# QUARKS-2014

18th International Seminar on High Energy Physics

Suzdal, Russia, June 2 – 8, 2014.


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# Construction of the NSVZ scheme for Abelian supersymmetric theories, regularized by higher derivatives

## NSVZ $\beta$-function in $\mathcal{N}=1$ supersymmetric theories

It is well known that the UV behavior of supersymmetric theories is better due to some non-renormalization theorems. In particular, in $\mathcal{N}=1$ supersymmetric theories the $\beta$-function is related with anomalous dimension of the matter superfields by the equation

$$\beta(\alpha) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}, \qquad \text{где}$$

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$

$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. B 277, (1986), 456.

The NSVZ $\beta$-function was obtained from different arguments: instantons, anomalies etc.

## NSVZ $\beta$-function for $\mathcal{N}=1$ SQED with $N_f$ flavors

Here we pay especial attention to the Abelian case, namely, to the $\mathcal{N}=1$ supersymmetric electrodynamics (SQED) with $N_f$ flavors, which (in the massless case) is described by the action

$$S = \frac{1}{4e_0^2}\mathrm{Re}\int d^4x\, d^2\theta\, W^a W_a + \sum_{i=1}^{N_f}\frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_i^* e^{2V}\phi_i + \widetilde{\phi}_i^* e^{-2V}\widetilde{\phi}_i\Big),$$

where $V$ is a real gauge superfield, $\phi_i$ and $\widetilde{\phi}_i$ with $i=1,\ldots,N_f$ are chiral matter superfields, and $W_a = \bar{D}^2 D_a V/4$. This case corresponds to

$$C_2 = 0; \qquad C(R) = I; \qquad T(R) = 2N_f \qquad r = 1,$$

where $I$ is the $2N_f \times 2N_f$ unit matrix. Therefore, for $\mathcal{N}=1$ SQED with $N_f$ flavors the NSVZ $\beta$-function has the form

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. 42, (1985), 224; Phys.Lett. 166B, (1986), 334.

## NSVZ $\beta$-function and calculations in the lowest loops

The NSVZ $\beta$-function can be compared with the results of calculations in the lowest orders. In order to make such calculations a theory should be regularized.

Dimensional regularization breaks the supersymmetry and is not convenient for calculations in supersymmetric theories. That is why supersymmetric theories are mostly regularized by the dimensional reduction. However, the dimensional reduction is not self-consistent.

W.Siegel, Phys.Lett. **B84** (1979) 193; **B94** (1980) 37.

Removing of the inconsistencies leads to the loss of explicit supersymmetry:

L.V.Avdeev, G.A.Chochia, A.A.Vladimirov, Phys.Lett. **B105** (1981) 272.

As a consequence, supersymmetry can be broken by quantum corrections in higher loops.

L.V.Avdeev, Phys.Lett. **B117**, (1982), 317;
L.V.Avdeev, A.A.Vladimirov, Nucl.Phys. **B219**, (1983), 262.

## NSVZ $\beta$-function and calculations in the lowest loops

Using the dimensional reduction and $\overline{\text{DR}}$-scheme a $\beta$-function of $\mathcal{N}=1$ supersymmetric theories was calculated up to the four-loop approximation:

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

The result coincides with the NSVZ $\beta$-function only in one- and two-loop approximations. In the higher loops it is necessary to make a special tuning of the coupling constant.

Thus, using of other regularizations is also interesting:

M.A.Shifman, A.I.Vainshtein, Sov.J.Nucl.Phys. **44** (1986) 321; J.Mas, M.Perez-Victoria, C.Seijas, JHEP, **0203**, (2002), 049.

Usually in supersymmetric theories other regularizations are used for calculations of a $\beta$-function only in one- and two-loop approxiamtions.

## Higher covariant derivative regularization

The higher covariant derivative regularization is a consistent regularization, which does not break supersymmetry.

A.A.Slavnov, Nucl.Phys., **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

In order to regularize a theory by higher derivatives it is necessary to add a term with higher degrees of covariant derivatives. Then divergences remain only in the one-loop approximation. These remaining divergences are regularized by inserting the Pauli–Villars determinants.

