Higher-Rank Fields and Currents

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Introduction

Using unfolding machinery we classify all $Sp(2M)$ invariant fields,

field equations and conserved currents.

This method is effective in describing conformal massless fields and

analysis of multiparticle states.

In particular, the duality between fields in higher dimensions and

currents in lower dimensions can be applied to (higher-spin) holography.

$Sp(2M)$ invariant space

Rank-one unfolded equation

$$\left(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-\right) C^{\pm}(Y|X) = 0\,, \qquad \sigma_- = \xi^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\,,$$

X^{AB} matrix coordinates of $\mathcal{M}_M$, $X^{AB} = X^{BA}$ $(A, B = 1, \ldots, M)$

Y^A - auxiliary commuting variables = twistor variables

$\xi^{MN} = dX^{MN}$ - anti-commuting variables $\xi^{MN} = \xi^{NM}$, $\xi^{MN}\xi^{AD} = -\xi^{AD}\xi^{MN}$.

Rank-one primary (dynamical) fields : $\sigma_- C(X|Y) = 0$: $C(X)$, $C_A(X)Y^A$

Unfolded equations $\Rightarrow$ dynamical equations

$$\frac{\partial}{\partial X^{\mathbf{A}\mathbf{E}}}\frac{\partial}{\partial X^{\mathbf{B}\mathbf{D}}}C(X) - \frac{\partial}{\partial X^{\mathbf{B}\mathbf{E}}}\frac{\partial}{\partial X^{\mathbf{A}\mathbf{D}}}C(X) = 0 \qquad \text{Klein-Gordon–like}\,,$$

$$\frac{\partial}{\partial X^{\mathbf{B}\mathbf{D}}}C_{\mathbf{A}}(X) - \frac{\partial}{\partial X^{\mathbf{A}\mathbf{D}}}C_{\mathbf{B}}(X) = 0 \qquad \text{Dirac–like}\,.$$

Rank-$\mathbf{r}$ dynamical fields in $\mathcal{M}_M$

Rank- $\mathbf{r}$ unfolded equations : $\mathbf{r}$ sets of twistor variables Y,

$$\left(\xi^{AB}\frac{\partial}{\partial X^{AB}} \pm i\sigma_-^{\mathbf{r}}\right) C^{\pm}(Y|X) = 0 \ ,$$

$$\sigma_-^{\mathbf{r}} = \xi^{AB}\sum_{j=1}^{\mathbf{r}} \frac{\partial^2}{\partial Y_j^A \partial Y_i^B}\,\delta_{ij}\,, \qquad i,j,\ldots = 1,\ldots,\mathbf{r}$$ **-color indices**

Rank-$\mathbf{r}$ primary fields : $\sigma_-^{\mathbf{r}}\, C(Y|X) = 0 \quad \Longrightarrow$

$$C(Y|X) = \sum_n C^{i_1;\ldots;i_n}_{A_1;\ldots;A_n}(X) Y_{i_1}^{A_1}\cdots Y_{i_n}^{A_n} \Rightarrow$$ **tracelessness:** $\delta_{\mathbf{i_1 i_2}} C^{\mathbf{i_1};\mathbf{i_2};\cdots}_{\cdots}(X) = 0.$

Y_i^A **commute** $\Rightarrow \quad C^{\cdots\mathbf{i_m}\cdots\ \mathbf{i_k}\cdots}_{\cdots\mathbf{A_m}\cdots\mathbf{A_k}\cdots}(X) = C^{\cdots\mathbf{i_k}\cdots\ \mathbf{i_m}\cdots}_{\cdots\mathbf{A_k}\cdots\mathbf{A_m}\cdots}(X) \Rightarrow$

rank-$\mathbf{r}$ primary fields are tensors $C_{\mathbf{Y}}(Y|X)$ described by traceless $\mathfrak{gl}_M$ Young diagrams $\mathbf{Y}[h_1,\ldots,h_m]$ with respect to indices $A,B = 1,\ldots,M$, *i.e.,* :

