BFKL Pomeron at Strong Coupling from the Quantum Spectral Curve of N=4 SYM

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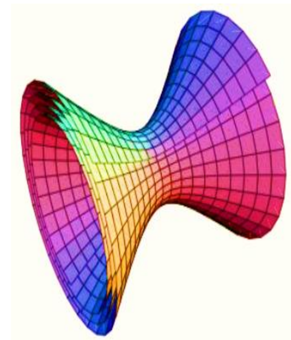
Based on arXiv:1402.0871
(N. Gromov, F. L.-M., G. Sizov, S. Valatka)

AdS/CFT duality

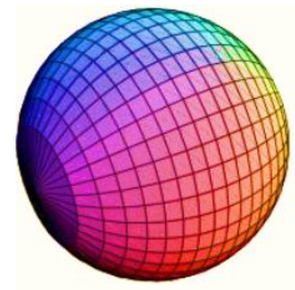
$\mathcal{N}=4$ Yang-Mills theory
in four dimensions
(planar limit)

$\Longleftrightarrow$

superstring theory in
$AdS_5 \times S^5$



$\times$



Operator conformal
dimensions Δ_i

$\Longleftrightarrow$

spectrum of
string energies E_i

The problem we study: finding the spectrum

Hope for exact solution of both theories!

Key tool: **integrability**

Motivation

- Understand gauge theory at strong coupling
- Similarities between N=4 SYM and QCD (BFKL, ….)
- Explore quantum strings, AdS/CFT

Lipatov, Faddeev, Korchemsky, …

N=4 Supersymmetric Yang-Mills (SYM)

$$S = \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 + \text{fermions} \right\}$$

$A_\mu, \;\; \Phi_i, \;\; \psi_j$ in adjoint of $SU(N_c)$

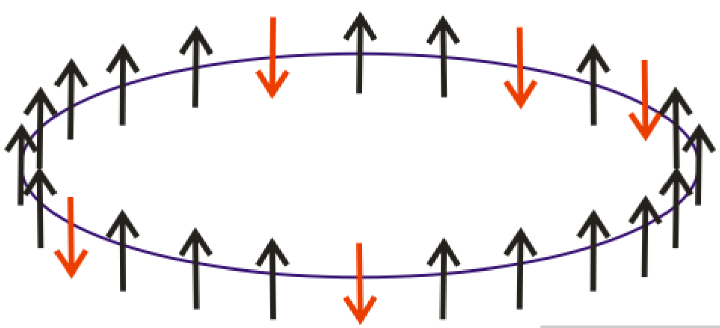
Planar limit: $N_c \to \infty$, $\lambda = g_{YM}^2 \, N_c$ is fixed – 't Hooft coupling

Theory is conformal $\Longrightarrow$ $\left\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \right\rangle = \frac{\text{const}}{|x-y|^{\Delta_i + \Delta_j}}$

$$\mathcal{O}(x) = \mathrm{Tr}\,(\Phi_1 \Phi_2 \Phi_1 \Phi_1 \Phi_2 \ldots \Phi_1 \Phi_2 \Phi_2)\,(x)$$ + permutations

$$\Delta_i = \Delta_i(\lambda)$$

At 1 loop – eigenvalues of integrable XXX spin chain!



Minahan, Zarembo 2002

String theory on AdS_5 x S^5

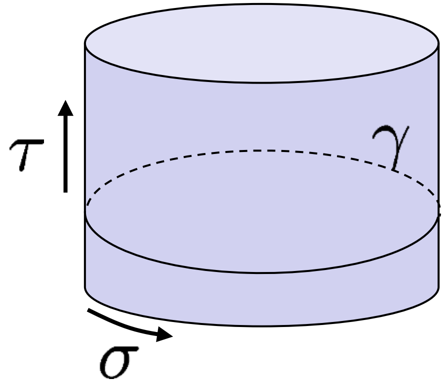
$$S = \frac{\sqrt{\lambda}}{4\pi} \int \partial_\mu \vec{X} \cdot \partial^\mu \vec{X} \;\; d\sigma d\tau + fermions$$

Metsaev,Tseytlin;Bena,Polchinski,Roiban;
Kazakov,Marshakov,Minahan,Zarembo

λ = gauge theory 't Hooft coupling

Coset sigma model $\dfrac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)}$

Infinitely many integrals of motion! Encoded in monodromy matrix.



$$\Omega(u,\tau) = P \exp \oint_\gamma \mathcal{A}(u,\tau,\sigma)$$

$$\partial_\tau \mathrm{Tr}\, \Omega(u,\tau) = 0$$

on E.O.M.