A.A.Slavnov, Theor.Math.Phys. **33** (1977) 977.

The higher covariant derivative regularization can be generalized to the $\mathcal{N}=1$ supersymmetric case

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

Also it can be constructed for $\mathcal{N}=2$ supersymmetric theories

V.K.Krivoshchekov, Phys.Lett. B **149** (1984) 128;
I.L.Buchbinder, K.S., Nucl.Phys. **B883** (2014) 20.

## $\mathcal{N}=1$ SQED with $N_f$ flavors, regularized by higher derivatives

In order to regularize the theory by higher derivatives it is necessary to add the higher derivative term to the action:

$$S_{\text{reg}} = \frac{1}{4e_0^2}\text{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a$$

$$+\sum_{i=1}^{N_f}\frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_i^* e^{2V}\phi_i + \widetilde{\phi}_i^* e^{-2V}\widetilde{\phi}_i\Big),$$

where $R(\partial^2/\Lambda^2)$ is a regulator, e.g. $R = 1 + \partial^{2n}/\Lambda^{2n}$.

Adding the higher derivative term allows to remove all divergences beyond the one-loop approximation. In order to remove one-loop divergencies we insert in the generating functional the Pauli–Villars determinants:

$$Z[J,\mathbf{\Omega}] = \int D\mu \prod_I \Big(\det PV(V, M_I)\Big)^{N_f c_I} \exp\Big\{iS_{\text{reg}} + iS_{\text{gf}} + \text{Sources}\Big\},$$

$$\sum_I c_I = 1;\;\; \sum_I c_I M_I^2 = 0;\;\; M_I = a_I\Lambda,\;\; \text{where } a_i \neq a_I(e_0).$$

## Renormalization

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta \Big( -\frac{1}{16\pi} V(-p)\, \partial^2\Pi_{1/2} V(p)\, d^{-1}(\alpha_0, \Lambda/p)$$

$$+\frac{1}{4}\sum_{i=1}^{N_f}\Big(\phi_i^*(-p,\theta)\phi_i(p,\theta) + \widetilde{\phi}_i^*(-p,\theta)\widetilde{\phi}_i(p,\theta)\Big)\, G(\alpha_0, \Lambda/p)\Big).$$

where $\partial^2\Pi_{1/2}$ is a supersymmetric transversal projection operator.

Then we defined the renormalized coupling constant $\alpha(\alpha_0, \Lambda/\mu)$, requiring that the inverse invariant charge $d^{-1}(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p)$ is finite in the limit $\Lambda \to \infty$. The renormalization constant $Z_3$ is defined by

$$\frac{1}{\alpha_0} \equiv \frac{Z_3(\alpha, \Lambda/\mu)}{\alpha}.$$

The renormalization constant $Z$ is constructed, requiring that the renormalized two-point Green function $ZG$ is finite in the limit $\Lambda \to \infty$:

$$G_{\text{ren}}(\alpha, \mu/p) = \lim_{\Lambda\to\infty} Z(\alpha, \Lambda/\mu) G(\alpha_0, \Lambda/p).$$

# The renormalization group functions defined in terms of the bare coupling constant

In most original papers

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. 166B, (1986), 334.

the NSVZ $\beta$-function was derived for the renormalization group functions defined in terms of the bare coupling constant

$$\beta\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv \frac{d\alpha_0(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}};$$

$$\gamma_i{}^j\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv -\frac{d\ln Z_i{}^j(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}$$

These renormalization group functions

1. are scheme independent for a fixed regularization;

2. depend on the regularization;

2. in all loops satisfy the NSVZ relation in the case of $\mathcal{N}=1$ SQED with $N_f$ flavors, regularized by higher derivatives.