$$h_1 + h_2 \leq \mathbf{r}\,, \qquad h_1 \leq M.$$

We use Young diagrams $\mathbf{Y}[\ldots]$ with manifest anti-symmetrization

Rank-$\mathbf{r}$ dynamical equations

Rank-$\mathbf{r}$ primary fields $C_{\mathbf{Y}^0}(Y|X)$ satisfy rank-$\mathbf{r}$ dynamical equations

$$\mathcal{E}^{A_1[\mathbf{r}-h_2+1]\,,\,A_2[\mathbf{r}-h_1+1]\,,\,A_3[h_3],\ldots,A_n[h_n]}_{i_1[h_1]\,,\quad i_2[h_2]\,,\quad i_3[h_3],\ldots,\,i_n[h_n]}\underbrace{\frac{\partial}{\partial Y^{A_1^1}_{i_1^1}}\cdots\frac{\partial}{\partial Y^{A_1^{h_1}}_{i_1^{h_1}}}}_{h_1}\cdots\underbrace{\frac{\partial}{\partial Y^{A_n^1}_{i_n^1}}\cdots\frac{\partial}{\partial Y^{A_n^{h_n}}_{i_n^{h_n}}}}_{h_n}$$

$$\underbrace{\frac{\partial}{\partial X^{A_1^{h_1+1}\,A_2^{h_2+1}}}\cdots\frac{\partial}{\partial X^{A_1^{\mathbf{r}-h_2+1}\,A_2^{\mathbf{r}-h_1+1}}}}_{\mathbf{r}+1-h_1-h_2}C_{\mathbf{Y}^0}(Y|X)=0\,.$$

The symmetry properties of the parameter $\mathcal{E}^{\cdots}_{\cdots}$ described by

$\mathbf{Y}^0[h_1,h_2,h_3,\ldots,h_n]$ **with respect to the lower indices**

and by its rank-$\mathbf{r}$ two-column dual

$$\mathbf{Y}^1[\mathbf{r}+1-h_2,\mathbf{r}+1-h_1,h_3,\ldots,h_n]$$

with respect to the upper ones.

$3d$ conformal fields and equations in $Sp(2)$ invariant space

Free $3d$ massless fields $C(t,x)$ can be described in terms of

two-component spinors y^α and symmetric matrix

$$x^{\alpha\beta} = x^{\beta\alpha}: \quad x^{\alpha\beta} = t\delta^{\alpha\beta} + x^1\sigma_1^{\alpha\beta} + x^2\sigma_3^{\alpha\beta} \quad , \qquad \alpha,\beta = 1,2\,,$$

where $\sigma_{1,3}^{\alpha\beta}$ **– traceless symmetric Pauli matrices.**

Conformal invariant massless equations = Rank-1 unfolded equations

Shaynkman, Vasiliev (2001)

$$dx^{\alpha\beta}\left(\frac{\partial}{\partial x^{\alpha\beta}} \pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right) C^\pm(y|x) = 0 \quad \Rightarrow$$

primaries :
$$\frac{\partial^2}{\partial y^\alpha \partial y^\beta} b(x) = \frac{\partial^2}{\partial y^\alpha \partial y^\beta} f_\beta(x) y^\beta \;=\; 0\,.$$

boson $b(x)$:
$$\varepsilon^{\beta\nu}\epsilon^{\alpha\gamma}\frac{\partial^2}{\partial x^{\alpha\beta}\partial x^{\gamma\nu}} b(x) \;=\; 0 \qquad \sim 3d \text{ Klein-Gordon}$$

fermion $f_\beta(x)$:
$$\epsilon^{\alpha\gamma}\frac{\partial}{\partial x^{\alpha\beta}} f_\gamma(x) \;=\; 0 \qquad \sim 3d \text{ Dirac}$$

$\epsilon^{\alpha\beta}$ - 2×2 **symplectic form .**

$3d$ conformal currents

Rank-2 equations = Current equations O.G, M.Vasiliev (2003)

$$dx^{\alpha\beta}\left\{\frac{\partial}{\partial x^{\alpha\beta}} - i\frac{\partial^2}{\partial v^\alpha \partial u^\beta}\right\}\mathcal{J}(u,\,v|x) = 0\,, \quad v = \frac{1}{2}(y_1+y_2),\;\; u = \frac{1}{2}(y_1-y_2)\,.$$

Doubled set of spinor=twistor variables.