Classical spectral curve

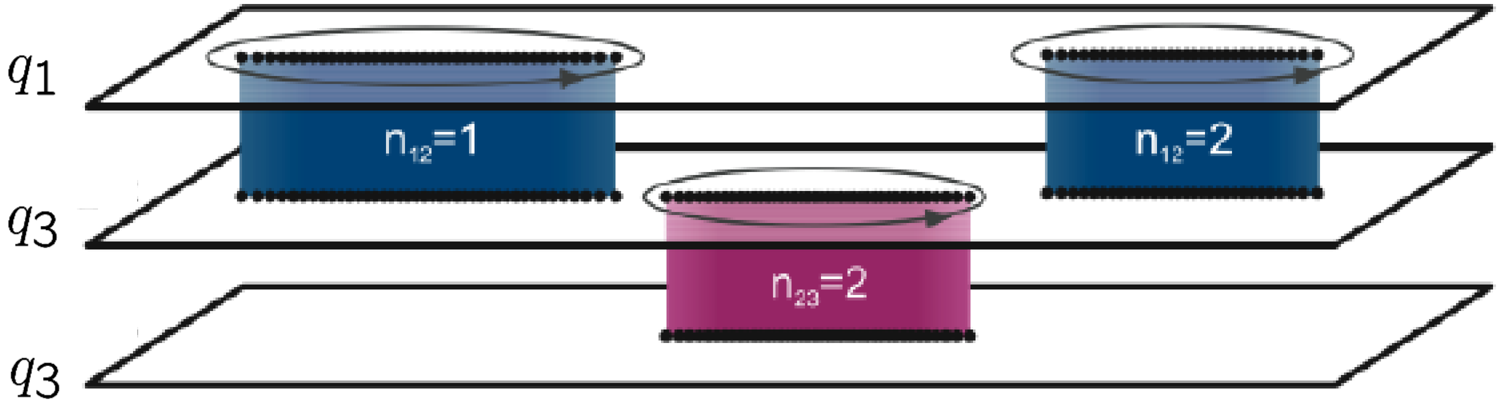
$$\Omega(u) \rightarrow \{e^{iq_1(u)}, e^{iq_2(u)}, e^{iq_3(u)}, e^{iq_4(u)}, e^{iq_5(u)}, e^{iq_6(u)}, e^{iq_7(u)}, e^{iq_8(u)}\}$$

Beisert,Kazakov,Sakai,Zarembo; ..

Eigenvalues are integrals of motion,
they define an 8-sheet algebraic curve

$$\det(\Omega(u) - z) = 0$$

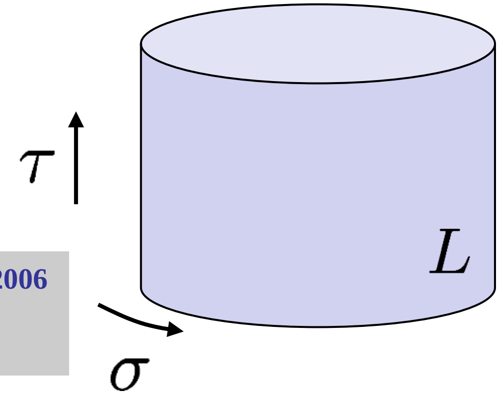
Branch cuts in the quasimomenta $q_i(u)$ connect sheets



Quantum integrability

String sigma model = integrable 2d QFT in finite volume L

L = # of fields in gauge theory operator



Exact S matrix

⇩

Asymptotic Bethe ansatz for energy levels at large L

Beisert, Staudacher 2005, 2006
Beisert,Hernandez,Lopez
Beisert,Eden,Staudacher

⇩

Thermodynamic Bethe ansatz (TBA) – infinite set of nonlinear integral equations.

