Bypassing no-go theorems for consistent interactions in gauge theories

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Suzdal, 4 June 2014

The talk is based on the articles

D.S. Kaparulin, S.L.Lyakhovich and A.A.Sharapov, `Consistent interactions and involution, 2013;`

S.L.Lyakhovich and A.A.Sharapov, `Multiple choice of gauge generators and consistency of interactions, 2014`

General Plan

1. **Introduction:** Gauge identities, symmetries and consistent inclusion of interactions.
2. **Involutivity and consistency of interactions.**
   - Involutive closure of dynamics. Gauge symmetries and (implicit) gauge identities;
   - Involution and control of degrees of freedom;
   - Perturbative inclusion of interactions, no-go theorems;
   - Examples: massive spins 1 and 2.
3. **By-passing perturbative no-go theorems.**
   - Example of by-passing the perturbative obstruction.
   - Multiple choice of free gauge generators and its impact on interaction.
   - One more example: Topological gravity in d dimensions.
   - Concluding remarks

Condensed notation, gauge symmetry and identities

The fields x^i are label by "condensed"index i. Summation over i implies space-time integration. The derivatives ∂_i are variational. The EoM's are the differential equations of a finite order:

$$T_a(x)=0 \tag{1}$$

In Lagrangian theory $a=i$, and $T_i(x)=\partial_i S(x)$.
The Jacobi matrix $\partial_i T_a=J_{ai}(x)$ in Lagrangian theory, $J=\partial_i\partial_j S(x)$ is symmetric, and its on-shell kernel defines the gauge symmetry.
Left on shell kernel corresponds to gauge identities between EoM's:

$$L^a_A J_{ai}(x)\big|_{T=0}=0 \quad \Leftrightarrow \qquad L^a_A T_a\equiv 0 \tag{2}$$

Right on shell kernel corresponds to gauge symmetry of the eqs:

$$J_{ai}(x)R^i_\alpha\big|_{T=0}=0 \quad \Leftrightarrow \quad \delta_\epsilon T_a\big|_{T=0}=0, \quad \delta_\epsilon x^i=R^i_\alpha\epsilon^\alpha \tag{3}$$

Example of gauge identities unrelated to gauge symmetries: Irreducible massive spin 2, without auxiliary fields.

For the massive symmetric traceless tensor, the EoM's read

$$T_{\mu\nu}\equiv(\Box-m^2)h_{\mu\nu}=0, \qquad T_{\mu}\equiv\partial^{\nu}h_{\mu\nu}=0. \tag{4}$$

The EoM's do not have any gauge symmetry, while they possess gauge identities:

$$\partial^{\nu}T_{\mu\nu}-(\Box-m^2)T_{\mu}\equiv 0 \tag{5}$$

Unlike spin 1, making use solely of the irreducible field $h_{\mu\nu}$, it is impossible to equivalently represent the equations (4) in the Lagrangian form.

Introducing auxiliary scalar field that might be interpreted as a trace of $h_{\mu\nu}$, these equation can be made Lagrangian.

The problem of consistent inclusion of interactions.

Given: Free/linear EoM's $T_a^{(0)}(x)=0$, or quadratic action $S_{(0)}(x)$.

Find: $T_a(x)=T_a^{(0)}(x)+T_a^{int}(x)$ or $S(x)=S_{(0)}(x)+S_{int}(x)$ such that the number of degrees of freedom does not change.

Solution by means of Noether procedure:

The idea is to deform action and gauge symmetry order by order

$$S=S_{(0)}+gS_{(1)}+g^2S_{(2)}+\cdots; \quad R=R_{(0)}+gR_{(1)}+g^2R_{(2)}+\cdots; \tag{6}$$

$$R_{(0)}{}^i_\alpha \partial_i S_{(1)} + R_{(1)}{}^i_\alpha \partial_i S_{(0)}=0 \; ; \tag{7}$$

$$R_{(1)}{}^i_\alpha \partial_i S_{(1)} + R_{(2)}{}^i_\alpha \partial_i S_{(0)} + R_{(0)}{}^i_\alpha \partial_i S_{(2)}=0 \tag{8}$$

$$\cdots$$

The procedure controls the mere fact that number of gauge symmetries does not change. This is insufficient to ensure consistency. As we will demonstrate, it is even unnecessary.

Definitions and terminology.

- The order of eq. is the maximal order of derivatives involved; The order of system is the maximal order of the eqs involved;
- A system of order n is said *involutive* if any differential consequence of the order less than or equal to n is already contained in the system.
- Any regular system can be brought to involution by inclusion of the lower order differential consequences. Then, it is said to be the *involutive closure* of the original system.
- The maximal order of derivative of gauge parameter ϵ^α is said the order of gauge symmetry generator R^i_α;
- The order of gauge identity is a sum maximal order of the identity generator L^a_A and the order of the eq. T_a it acts on.

Involutive closure and implicit gauge identities.

Remark 1. If the system is not involutive, it is equivalent to its involutive closure. The involutive closure has the same gauge symmetry, while it may have extra *implicit* gauge identities.

