x}\,, & \mathbf{p} \\ \mathbf{x}\cdot\mathbf{S} - \overline{\mathbf{S}}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x} + (u_+ - u_-)\cdot\mathbf{x}\,, & u_- \cdot \mathbf{1} + \overline{\mathbf{S}} + \mathbf{x}\cdot\mathbf{p} \end{pmatrix},$$

where T_s is the spinor representation and $\rho_{\Delta,\ell,\dot{\ell}}$ is the differential representation of the conformal algebra $so(p+1,q+1)$ which acts on the conformal fields $\Phi_{\Delta,\ell,\dot{\ell}}(x)$;

$$u_+ = u + \frac{\Delta - n}{2}\,, \quad u_- = u - \frac{\Delta}{2}\,, \quad n = p + q\,,$$

We have used the expression for the ”polarized” Casimir operator $\frac{1}{2}T_s(M^{ab}) \otimes \rho_{\Delta,\ell,\dot{\ell}}(M_{ab})$ which was discussed in context of the differential representation of the conformal algebra.

Proposition 1.
The operator $\mathrm{L}^{(\Delta)}(u_+, u_-)$ satisfies the *RLL* relation

$$\mathcal{R}_{23}(u-v)\,(\mathrm{L}_2^{(\Delta_1)})^\alpha_\beta(u)\,(\mathrm{L}_3^{(\Delta_2)})^\beta_\gamma(v) =$$
$$= (\mathrm{L}_2^{(\Delta_2)})^\alpha_\beta(v)\,(\mathrm{L}_3^{(\Delta_1)})^\beta_\gamma(u)\,\mathcal{R}_{23}(u-v) \quad \in End(V \otimes V_{\Delta_1} \otimes V_{\Delta_2})\ ,$$

with $\mathcal{R}$-operator $\in End(V_{\Delta_1} \otimes V_{\Delta_2})$

$$\mathcal{R}_{12}(u-v) = q_{12}^{2(u_- - v_+)} \cdot \hat{p}_2^{\,2(u_+ - v_+)} \cdot \hat{p}_1^{\,2(u_- - v_-)} \cdot q_{12}^{2(u_+ - v_-)}\ ,$$

if $\mathbf{S} = 0 = \overline{\mathbf{S}}$ (for any dimension $n = p + q$), i.e. in the case of the scalar propagators.

The operator $\mathrm{L}^{(\Delta)}(u_+, u_-)$ is also intertwined by the matrix R which acts in $End(V \otimes V)$.

Proposition 2.
For two special cases the operator $\mathrm{L}^{(\Delta)}(u)$ satisfies the *RLL* relation

$$\mathrm{R}_{12}(u-v)\,\mathrm{L}_1^{(\Delta)}(u)\,\mathrm{L}_2^{(\Delta)}(v) = \mathrm{L}_1^{(\Delta)}(v)\,\mathrm{L}_2^{(\Delta)}(u)\,\mathrm{R}_{12}(u-v) \in End(V \otimes V \otimes V_\Delta)$$

with the R-matrix $\mathrm{R}_{12}(u) \in \mathrm{End}(V \otimes V)$, where V is the $2^{\frac{n}{2}}$-dimensional space of spinor representation T_s of $\mathrm{conf}(\mathbb{R}^{p,q})$ and indices $1,2$ are numbers of spaces V.
The two cases are:

- Dimension $n = p + q$ of the space $\mathbb{R}^{p,q}$ is arbitrary and representation ρ_Δ of $\mathrm{conf}(\mathbb{R}^{p,q})$ is special and corresponds to the scalars: $\mathbf{S} = 0$ and $\bar{\mathbf{S}} = 0$.
- Dimension $n = p + q$ of the space $\mathbb{R}^{p,q}$ is fixed by $n = 4$ and representation ρ_Δ of $\mathrm{conf}(\mathbb{R}^{p,q})$ is arbitrary: $\mathbf{S} \neq 0$ and $\bar{\mathbf{S}} \neq 0$.

Remark. The *RLL* relations look like defining relations for the Yangian $Y(spin(p+1, q+1))$ and $\mathrm{L}^{(\Delta)}(u)$ the image of the evaluation repr. of this Yangian.

