Phase transition in matrix model and knot invariants

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Knot invariants: Jones polynomial

Jones polynomial:

$$qJ(L_1) - q^{-1}J(L_2) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J(L_3), \qquad J(\bigcirc) = q^{\frac{1}{2}} + q^{-\frac{1}{2}}. \tag{1}$$

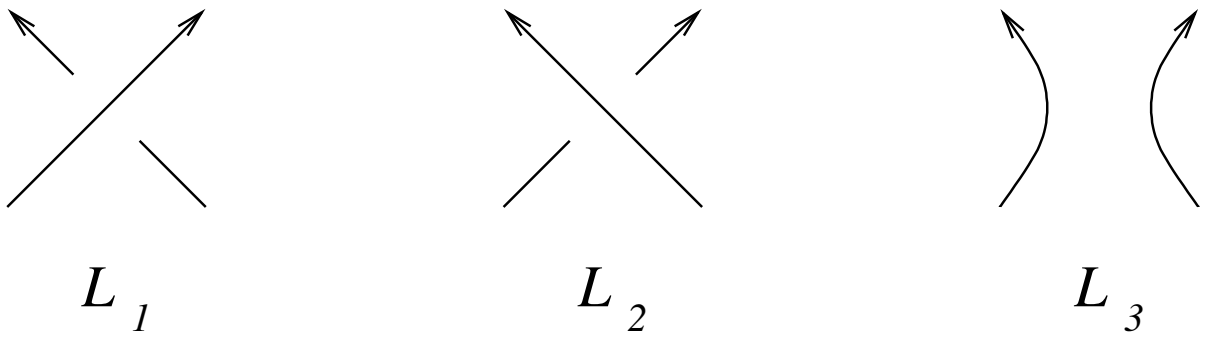


Figure: Skein relation for the Jones polynomial

HOMFLY polynomial:

$$aP(L_1) - a^{-1}P(L_2) = (q^{\frac{1}{2}} - q^{\frac{1}{2}})P(L_3). \tag{2}$$

Knot invariants: Kauffman bracket

$$\langle L \rangle = \langle L_0 \rangle + q^2 \langle L_1 \rangle, \qquad \langle \bigcirc \rangle = q^{\frac{1}{2}} + q^{-\frac{1}{2}}. \tag{3}$$

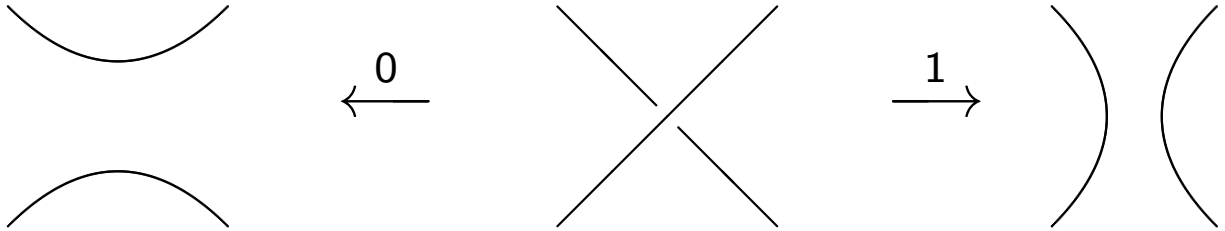


Figure: Two possible resolutions of a "positive" intersection in a knot.

Random walks and $T_{2,n}$ knot invariants

0-move merges red areas, 1-move separates them:

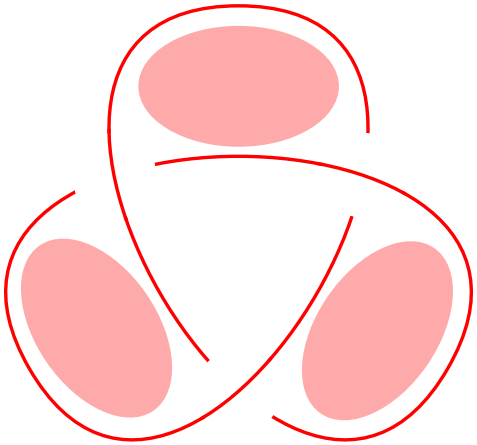


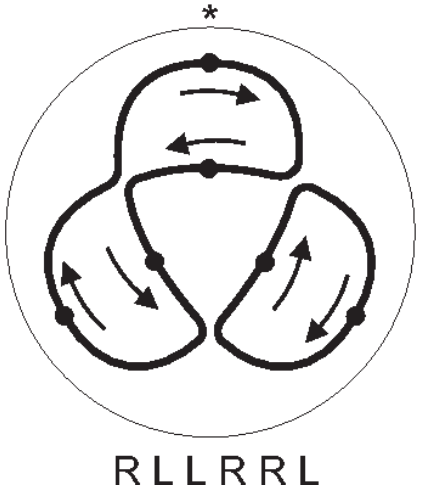
Figure: Trefoil knot with "chessboard" coloring

Random walk of 6 steps on a cirle with circumference 3:

$$000 \rightarrow RRRRRR, \tag{4}$$
$$001 \rightarrow RRRLLL, \tag{5}$$
$$011 \rightarrow RLRRLL, \tag{6}$$
$$111 \rightarrow RLRLRL. \tag{7}$$

Random walks and $T_{2,n}$ knot invariants

All possible resolutions give all walks on the knot diagram:



Jones polynomial is a partition function for the random walks:

$$J(T_{2,n}) = q^{\#} \sum_{\text{closed paths of length } 2n} (q^2)^{\#\text{ of turns}} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{\#\text{ of walks through } *}. \tag{8}$$

Refining polynomial invariants

Jones polynomial:

$$qJ(L_1) - q^{-1}J(L_2) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J(L_3). \tag{9}$$

HOMFLY polynomial:

$$aP(L_1) - a^{-1}P(L_2) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})P(L_3). \tag{10}$$

Superpolynomial:

$$\mathcal{P}(a, q, t) : \mathcal{P}(a, q, -1) = P(a, q). \tag{11}$$

no skein relation

Categorification:

$$\mathcal{P}(a, q, t) = \sum a^i q^j t^k \mathcal{H}_{i,j,k} \tag{12}$$

Random walks and $T_{n,n+1}$ knots: Dyck path

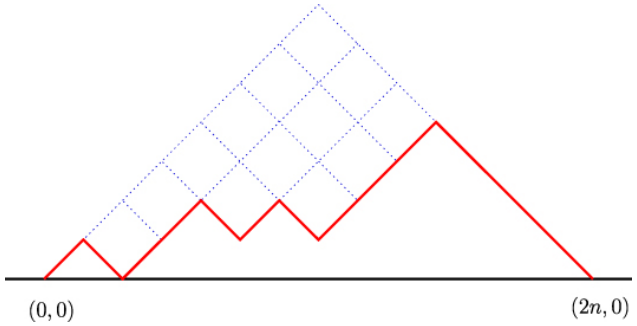


Figure: Dyck path

Number of Dyck paths of length $2n$ is given by the Catalan number C_n:

$$C_n = \sum_{k=0}^{n-1} C_i C_{n-1-i}. \tag{13}$$

$$\mathcal{P}(a,q,t)|_{q=1,a^2t=1} = \sum_{\pi \in D_n} t^{2S(\pi)} \tag{14}$$

Catalan numbers

$T = (\mathbb{C}^*)^2$, V is a tautological bundle on Hilbert scheme:

$$C_n(q_1, q_2) = \chi^T(\mathrm{Hilb}^n(\mathbb{C}^2), \Lambda^n V) \tag{15}$$

q–Catalan numbers are "Nekrasov–Shatashvili" limit of $C_n(q_1, q_2)$:

$$C_n(q) = C_n(1, q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q), \qquad C_0(q) = 1. \tag{16}$$

q–Catalan numbers can be expressed via Dyck paths:

$$C_n(q) = \sum_{\pi \in D_n} q^{2S(\pi)}. \tag{17}$$

Dyck paths weighted with area and length

Generating function for Dyck paths with fixed area and length:

$$Z(s,q) = \sum_{n=1}^{\infty} s^n \sum_{\pi \in D_n} q^{2S(\pi)}. \tag{18}$$

Functional equation:

$$Z(s,q) = 1 + sZ(s,q)Z(qs,q). \tag{19}$$

At $q = 1$:

$$Z(s,1) = \frac{1-\sqrt{1-4s}}{2s}. \tag{20}$$

At $q \neq 1$:

$$Z(s,q) = \frac{A_q(s)}{A_q(s/q)}, \qquad A_q(s) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-s)^n}{(q;q)_n}, \qquad (t;q)_n = \prod_{k=0}^{n-1}(1-tq^k). \tag{21}$$

Generating function for Dyck paths: Singularity

$$Z(s \to \frac{1}{4}^{-}, q \to 1^{-}) \sim Z_{reg}(s,q) + (1-q)^{\frac{1}{3}} F\left(\frac{\frac{1}{4}-s}{(1-q)^{\frac{2}{3}}}\right), \tag{22}$$

where

$$F(z) = -2\frac{d}{dz}\ln \mathrm{Ai}(4z), \tag{23}$$

$\mathrm{Ai}(z)$ being the Airy function,

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + tz\right) dt. \tag{24}$$

$F(z)$ solves the Painlevé II equation,

$$F''(z) = 2F^3(z) + 4zF(z) + 2. \tag{25}$$

Third–order phase transition in a matrix model

Yang–Mills theory on a sphere:

$$Z_2\left(g_{YM}, A\right) = \int [dA]\,[d\Phi] \exp\left\{\int_{S_2} d^2x \left(\mathrm{Tr}\Phi F + g_{YM}^2 \mathrm{Tr}\Phi^2\right)\right\}. \quad (26)$$

In terms of representations:

$$Z_2\left(g_{YM}, A\right) = \sum_R (\mathrm{dim} R)^2 e^{-\frac{g_{YM}^2 A}{2N} C_2(R)}, \quad (27)$$

where

$$C_2(R) = \sum_{i=1}^{N} n_i \left(n_i - 2i + N + 1\right), \qquad \mathrm{dim} R = \prod_{i>j} \left(1 - \frac{n_i - n_j}{i - j}\right). \quad (28)$$

$2d$ Yang–Mills theory: large N limit

Introduce continuous variables:

$$h(x) = -\frac{1}{2} + \frac{i - n_i}{N}, \qquad x = \frac{i}{N} \tag{29}$$

Effective action in terms of $h(x)$:

$$S_{eff}[h(x)] = -\int_0^1 \int_0^1 dx dx' \, \log \left| h(x) - h(x') \right| + \frac{g_{YM}^2 A}{2} \int_0^1 dx \, h^2(x) - \frac{A}{24}. \tag{30}$$

Introduce $\rho(h) = \frac{dx(h)}{dh}$, $\rho(h) \leq 1$.

$2d$ Yang–Mills theory: phase transition

$$\frac{\delta S_{eff}}{\delta h} = 0 \quad \Rightarrow \quad \rho = \frac{g_{YM}^2 A}{2\pi}\sqrt{\frac{4}{g_{YM}^2 A} - h^2}. \tag{31}$$

Douglas–Kazakov phase transition:

$$A_{cr} = \frac{\pi^2}{g_{YM}^2}. \tag{32}$$

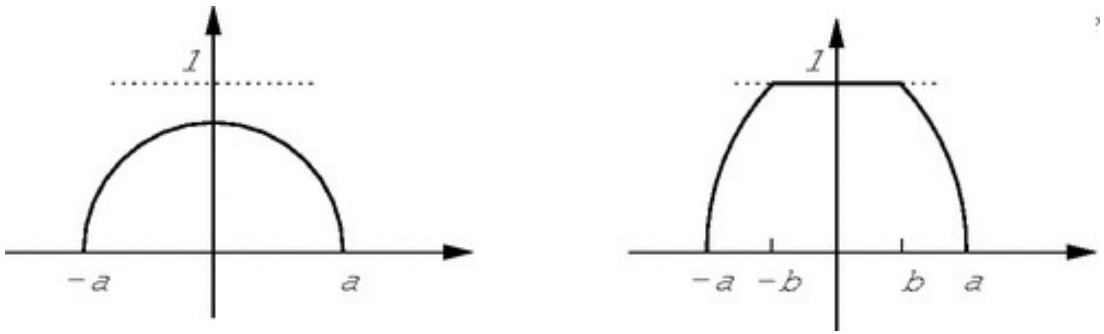


Figure: $\rho(h)$ before and after Douglas–Kazakov phase transition

Yang–Mills action via continuous random walks

Slater determinant for fermions on a circle:

$$\Psi\left(\mathbf{n}|\boldsymbol{\theta}\right) = \frac{1}{\sqrt{N!}} \det_{1\leq j,k\leq N} \phi_{n_j}\left(\theta_k\right), \qquad \phi_{n_j}(\theta) = \frac{1}{\sqrt{2\pi}} e^{in_j\theta}. \tag{33}$$

Reunion probability:

$$P_N^R\left(t, L|\boldsymbol{\theta} \to 0\right) = \langle\Psi|e^{-tH_L}|\Psi\rangle = \sum_{\mathbf{n}} \Psi\left(\mathbf{n}|\boldsymbol{\theta}\right) \Psi^*\left(\mathbf{n}|\boldsymbol{\theta}\right) e^{-tE(\mathbf{n})}\Big|_{\boldsymbol{\theta}\to 0}, \tag{34}$$

where $H_L = -\frac{2\pi^2}{L^2} \sum \frac{\partial^2}{\partial\theta_k^2}$.

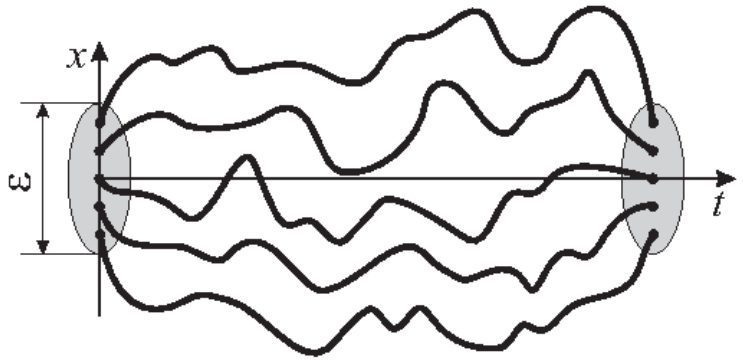


Figure: Vicious walkers returning to the starting point

Yang–Mills action via continuous random walks

$$P_N^R\left(t, L|\boldsymbol{\theta} \to 0\right) = C\delta^{N(N-1)/2} \sum_{\mathbf{n}} \Delta^2\left(\mathbf{n}\right) e^{-\frac{2\pi^2 t}{NL^2}\sum n_i^2}, \tag{35}$$

where $\Delta\left(\mathbf{n}\right) = \prod_{i<j}(n_i - n_j)$, $\delta = \theta_i - \theta_{i+1}$. The reunion probability coincides with the partition function on a sphere:

$$g_{YM}^2 A = \left(\frac{2\pi}{L}\right)^2 t. \tag{36}$$

Critical time $t_c = L^2/4$ separates regions with different probability to have certain winding number:

$$\mathbb{P}(w = 0, t < t_c) = 1 - \mathcal{O}(e^{-cN}). \tag{37}$$

$$\mathbb{P}(w, t > t_c) = Ce^{-\kappa w^2} + \mathcal{O}(N^{-1}). \tag{38}$$

Future directions

1. Another statistics in Dyck path (generating functions for Jones, HOMFLY polynomials)

$$Z(t,q) = \sum t^n J_{n,n+1}(q) \sim \vartheta(t;q). \tag{39}$$

2. $T_{kn+1,n}$ series of knots
   Dyck paths made of $1 \times k$ tiles
3. Conjectural relation to the Hofstadter problem