Gromov, Kazakov, Kozak, Vieira 2009
Arutyunov, Frolov 2009
Bombardelli, Fiorovanti, Tateo 2009

$$\log\frac{Y_{1,1}}{\mathbf{Y}_{1,1}} = K_{m-1} * \log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1a} * \log(1+Y_{a,0})$$

$$\log\frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} = K_{m-1} * \log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{B}^{(01)}_{1a} * \log(1+Y_{a,0})$$

$$\log\frac{\overline{Y}_{1,s}}{\overline{\mathbf{Y}}_{1,s}} = -K_{s-1,t-1} * \log\frac{1+\overline{Y}_{1,t}}{1+\overline{\mathbf{Y}}_{1,t}} - K_{s-1}\hat{*}\log\frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}$$

$$\log\frac{Y_{a,1}}{\mathbf{Y}_{a,1}} = -K_{a-1,b-1} * \log\frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} - K_{a-1}\hat{*}\log\frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} + \left[\mathcal{R}^{(01)}_{ab} + \mathcal{B}^{(01)}_{a-2,b}\right] * \log(1+Y_{b,0})$$

$$\log\frac{Y_{a,0}}{\mathbf{Y}_{a,0}} = \left[2\mathcal{S}_{ab} - \mathcal{R}^{(11)}_{ab} + \mathcal{B}^{(11)}_{ab}\right] * \log(1+Y_{b,0}) + 2\left[\mathcal{R}^{(10)}_{ab} + \mathcal{B}^{(10)}_{a,b-2}\right] \underset{\text{sym}}{*} \log\frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} + 2\mathcal{R}^{(10)}_{a1} \underset{\text{sym}}{\hat{*}} \log\frac{1+Y_{1,1}}{1+\mathbf{Y}_{1,1}} - 2\mathcal{B}^{(10)}_{a1} \underset{\text{sym}}{\hat{*}} \log\frac{1+\overline{Y}_{2,2}}{1+\overline{\mathbf{Y}}_{2,2}}$$

Gives spectrum at any L and any coupling!

But very complicated.

TBA equations

$$\log \frac{Y_{1,1}}{\mathbf{Y}_{1,1}} = K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1a} * \log(1+Y_{a,0})$$

$$\log \frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} = K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{B}^{(01)}_{1a} * \log(1+Y_{a,0})$$

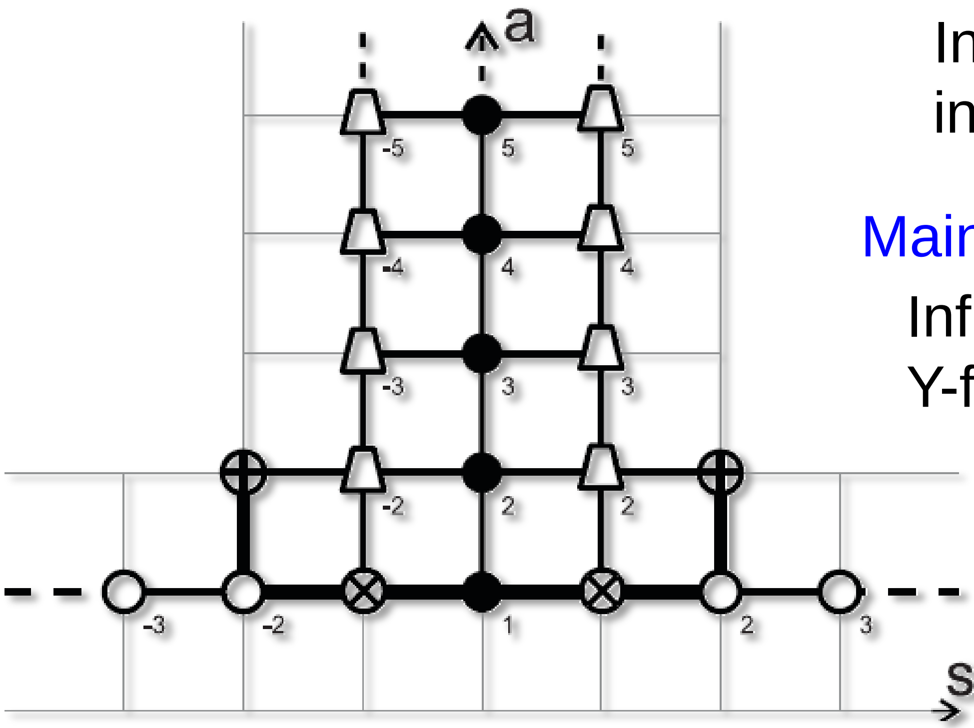
$$\log \frac{\overline{Y}_{1,s}}{\overline{\mathbf{Y}}_{1,s}} = -K_{s-1,t-1} * \log \frac{1+\overline{Y}_{1,t}}{1+\overline{\mathbf{Y}}_{1,t}} - K_{s-1} \hat{*} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}$$