## The renormalization group functions defined in terms of the bare coupling constant

The above RG functions do not depend on the renormalization prescription, because they can be expressed via unrenormalized Green functions:

$$0 = \lim_{p\to 0} \frac{d d^{-1}(\alpha_0, \Lambda/p)}{d\ln\Lambda}\Big|_{\alpha=\mathrm{const}} = \lim_{p\to 0}\Big(\frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\ln p}\Big)$$

where in the last equality $\alpha_0$ and $p$ are considered as independent variables. Similarly, differentiating

$$\ln G(\alpha_0, \Lambda/q) = \ln G_{\mathrm{ren}}(\alpha, \mu/q) - \ln Z(\alpha, \Lambda/\mu) \\ + (\text{terms vanishing in the limit } q\to 0)$$

with respect to $\ln\Lambda$ at a fixed value of $\alpha$, in the limit $q\to 0$ we obtain

$$\gamma(\alpha_0) = \lim_{q\to 0}\Big(\frac{\partial \ln G(\alpha_0,\Lambda/q)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial\ln G(\alpha_0,\Lambda/q)}{\partial\ln q}\Big).$$

Therefore, $\beta(\alpha_0)$ and $\gamma(\alpha_0)$ do not depend on an arbitrariness of choosing the renormalization constants.

## NSVZ relation with the HD regularization

With the higher covariant derivative regularization loop integrals giving a $\beta$-function defined in terms of the bare coupling constant are integrals of total derivatives

A.Soloshenko, K.S., hep-th/0304083.

and even integrals of double total derivatives

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704**, (2005), 445.

This allows to calculate one of the loop integrals analytically and obtain the NSVZ relation for the RG functions defined in terms of the bare coupling constant (see below). In the Abelian case this has been done in all loops

K.S., Nucl.Phys. B **852** (2011) 71; arXiv:1404.6717 [hep-th].

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0}$$

$$= \frac{N_f}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

In the non-Abelian case the calculations have been done only in the two-loop approximation and reveal the same features.

## Three-loop calculation for $\mathcal{N}=1$ SQED

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = 2\pi N_f \frac{d}{d\ln\Lambda}\Big\{ \sum_I c_I \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu} \frac{\ln(q^2+M^2)}{q^2} + 4\pi \int \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{e^2}{k^2R_k^2}$$

$$\times \frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big(\frac{1}{q^2(k+q)^2} - \sum_I c_I \frac{1}{(q^2+M_I^2)((k+q)^2+M_I^2)}\Big)\Big[R_k\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$

$$-2e^2N_f\Big(\int\frac{d^4t}{(2\pi)^4}\frac{1}{t^2(k+t)^2} - \sum_J c_J\int\frac{d^4t}{(2\pi)^4}\frac{1}{(t^2+M_J^2)((k+t)^2+M_J^2)}\Big)\Big]$$

$$+4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{e^4}{k^2R_kl^2R_l}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big\{\Big(-\frac{2k^2}{q^2(q+k)^2(q+l)^2(q+k+l)^2}$$

$$+\frac{2}{q^2(q+k)^2(q+l)^2}\Big) - \sum_I c_I\Big(-\frac{2(k^2+M_I^2)}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)}$$

$$\times\frac{1}{((q+k+l)^2+M_I^2)} + \frac{2}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)} - \frac{1}{(q^2+M_I^2)^2}$$

$$\times\frac{4M_I^2}{((q+k)^2+M_I^2)((q+l)^2+M_I^2)}\Big)\Big\}$$

## The RG functions defined in terms of the renormalized coupling constant

RG function defined in terms of the bare coupling constant are scheme independent for a fixed regularization. However, RG functions are usually defined by a different way, in terms of the renormalized coupling constant:

$$\widetilde\beta\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\alpha(\alpha_0,\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\mathrm{const}};$$

$$\widetilde\gamma_i{}^j\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\ln Z_i{}^j(\alpha(\alpha_0,\Lambda/\mu),\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\mathrm{const}}.$$

These RG functions are scheme-dependent. It is possible to prove that they coincide with the RG functions defined in terms of the bare coupling constant, if the boundary conditions

$$Z_3(\alpha,x_0)=1;\qquad Z_i{}^j(\alpha,x_0)=1$$

are imposed on the renormalization constants, where $x_0$ is an arbitrary fixed value of $\ln\Lambda/\mu$.