Closed differential forms = $3d$ conserved current

$$\left(i\,dx^{\alpha\beta}\frac{\partial}{\partial u^\beta} + d\,v^\alpha\right)^2 \mathcal{J}(u\,,v|x)\Big|_{u=0}\,, \qquad \left(i\,dx^{\alpha\beta}\frac{\partial}{\partial v^\beta} + d\,u^\alpha\right)^2 \widetilde{\mathcal{J}}(u\,,v|x)\Big|_{v=0}\,.$$

Current equations are obeyed by generalized bilinear stress tensors

$$T^{k\,l}_{\alpha_1\ldots\alpha_n}(x) = \frac{\partial}{\partial u^{\alpha_1}}\cdots\frac{\partial}{\partial u^{\alpha_n}}\Big(C^k_+(v-u|x)C^l_-(v+u|x)\Big)|_{u=0} \;:$$

$C_\pm(y|x)$ **– rank-1 fields.**

$$\widetilde{T}^{k\,l}_{\alpha_1\ldots\alpha_n}(x) = \frac{\partial}{\partial v^{\alpha_1}}\cdots\frac{\partial}{\partial v^{\alpha_n}}\Big(C^k_+(v-u|x)C^l_-(v+u|x)\Big)|_{v=0}$$

σ_--cohomology analysis

Rank-$\mathbf{r}$ primary fields and field equations are represented by

the cohomology groups $H^0(\sigma_-^{\mathbf{r}})$ and $H^1(\sigma_-^{\mathbf{r}})$, respectively. M.A.Vasiliev 1989

Examples

The full lists of YD associated with rank-1 and rank-2 fields and

equations :

$H^0(\sigma_-) : \mathbf{Y}^0$	$\bullet$	$\square$
$H^1(\sigma_-) : \mathbf{Y}^1$	$\square\square$	$\square\square$

,

$H^0(\sigma_-^2) : \mathbf{Y}^0$	$\bullet$	$\square$	$\square$	$2+n$
$H^1(\sigma_-^2) : \mathbf{Y}^1$	$\square\square$	$\square\square$	$\square\square$	$2+n$

.

Homotopy operator

Standard homotopy trick : Conjugated linear operators

Ω and Ω^* , $\Omega^2 = 0 \Rightarrow \Delta = \{\Omega, \Omega^*\}$ - **semi-positive homotopy operator .**

If Δ is diagonalizable $\Rightarrow H(\Omega) \subset \ker \Delta \cap \ker \Omega$.

$$\Omega := \sigma_-^{\mathbf{r}} = T_{AB}\xi^{AB} , \qquad \Omega^* = T^{AB}\frac{\partial}{\partial \xi^{CD}} ,$$

$$T_{AB} = \frac{\partial}{\partial Y_i^A}\frac{\partial}{\partial Y_j^B}\delta^{ij}, \quad T^{CD} = Y_i^C Y_j^D \delta^{ij} , \quad T_B^A = Y_j^A \frac{\partial}{\partial Y_j^B} \qquad = \mathfrak{sp}(2M)$$

$$\boxed{\Delta = \{\Omega , \Omega^*\} = \tfrac{1}{2}\tau_{mk}\tau^{mk} + \nu_B^A \nu_B^A - (M + 1 - \mathbf{r})\nu_A^A}$$

$\tau_{mk} = Y_m{}^A \dfrac{\partial}{\partial Y^{kA}} - Y_k{}^A \dfrac{\partial}{\partial Y^{mA}}$ **– generators of $\mathfrak{o}(\mathbf{r})$,**