$$\log \frac{Y_{a,1}}{\mathbf{Y}_{a,1}} = -K_{a-1,b-1} * \log \frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} - K_{a-1} \hat{*} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} + \left[\mathcal{R}^{(01)}_{ab} + \mathcal{B}^{(01)}_{a-2,b}\right] * \log(1+Y_{b,0})$$

$$\log \frac{Y_{a,0}}{\mathbf{Y}_{a,0}} = \left[2\mathcal{S}_{ab} - \mathcal{R}^{(11)}_{ab} + \mathcal{B}^{(11)}_{ab}\right] * \log(1+Y_{b,0}) + 2\left[\mathcal{R}^{(10)}_{ab} + \mathcal{B}^{(10)}_{a,b-2}\right] \underset{\text{sym}}{*} \log \frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} + 2\mathcal{R}^{(10)}_{a1} \underset{\text{sym}}{\hat{*}} \log \frac{1+Y_{1,1}}{1+\mathbf{Y}_{1,1}} - 2\mathcal{B}^{(10)}_{a1} \underset{\text{sym}}{\hat{*}} \log \frac{1+\overline{Y}_{2,2}}{1+\overline{\mathbf{Y}}_{2,2}}$$

Can this be the final solution to the spectral problem in N=4 SYM?!

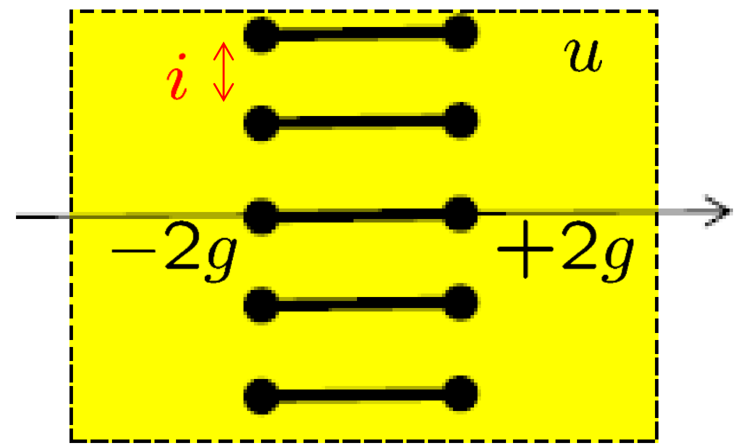
Y-functions and analyticity



Infinite set of unknown functions in TBA equations – $Y_{a,s}(u)$

Main difficulty:

Infinite ladder of branch points in Y-functions



$g = \frac{\sqrt{\lambda}}{4\pi}$

TBA ⇨ Simple functional eqns (Y-system, T-system, Hirota)

Gromov, Kazakov, Vieira 2009

Solution is known, but need to ensure correct analytical properties

The Pμ system

Thermodynamic Bethe Ansatz

$$
\begin{aligned}
\log\frac{Y_{1,1}}{\mathbf{Y}_{1,1}} &= K_{m-1}*\log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}}+\mathcal{R}^{(01)}_{1a}*\log(1+Y_{a,0})\\
\log\frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} &= K_{m-1}*\log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}}+\mathcal{B}^{(01)}_{1a}*\log(1+Y_{a,0})\\
\log\frac{\overline{Y}_{1,s}}{\overline{\mathbf{Y}}_{1,s}} &= -K_{s-1,t-1}*\log\frac{1+\overline{Y}_{1,t}}{1+\overline{\mathbf{Y}}_{1,t}}-K_{s-1}\hat{*}\log\frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}\\
\log\frac{Y_{a,1}}{\mathbf{Y}_{a,1}} &= -K_{a-1,b-1}*\log\frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}}-K_{a-1}\hat{*}\log\frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}\\
&\quad+\left[\mathcal{R}^{(01)}_{ab}+\mathcal{B}^{(01)}_{a-2,b}\right]*\log(1+Y_{b,0})\\
\log\frac{Y_{a,0}}{\mathbf{Y}_{a,0}} &= \left[2\mathcal{S}_{ab}-\mathcal{R}^{(11)}_{ab}+\mathcal{B}^{(11)}_{ab}\right]*\log(1+Y_{b,0})+2\left[\mathcal{R}^{(10)}_{ab}+\mathcal{B}^{(10)}_{a,b-2}\right]\underset{\text{sym}}{*}\log\frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}}\\
&\quad+2\mathcal{R}^{(10)}_{a1}\underset{\text{sym}}{\hat{*}}\log\frac{1+Y_{1,1}}{1+\mathbf{Y}_{1,1}}-2\mathcal{B}^{(10)}_{a1}\underset{\text{sym}}{\hat{*}}\log\frac{1+\overline{Y}_{2,2}}{1+\overline{\mathbf{Y}}_{2,2}}
\end{aligned}
$$