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459; Phys.Lett. **B730** (2014) 184.

## The NSVZ-scheme with the higher derivatives

$$\widetilde{\gamma}\left(\alpha(\alpha_0,x)\right) = -\frac{d\ln Z\left(\alpha(\alpha_0,x),x\right)}{dx}$$
$$= -\frac{\partial \ln Z(\alpha,x)}{\partial\alpha}\cdot\frac{\partial\alpha(\alpha_0,x)}{\partial x} - \frac{\partial\ln Z\left(\alpha(\alpha_0,x),x\right)}{\partial x},$$

where the total derivative with respect to $x = \ln\Lambda/\mu$ also acts on $x$ inside $\alpha$. Calculating these expressions at the point $x = x_0$ and taking into account that $\partial\ln Z(\alpha,x_0)/\partial\alpha = 0$ we obtain

$$\widetilde{\gamma}(\alpha_0) = \gamma(\alpha_0).$$

The equality for the $\beta$-functions can be proved similarly.

The RG functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ (defined in terms of the renormalized coupling constant) are scheme-dependent. They satisfy the NSVZ relation only in a certain subtraction scheme, called the NSVZ scheme, which is evidently fixed in all loops by the boundary conditions

$$(Z_3)_{\mathrm{NSVZ}}(\alpha_{\mathrm{NSVZ}},x_0) = 1; \qquad Z_{\mathrm{NSVZ}}(\alpha_{\mathrm{NSVZ}},x_0) = 1,$$

if the theory is regularized by higher derivatives.

## The scheme dependence in the three-loop approximation

The two-loop Green function of the matter superfields is given by

$$G(\alpha_0,\Lambda/p) = 1 - \int \frac{d^4k}{(2\pi)^4}\frac{2e_0^2}{k^2 R_k (k+p)^2} + \int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4}{k^2 R_k l^2 R_l}$$

$$\times\left(\frac{1}{(k+p)^2(l+p)^2} + \frac{1}{(l+p)^2(k+l+p)^2} - \frac{(k+l+2p)^2}{(k+p)^2(l+p)^2(k+l+p)^2}\right)$$

$$+\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4 N_f}{k^2 R_k^2 (k+p)^2}\left(\frac{1}{l^2(k+l)^2} - \sum_{I=1}^{n} c_I \frac{1}{(l^2+M_I^2)\left((k+l)^2+M_I^2\right)}\right)$$

$$+O(e_0^6),$$

For $R_k = 1 + k^{2n}/\Lambda^{2n}$ it is possible to find a divergent part of this expression and the corresponding renormalization constant. Then the three-loop renormalization of the coupling constant can be found using the relation

$$\frac{d}{d\ln\Lambda}\left(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\right)\Big|_{p=0} = \frac{N_f}{\pi}\left(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\right).$$

# The scheme dependence in the three-loop approximation

The (three-loop) result for the renormalized coupling constant is not uniquely defined:

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big) - \frac{\alpha^2 N_f}{\pi^3}\Big(\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu}$$
$$- \ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} - N_f b_1\Big) + b_3\Big) + O(\alpha^3),$$

where $b_i$ are arbitrary finite constants.

Similarly, the renormalization constant $Z$ (in the two-loop approximation) for the matter superfields is not also uniquely defined:

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{\alpha^2(N_f+1)}{2\pi^2}\ln^2\frac{\Lambda}{\mu}$$
$$-\frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I - N_f b_1 + N_f + \frac{1}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3),$$

where $g_i$ are other arbitrary finite constants.

The subtraction scheme is fixed by fixing values of the constants $b_i$ and $g_i$.

## The scheme dependence in the three-loop approximation

The RG functions defined in terms of the bare coupling constant are

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3);$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3).$$

They do not depend on the finite constants $b_i$ and $g_i$ (i.e. they are scheme-independent) and satisfy the NSVZ relation.