Remark 2. The involutive closure of Lagrangian system is not necessarily Lagrangian.

Example of involutive closure and implicit identity: Proca.

$$T_\mu \equiv (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu - m^2\,\eta_{\mu\nu})A^\nu = 0\,, \qquad \mathrm{ord}(T_\mu) = 2\,. \tag{9}$$

Involutive closure is got by inclusion of the first order consequence:

$$T_\perp \equiv \partial_\mu A^\mu = 0\,, \qquad \mathrm{ord}(T_\perp) = 1. \tag{10}$$

The involutive closure has the third order gauge identity:

$$L^a T_a \equiv 0\,, \quad a = (\mu, \perp), \quad L = (\partial^\mu, m^2)\,, \quad \mathrm{ord}(L) = 3 \tag{11}$$

Involutive closure and covariant degree of freedom count

The number of physical degrees of freedom $\mathcal{N}$ is understood as the number of independent Cauchy data modula gauge transformations.

Given the involutive system with gauge symmetries and identities, $\mathcal{N}$ reads:

$$\mathcal{N}=\sum_{k=0}^{\infty}k(t_k-l_k-r_k). \tag{12}$$

- t_k is a number of equations of order k;
- l_k is the number of gauge identities of k-th order;
- r_k is the number of gauge symmetries of kth order

Example - Proca: $t_2=4$, $t_1=1$, $l_3=1$, hence $\mathcal{N}=2{\cdot}4+1{\cdot}1-3{\cdot}1=6$, that corresponds to 3 polarizations of massive spin 1 in $d=4$.

Involution and perturbative inclusion of consistent interactions

Given the *free involutive gauge system*,

$$T^{(0)}_a = 0 \,, \qquad L^{(0)a}_A T^{(0)}_a \equiv 0 \,, \qquad R^{(0)i}_\alpha \partial_i T^{(0)}_a \equiv 0 \,, \tag{13}$$

perturbative inclusion of interaction is a deformation of the equations, identities and gauge symmetries by nonlinear terms,

$$T^{(0)}_a \;\to\; T_a = T^{(0)}_a + g T^{(1)}_a + g^2 T^{(2)}_a \;+ \dots \,, \tag{14}$$

$$R^{(0)i}_\alpha \;\to\; R^i_\alpha = R^{(0)i}_\alpha + g R^{(1)i}_\alpha + g^2 R^{(2)i}_\alpha + \dots \,, \tag{15}$$

$$L^{(0)a}_A \;\to\; L^a_A = L^{(0)a}_A + g L^{(1)a}_A + g^2 L^{(2)a}_A + \dots \,. \tag{16}$$

Here g is a coupling constant, generators $L^{(1)a}_A$ and $R^{(1)i}_\alpha$ are linear in fields; $T^{(1)}_a$, $L^{(2)a}_A$, and $R^{(2)i}_\alpha$ are bi-linear, etc.
Notice that in each order of the deformation, the orders of equations, identities and symmetries can never decrease.

Involution and perturbative inclusion of interactions

The perturbative consistency implies that deformed EoM's posses deformed gauge symmetries and identities in every order in g:

$$L_A^a T_a \equiv 0 \ , \quad R_\alpha^i \partial_i T_a = U_\alpha^a T_a \tag{17}$$

The expansion in g reads:

$$\begin{gathered} R_\alpha^{(0)i} \partial_i T_a^{(1)} = U_{\alpha a}^{(1)b} T_b^{(0)} - R_\alpha^{(1)i} \partial_i T_a^{(0)}, \\ L_A^{(0)a} T_a^{(1)} + L_A^{(1)a} T_a^{(0)} = 0. \end{gathered} \tag{18}$$

$$\begin{gathered} R_\alpha^{(0)i} \partial_i T_a^{(2)} + R_\alpha^{(1)i} \partial_i T_a^{(1)} + R_\alpha^{(2)i} \partial_i T_a^{(0)} = U_{\alpha a}^{(1)b} T_b^{(1)} + U_{\alpha a}^{(2)b} T_b^{(0)}, \\ L_A^{(0)a} T_a^{(2)} + L_A^{(1)a} T_a^{(1)} + L_A^{(2)a} T_a^{(0)} = 0 \ , \end{gathered} \tag{19}$$

$$\cdot \quad \cdot \quad \cdot$$

The relations (18), (19) impose restrictions on interaction even if there is no gauge symmetry. Resolving the relations above order by order one constructs all the consistent interactions. If any obstruction arise in some order, it is a no-go theorem.

Summary of the procedure for perturbative inclusion of interactions

1. The free system is brought to the involutive form.
2. All the gauge symmetries and identities are identified.
3. The interaction vertices are iteratively included to comply with three basic requirements in every order of coupling constant:

- The field equations have to remain involutive;
- The gauge algebra of the involutive system can be deformed, though the number of gauge symmetry and gauge identity generators remains the same as it has been in the free theory;
- The number of physical degrees of freedom, being defined by n_k,l_k,r_k, cannot change, while all the these numbers can.

This procedure ensures finding all the consistent interaction vertices, for any regular system of free field equations.