simplification

Pμ system

Riemann-Hilbert problem
for 4+5 functions
with simple analytical properties

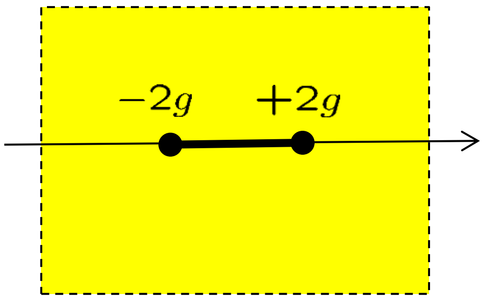
Gromov, Kazakov,Leurent,Volin 2013

Pμ-system/Quantum Spectral Curve

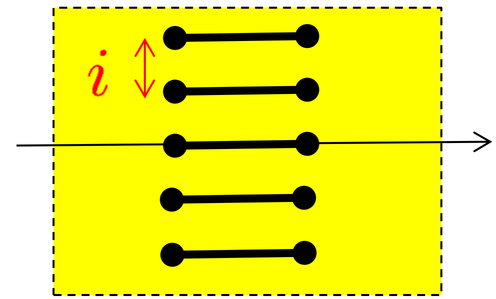
TBA equations reduced to only 4+5 functions

$a, b = 1, \ldots, 4$

$$\mathbf{P}_a(u)$$



$$\mu_{ab}(u) = -\mu_{ba}(u)$$

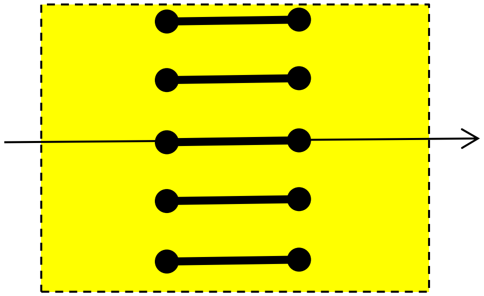


Branchpoints are quadratic

$\mu_{12}\mu_{34} - \mu_{13}\mu_{24} + \mu_{14}^2 = 1 \quad \mu_{14} = \mu_{23}$

Analytic continuation around branchpoint

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c$$



$\chi^{14} = -\chi^{23} = \chi^{32} = -\chi^{41} = 1$

$$\tilde{\mu}_{a,b} = \mu_{a,b} + \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a$$

$$\tilde{\mu}_{a,b} = \mu_{a,b}(u+i)$$

The energy from Pμ-system

Conserved charges are encoded in asymptotics

E.g. for twist operators $\mathrm{Tr}(ZD^S Z^{L-1})$ (where $Z = \Phi_1 + i\Phi_2$)

$$\mathbf{P}_1 \simeq A_1 u^{-L/2}$$
$$\mathbf{P}_2 \simeq A_2 u^{-L/2-1}$$
$$\mathbf{P}_3 \simeq A_3 u^{+L/2}$$
$$\mathbf{P}_4 \simeq A_4 u^{+L/2-1}$$

And anomalous dimension Δ is found from

$$A_2 A_3 = \frac{[(L-S+2)^2-\Delta^2][(L+S)^2-\Delta^2)]}{16iL(L+1)}$$
$$A_4 A_1 = \frac{[(L+S-2)^2-\Delta^2][(L-S)^2-\Delta^2]}{16iL(L-1)}\,.$$

Relation to classical spectral curve

In the classical limit $$P_a(u) \simeq e^{\int^u q_a(v)dv}$$

Thus P-mu system may be viewed as a quantum version of the curve

Strongly reminds WKB wavefunction $$\psi(x) \simeq e^{\int^x p(y)dy}$$

We expect P_a should be the exact Baxter Q-functions
= wavefunctions in separated variables