The RG functions defined in terms of the renormalized coupling constant are

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} + N_f(b_2 - b_1)\Big) + O(\alpha^3)$$

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I - N_f b_1 + N_f g_1\Big) + O(\alpha^3)$$

and depend on a subtraction scheme.

## The NSVZ scheme in the three-loop approximation

The NSVZ scheme is determined by the conditions

$$\alpha_0(\alpha_{\textsf{NSVZ}}, x_0) = \alpha_{\textsf{NSVZ}}; \qquad Z_{\textsf{NSVZ}}(\alpha_{\textsf{NSVZ}}, x_0) = 1$$

For simplicity we set $g_1 = 0$ (this constant can be excluded by a redefinition of $\mu$). In this case $x_0 = 0$ and the above conditions (for the NSVZ scheme) give

$$g_2 = b_1 = b_2 = b_3 = 0.$$

In this case in the considered approximations

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha^3) = \frac{\beta(\alpha)}{\alpha^2};$$

$$\widetilde{\gamma}(\alpha) = \frac{d\ln Z}{d\ln\mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I\Big) + O(\alpha^3) = \gamma(\alpha).$$

As a consequence, in this scheme the NSVZ relation is satisfied.

# RG function for $\mathcal{N}=1$ SQED in different subtraction schemes

NSVZ-scheme with the higher derivatives

$$\widetilde{\gamma}_{\textsf{NSVZ}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\textsf{NSVZ}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f\sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3)\Big).$$

MOM-scheme (The results with the dimensional reduction and with the higher derivative regularization coincide.)

$$\widetilde{\gamma}_{\textsf{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\textsf{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

$\overline{\textsf{DR}}$-scheme

$$\widetilde{\gamma}_{\overline{\textsf{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\textsf{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

## Finite renormalizations

Under a finite renormalization

$$\alpha \to \alpha'(\alpha); \qquad Z'(\alpha', \Lambda/\mu) = z(\alpha) Z(\alpha, \Lambda/\mu)$$

the $\beta$-function and the anomalous dimension defined in terms of the renormalized coupling constant are changed according to the following rules:

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\ln\mu}\Big|_{\alpha_0=\text{const}} = \frac{d\alpha'}{d\alpha}\widetilde{\beta}(\alpha);$$

$$\widetilde{\gamma}'(\alpha') = \frac{d\ln Z'}{d\ln\mu}\Big|_{\alpha_0=\text{const}} = \frac{d\ln z}{d\alpha}\cdot\widetilde{\beta}(\alpha) + \widetilde{\gamma}(\alpha).$$

Using these equations it is easy to see that if $\widetilde{\beta}(\alpha)$ and $\widetilde{\gamma}(\alpha)$ satisfy the NSVZ relation, then

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2 N_f}{\pi}\frac{1-\widetilde{\gamma}'(\alpha')}{1-\alpha^2 N_f (d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}.$$

## Scheme-independence of terms proportional to $(N_f)^1$

Under finite renormalizations

$$\widetilde\beta'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2 N_f}{\pi}\frac{1-\widetilde\gamma'(\alpha')}{1-\alpha^2 N_f (d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}.$$

Quantum corrections to the coupling constant are produced by diagrams which contain at least one loop of the matter superfields. Such a loop gives a factor $N_f$. Thus, it is reasonable to make finite renormalizations of the coupling constant proportional to $N_f$:

$$\alpha'(\alpha)-\alpha = O(N_f);\qquad z(\alpha) = O\left((N_f)^0\right).$$

Then we see that all scheme dependent terms in the $\beta$-function are proportional at least to $(N_f)^2$ in all orders of the perturbation theory. Moreover, it is evident that the terms proportional to $(N_f)^0$ in the anomalous dimension are scheme independent. Also we know that the NSVZ scheme exists. Therefore, the NSVZ relation is satisfied for terms proportional to $(N_f)^1$ in all orders, while terms proportional to $(N_f)^\alpha$ with $\alpha\ge 2$ are scheme dependent.