Let Γ_a be $2^{\frac{n}{2}+1}$-dim. gamma-matrices in $\mathbb{R}^{p+1,q+1}$ $(n = p+q)$ which generate the Clifford algebra with the basis

$$\Gamma_{a_1 \dots a_k} = \frac{1}{k!} \sum_{s \in \mathcal{S}_k} (-1)^{\mathrm{p}(s)} \Gamma_{s(a_1)} \cdots \Gamma_{s(a_k)} \;\; (k \leq n+2) \, ,$$

where $\mathrm{p}(s)$ denote the parity of s. The $SO(p+1, q+1)$-invariant R-matrix is (it is necessary to take Weyl projection)

$$\mathrm{R}(u) = \sum_{k=0}^{n+2} \frac{\mathrm{R}_k(u)}{k!} \cdot \Gamma_{a_1 \dots a_k} \otimes \Gamma^{a_1 \dots a_k} \quad \in \quad \mathrm{End}(V \otimes V) \, ,$$

where V is the $2^{\frac{n}{2}+1}$-dimensional space of spinor representation T of $SO(p+1, q+1)$. To satisfy the Yang-Baxter equation the functions $\mathrm{R}_k(u)$ have to obey the recurrent relations (R.Shankar and E.Witten (1978), Al.B.Zamolodchikov (1981), M.Karowsky and H.Thun (1981))

$$\mathrm{R}_{k+2}(u) = -\frac{u+k}{u+n-k} \, \mathrm{R}_k(u) \, .$$

Proposition 3.

For any spin $\mathbf{S}$, $\bar{\mathbf{S}}$ and $n = p + q = 4$ the operator $\mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u)$ satisfies the *RLL* relation

$$\mathcal{R}_{12}(u-v)\cdot(\mathrm{L}_1^{(\Delta_1,\ell_1,\dot{\ell}_1)})^{\alpha}_{\beta}(u)\cdot(\mathrm{L}_2^{(\Delta_2,\ell_2,\dot{\ell}_2)})^{\beta}_{\gamma}(v) =$$

$$= (\mathrm{L}_1^{(\Delta_2,\ell_2,\dot{\ell}_2)})^{\alpha}_{\beta}(v)\cdot(\mathrm{L}_2^{(\Delta_1,\ell_1,\dot{\ell}_1)})^{\beta}_{\gamma}(u)\cdot\mathcal{R}_{12}(u-v) \in$$

$$\in End\big(V\otimes V_{\Delta_1,\ell_1,\dot{\ell}_1}\otimes V_{\Delta_2,\ell_2,\dot{\ell}_2}\big)\,,$$

with special Yang-Baxter R-operator

$$[\mathcal{R}_{12}\,\Phi](x_1\,,\,\lambda_1\,,\,\tilde{\lambda}_1\,;x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_2\,) =$$

$$= \int \frac{\mathrm{d}^4q\,\mathrm{d}^4k\,\mathrm{d}^4y\,\mathrm{d}^4z\;\; e^{i\,(q+k)\,x_{21}}\,e^{i\,k\,(y-z)}}{q^{2(u_--v_++2)}z^{2(u_+-v_++2)}y^{2(u_--v_-+2)}k^{2(u_+-v_-+2)}}\cdot \tag{2}$$

$$\cdot\Phi(x_1-y\,,\,\lambda_2\,\mathbf{z}\,\bar{\mathbf{k}}\,,\,\tilde{\lambda}_2\,\bar{\mathbf{q}}\,\mathbf{y}\,;\,x_2-z\,,\,\lambda_1\,\mathbf{q}\,\bar{\mathbf{z}}\,,\,\tilde{\lambda}_1\,\bar{\mathbf{y}}\,\mathbf{k}\,)\,,$$

where we have used compact notation

$$\mathbf{x} = \sigma_\mu x^\mu/|x|\,,\quad \bar{\mathbf{x}} = \bar{\sigma}_\mu\, x^\mu/|x|\,.$$

Remark. The integrable model of the type of Zamolodchikov's "Fishnet" diagram Integrable System is not known for $\mathcal{R}$ given in (2).

Green function for two fields of the types $(\ell, \dot{\ell})$ and $(\dot{\ell}, \ell)$ in conformal field theory and the solution is well known

$$(\Phi(X), \Phi(Y)) = \frac{1}{(2\ell)!} \frac{1}{(2\dot{\ell})!} \frac{\left(\tilde{\lambda}\,(\overline{\mathbf{x}-\mathbf{y}})\,\eta\right)^{2\ell} \left(\lambda\,(\mathbf{x}-\mathbf{y})\,\tilde{\eta}\right)^{2\dot{\ell}}}{(x-y)^{2(4-\Delta)}} \,.$$

In this Section for simplicity we shall use compact notation

$$\mathbf{x} = \sigma_\mu \frac{x^\mu}{|x|} \;;\; \overline{\mathbf{x}} = \overline{\sigma}_\mu \frac{x^\mu}{|x|} \qquad (3)$$