An example of by-passing the perturbative no-go theorem for consistent interactions.

Consider the following action in $2d$ Minkowski space:

$$S[\phi,A]=\int d^2x\phi\left(\partial_\mu A^\mu+\frac{g}{2}A_\mu A^\mu\right)\ . \tag{20}$$

The field equations read

$$\partial_\mu A^\mu+\frac{g}{2}A_\mu A^\mu=0\ ,\qquad D^-_\mu\phi=0, \tag{21}$$

where $D^\pm_\mu=\partial_\mu\pm gA_\mu$, and $\epsilon^{\mu\nu}D^-_\mu D^-_\nu=g\epsilon^{\mu\nu}\partial_\mu A_\nu\equiv gF$.
Unless $F\neq0$, it is a topological theory, as there is a consequence $\phi=0$, while the two components of A_μ are subject to a single equation, so they are pure gauge.
In the free limit $g\to0$, ϕ is still fixed, while A_μ should be pure gauge for the same reason as with $g\neq0$.
However, the Noether procedure leads to the no-go theorem.

An example of by-passing the perturbative no-go theorem for consistent interactions.

Free Lagrangian $L = \phi\partial_\mu A^\mu$ has an *irreducible gauge symmetry*:

$$\delta_\varrho \phi = 0, \qquad \delta_\varrho A^\mu = \epsilon^{\mu\nu}\partial_\nu \varrho, \tag{22}$$

that gauges out A^μ, while $\phi = 0$ shell.The free model is topological. The cubic vertex ϕA^2 is *not invariant* w.r.t. (22) even modulo a total divergence and the free equations,

$$\delta_\varrho \int d^2x \phi A^2 = -2\int d^2x \phi F \varrho \neq 0 \,. \tag{23}$$

It is a standard no-go theorem for the cubic interaction.
The interaction is consistent, however, in the sense that it does not change the degree of freedom number.
The explanation is that the interacting theory has the *reducible gauge symmetry* with a smooth limit that differs from (22)

Multiple choice of gauge symmetries and consistency of interactions.

Lagrangian $L = \phi(\partial_\mu A^\mu + \frac{g}{2} A^2)$ enjoys gauge symmetry

$$\delta_\varepsilon \phi = 0 \ , \qquad \delta_\varepsilon A^\mu = g\varepsilon^\mu - \epsilon^{\mu\nu} D^+{}_\nu (F^{-1} D^+_\lambda \varepsilon^\lambda) \ , \tag{24}$$

where ε^μ is gauge parameter. The gauge-for-gauge transform reads

$$\delta_\varkappa \varepsilon^\mu = \epsilon^{\mu\nu} D^+_\nu \varkappa \ , \tag{25}$$

The free limit of gauge transformations (24), (25) reads

$$\delta_\varepsilon \phi = 0, \quad \delta_\varepsilon A^\mu = -\epsilon^{\mu\nu} \partial_\nu (F^{-1} \partial_\lambda \varepsilon^\lambda), \quad \delta_\varkappa \varepsilon^\lambda = \epsilon^{\lambda\nu} \partial_\nu \varkappa . \tag{26}$$

These transformations reproduce the irreducible free transformation $\delta_\varrho A^\mu = \epsilon^{\mu\nu} \partial_\nu \varrho$ with $\varrho = -F^{-1} \partial_\lambda \varepsilon^\lambda$.
At the free level the reducible and irreducible transformations are equivalent, as each of them spans the on-shell kernel of the $d^2 S$.
The reducible symmetry ***is compatible*** *with interaction,*
while the irreducible one ***is not.***

One more example and conclusions.

Topological gravity in $d>2$?
Consider the action and equations

$$S[\phi,g]=\int \phi R\sqrt{-g}d^4x, \tag{27}$$

$$R=0, \qquad (\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box-R_{\mu\nu})\phi=0. \tag{28}$$

Linearizing over the background $g_{\mu\nu}=\eta_{\mu\nu}$ one can find the gauge symmetry that gauges out all the degrees of freedom of g, while ϕ is not dynamical.
The field ϕ remains non-dynamical in general, so $\frac{d(d+1)}{2}$ components of $g_{\mu\nu}$ are subject to a single scalar equation $R=0$, that means there are no physical degrees of freedom.
The linearized symmetry again does not admit deformation, while the complete theory remains topological. It is one more less trivial example of bypassing the no-go theorem.

Summary of the proposed procedure for inclusion of interactions.

- The free system is brought to the involutive form;
- The generating set is chosen (the choice isn't unique) for gauge symmetries and identities of the involutive system;
- The deformations are iterated for EoM's, identities and symmetries in a consistent way with the DoF count relation.
- If one generating set of symmetries and identities obstructs interaction, another resolution can by-pass the obstruction.

Advantages in comparison with Noether procedure

- It controls DoF number, not just gauge symmetry. All the vertices are identified once they comply with the DoF number;
- It applies to Lagrangian and non-Lagrangian systems;
- It allows one to by-pass the no-go theorems for certain generating set by switching to another generating set.

LET US INTERACT CONSISTENTLY!

THANK YOU!