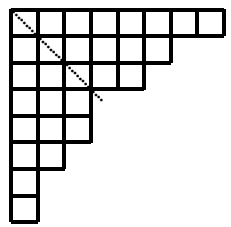
$\nu_B^A = \chi_B^A + T_B^A$ **– generators of $\mathfrak{gl}_M^{tot}$ that acts on Y_i^A and ξ^{AB}**

$\chi_B^A = 2\xi^{AD}\dfrac{\partial}{\partial \xi^{BD}}$ **– generators of $\mathfrak{gl}_M$.**

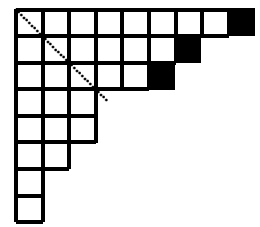
Almost symmetric Young diagrams

N-form in ξ^{AB} = anti-symmetrized tensor product $\left[\otimes^N \mathbf{Y}[1,1]\right] = \left[\otimes^N \square\square\right]$

= linear combinations of almost symmetric $\mathbf{Y}_A[a_1, a_2, \ldots]$, $\sum a_j = 2N$.



Symmetric $\mathbf{Y}_S$ $\Rightarrow$



Almost symmetric $\mathbf{Y}_A$

Young diagrams and Cazimir operators

$$\mathbf{Y}'[B_1 \ldots] \subset \mathbf{Y}[h_1 \ldots] \otimes (\otimes_n \mathbf{Y}_\delta[1,1]) \otimes \mathbf{Y}_A[a_1, \ldots]$$

where n is a number of $\mathfrak{o}(\mathbf{r})$ metric tensors δ_{ij},

$\mathbf{Y}[h_1 \ldots, h_k]$ – $\mathfrak{o}(\mathbf{r})$, $\mathbf{Y}'[B_1 \ldots, B_m]$ – $\mathfrak{gl}_M$ YD, $\mathbf{Y}_A[a_1, \ldots]$ – AS YD

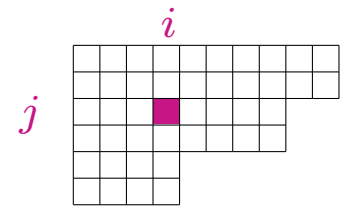
Cazimir operators:

$$\tau_{mk}\tau^{mk} = 2\sum_j h_j(h_j - \mathbf{r} - 2(i-1)), \qquad \nu^A_B \nu^A_B = -\sum_i B_i(B_i - M - 1 - 2(i-1)),$$

$$\Rightarrow \qquad \Delta = -\sum_i B_i(B_i - 2(i-1)) + \sum_j h_i(h_i - 2(i-1)) + \mathbf{r}\sum_i (B_i - h_i).$$

Southwest principle

$\mathcal{S}(i,j)$ – a sell on the intersection of $j-th$ row and $i-th$ column



$$\mathbf{Y} = \bigcup_{\mathcal{S}(i,j)\in\mathbf{Y}} \mathcal{S}(i,j)$$

Numerical characteristic $\chi^a(\mathcal{S}(i,j)) = i - j + a\,, \quad a \in \mathbb{R}$

$\chi^a(\mathbf{Y}) = -\frac{1}{2}\sum_i h_i(h_i - 2i + 1 - 2a)$ since $\mathbf{Y}[h_1, h_2, \ldots]$ = unification of columns

$$\Rightarrow \quad \ker\Delta : \quad \chi^{\frac{1}{2}(\mathbf{r}-1)}(\mathbf{Y}' \setminus \mathbf{Y}) = 0.$$

$\chi^a(\mathcal{S}_1) < \chi^a(\mathcal{S}_2) \quad \Leftrightarrow \mathcal{S}_1$ is situated the more southwest then $\mathcal{S}_2$

Δ semi-positive $\Rightarrow$ $\min(\Delta)$ is reached when all cells of $\mathbf{Y}'$

are maximally south-west . It allows us to find $H^N(\sigma^{\mathbf{r}}_{-})\ \forall N$