## Relation between the NSVZ and $\overline{\text{DR}}$ schemes in the three-loop approximation

Using the dimensional reduction and the $\overline{\text{DR}}$ scheme the three-loop $\beta$-function and the two-loop anomalous dimension for $\mathcal{N}=1$ SUSY theories was found in

I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138.

The NSVZ relation can be obtained after a finite renormalization

$$\alpha_{\overline{\text{DR}}} = \alpha_{\text{NSVZ}} - \frac{N_f\alpha^3_{\text{NSVZ}}}{4\pi^2} + O(\alpha^4),$$

implicitly assuming that $Z_{\overline{\text{DR}}}(\alpha_{\overline{\text{DR}}},\Lambda/\mu) = Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}},\Lambda/\mu)$.

If the dimensional reduction and $\overline{\text{DR}}$-scheme is used for a regularization, then (after making the calculations) such redefinitions should be made in all orders of the perturbation theory in order to obtain the NSVZ relation.

With the higher derivative regularization the NSVZ relation is obtained automatically for the above discussed boundary conditions.

## Finite renormalizations in the non-Abelian case

In the non-Abelian case one should take into account the dependence on the Yukawa couplings $\lambda^{ijk}$. Under the finite renormalizations

$$\alpha \to \alpha'(\alpha,\lambda);\quad \lambda \to \lambda'(\alpha,\lambda);\quad {Z'}_i{}^j(\alpha',\lambda',\Lambda/\mu) = z_i{}^k(\alpha,\lambda) Z_k{}^j(\alpha,\lambda,\Lambda/\mu),$$

where we assume that $z$ and $Z$ commute, the NSVZ relation is changed as follows:

$$\widetilde{\beta}'(\alpha',\lambda') = -\frac{\alpha^2}{2\pi(1-C_2\alpha/2\pi)\partial\alpha/\partial\alpha' - \alpha^2 C(R)_l{}^k \partial\ln z_k{}^l/\partial\ln\alpha'}\Big\{3C_2$$

$$-T(R) + \frac{1}{r}C(R)_m{}^n\Big[{\widetilde{\gamma}'}_n{}^m(\alpha',\lambda') - \frac{3}{2}\Big((\lambda')^{ljk}\,{\widetilde{\gamma}'}_l{}^i(\alpha',\lambda')\,\frac{\partial\ln z_n{}^m}{\partial(\lambda')^{ijk}} + \mbox{c.c.}\Big)\Big]$$

$$+\frac{3}{2}\cdot\frac{2\pi}{\alpha^2}\Big(1-C_2\frac{\alpha}{2\pi}\Big)\Big((\lambda')^{ljk}\,{\widetilde{\gamma}'}_l{}^i(\alpha',\lambda')\,\frac{\partial\alpha}{\partial(\lambda')^{ijk}} + \mbox{c.c}\Big)\Big\}_{\alpha=\alpha(\alpha',\lambda')}.$$

We observe that in $L$ loops the terms proportional to tr $\left(C(R)^L\right)$ are the same in both sides of this equation for an arbitrary renormalization prescription.

## Conclusion

- ✓ For $\mathcal{N}=1$ SQED with $N_f$ flavors, regularized by higher derivatives, the NSVZ $\beta$-function is naturally obtained for the renormalization group functions defined in terms of the bare coupling constant. These functions do not depend on the renormalization prescription.
- ✓ The NSVZ $\beta$-function appears because integrals which determine the $\beta$-function defined in terms of the bare coupling constant are factorized into integrals of double total derivatives.
- ✓ If the renormalization group functions are defined in terms of the renormalized coupling constant, the NSVZ $\beta$-function is obtained in a special subtraction scheme, called the NSVZ scheme. In case of using the higher derivative regularization this scheme is obtained by imposing the boundary conditions $(Z_3)_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1$ and $Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1$.
- ✓ Terms proportional to $(N_f)^1$ (or proportional to $\text{tr}\, C(R)^L$ in $L$ loops in the non-Abelian case) are scheme independent and satisfy the NSVZ relation in all schemes.

Thank you for the attention!