Higher $\sigma^{\mathbf{r}}_{-}$- cohomology groups

The full list of YD associated with $H^N(\sigma^{\mathbf{r}}_{-})$ **:**

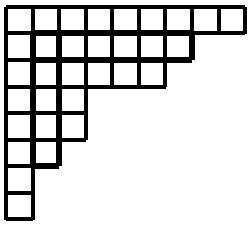
$$\mathbf{Y}'_{(\mathbf{Y},\mathbf{Y}_A)}[H_1, H_2, \ldots] : \quad H_j = h_j + a_j + \sum_j \mathbf{s}_{i\ j}(\mathbf{Y}, \mathbf{Y}_A)$$

$\mathbf{Y}[h_1, h_2, \ldots,]$ – **traceless YD ,**

$\mathbf{Y}_A[a_1, \ldots a_k]$ – **almost symmetric YD** $\sum a_i = 2N$,

$\mathbf{s}(\mathbf{Y}, \mathbf{Y}_A)$ – **shift matrix** .

For almost symmetric $\mathbf{Y}_A[8, 6, 5, 3, 3, 3, 2, 1, 1] =$



$$\mathbf{s}(\mathbf{Y}, \mathbf{Y}_A) = \begin{pmatrix} \Delta_0 & \Delta_0 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ \Delta_2 & \Delta_1 & \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & 0 & 0 \\ \Delta_3 & \Delta_2 & \Delta_0 & \Delta_0 & \Delta_2 & \Delta_3 & 0 & 0 & 0 \\ \Delta_4 & \Delta_3 & \Delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_5 & \Delta_4 & \Delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_6 & \Delta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_0 = \mathbf{r} - h_1 - h_2$$

$$\Delta_k = h_k - h_{k+1} \quad (k > 0)\,.$$

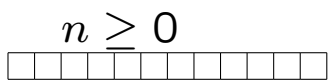
Minkowski-like reduction.

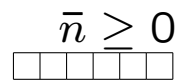
$4d$ Minkowski unfolded equations is a subsystem of rank-2 ones in $\mathcal{M}_4$ with $Y^A = (y^\alpha,\ \bar{y}^{\beta'})\,, \qquad X^{AB} = (x^{\alpha\beta'},\ x^{\alpha\beta},\ \overline{x}^{\alpha'\beta'}) \quad (\ \overline{y^\alpha} = \bar{y}^{\alpha'})$.

Rank-$\mathbf{r}$ primary fields $C_{\mathbf{Y},\overline{\mathbf{Y}}}(y,\bar{y}|x)$ are described by pairs of the mutually traceless Young diagrams

$\mathbf{Y}[h_1,\ldots,h_k]$ and $\overline{\mathbf{Y}}[\overline{h}_1,\ldots,\overline{h}_n]$: $h_1+\overline{h}_1 \leq \mathbf{r}$.

For example, Minkowsky primary free currents =rank-2 Minkowsky primary fields are described by

$\mathbf{Y}(n) =$ 

, $\qquad \overline{\mathbf{Y}}(\bar{n}) =$ 

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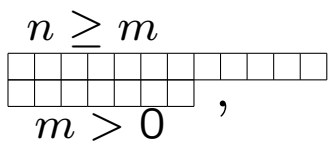
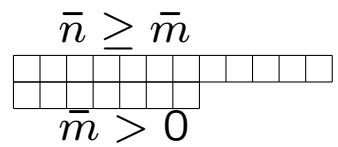
$\mathbf{Y}(n,m) =$ 

, $\qquad \overline{\mathbf{Y}} = \bullet$;

$\mathbf{Y} = \bullet$, $\qquad \overline{\mathbf{Y}}(\bar{n},\bar{m}) =$ 

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Conclusions

All $Sp(2M)$ invariant field and field equations in the space $\mathcal{M}_M$,

including usual $3d$ space and $4d$ Minkowsky space.

All higher cohomologies are found $\Rightarrow$

All primary differential forms and their equations are described.

This can be used in the construction of interacting theories.