Gromov, Kazakov,Leurent,Volin 2013

Application:

small spin limit
and pomeron intercept

Twist operators at small spin

$$\mathcal{O} = \mathrm{Tr}(ZD^S Z^{L-1})$$

Gromov, F.L.-M. Sizov, Valatka '14

For S=0 this operator is protected (BPS)
We consider the peculiar limit when $S \to 0$

1. $$\begin{cases} \tilde{\mathbf{P}}_1 = -\mu_{1,2}\mathbf{P}_3 + \mu_{1,3}\mathbf{P}_2 - \mu_{1,4}\mathbf{P}_1, \\ \tilde{\mathbf{P}}_2 = -\mu_{1,2}\mathbf{P}_4 + \mu_{2,3}\mathbf{P}_2 - \mu_{2,4}\mathbf{P}_1, \\ \tilde{\mathbf{P}}_3 = -\mu_{1,3}\mathbf{P}_4 + \mu_{2,3}\mathbf{P}_3 - \mu_{3,4}\mathbf{P}_1, \\ \tilde{\mathbf{P}}_4 = -\mu_{1,4}\mathbf{P}_4 + \mu_{2,4}\mathbf{P}_3 - \mu_{3,4}\mathbf{P}_2. \end{cases} \Rightarrow \begin{cases} \tilde{\mathbf{P}}_1 = -\mathbf{P}_3 + \mathbf{P}_1, \\ \tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_2 - \mathbf{P}_1 \sinh(2\pi u), \\ \tilde{\mathbf{P}}_3 = -\mathbf{P}_3, \\ \tilde{\mathbf{P}}_4 = +\mathbf{P}_4 + \mathbf{P}_3 \sinh(2\pi u). \end{cases}$$

2. $$\tilde{\mu}_{ab}(u) = \mu_{ab}(u+i)$$

3. $$\tilde{\mu}_{a,b} = \mu_{a,b} + \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a$$

Key simplification: all $\mathbf{P}_a$ are small $\Rightarrow$ μ_{ab} are trivial

Easy to solve in terms of

$$x + \frac{1}{x} = \frac{u}{g}$$

e.g. $\mathbf{P_1} = C/x,\ \mathbf{P_3} = Cx - C/x$

As a result we get

$$\Delta = L + S\frac{4\pi g I_{L+1}(4\pi g)}{L I_L(4\pi g)} + O(S^2)$$

Matches result of [Basso 2011] !

S^2 order

P-mu system can be solved order by order in S

$$\Delta = L + S\Delta^{(1)}(g) + S^2\Delta^{(2)}(g) + \ldots$$

We computed the S^2 term at any coupling for L=2,3,4

$$\Delta^{(2)}_{L=2} = \oint \frac{du_x}{2\pi i} \oint \frac{du_y}{2\pi i} \left[\frac{8\pi^3 I_1(\sqrt{\lambda})^2 \left(x^3 - \left(x^2+1\right) y\right) \left(2\pi g I_1(\sqrt{\lambda}) - I_2(\sqrt{\lambda})\right)}{I_2(\sqrt{\lambda})^3 \left(x^3 - x\right) y^2} \right.$$

$$\left. + \ldots - \frac{4\pi^3 (\mathrm{sh}^x_-)^2 \left(x^2+1\right) y^2}{I_2(\sqrt{\lambda})^2 \left(x^2-1\right)} \right] \frac{1}{4\pi i} \partial_u \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)}$$

At weak coupling – matches known predictions to 4 loops!

$$\gamma^{(2)}_{J=2} = -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32\pi^2\zeta_3}{3}\right) + g^6\left(200\pi^2\zeta_5 - 2016\zeta_7\right)$$

$$+ g^8\left(-\frac{16\pi^6\zeta_3}{45} - \frac{88\pi^4\zeta_5}{9} - \frac{9296\pi^2\zeta_7}{3} + 27720\zeta_9\right) + \ldots$$

Konishi operator at strong coupling

Re-expansion of small S result ⇒ predictions for operators with finite S

Simplest unprotected operator, L=2, S=2

$$\mathcal{O} = \mathrm{Tr}(Z D^2 Z)$$

Our 3-loop prediction for string theory

$$\Delta_{Konishi} = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta(3)+\frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15\,\zeta(5)}{2} + 6\,\zeta(3) - \frac{1}{2}}{\lambda^{5/4}}$$

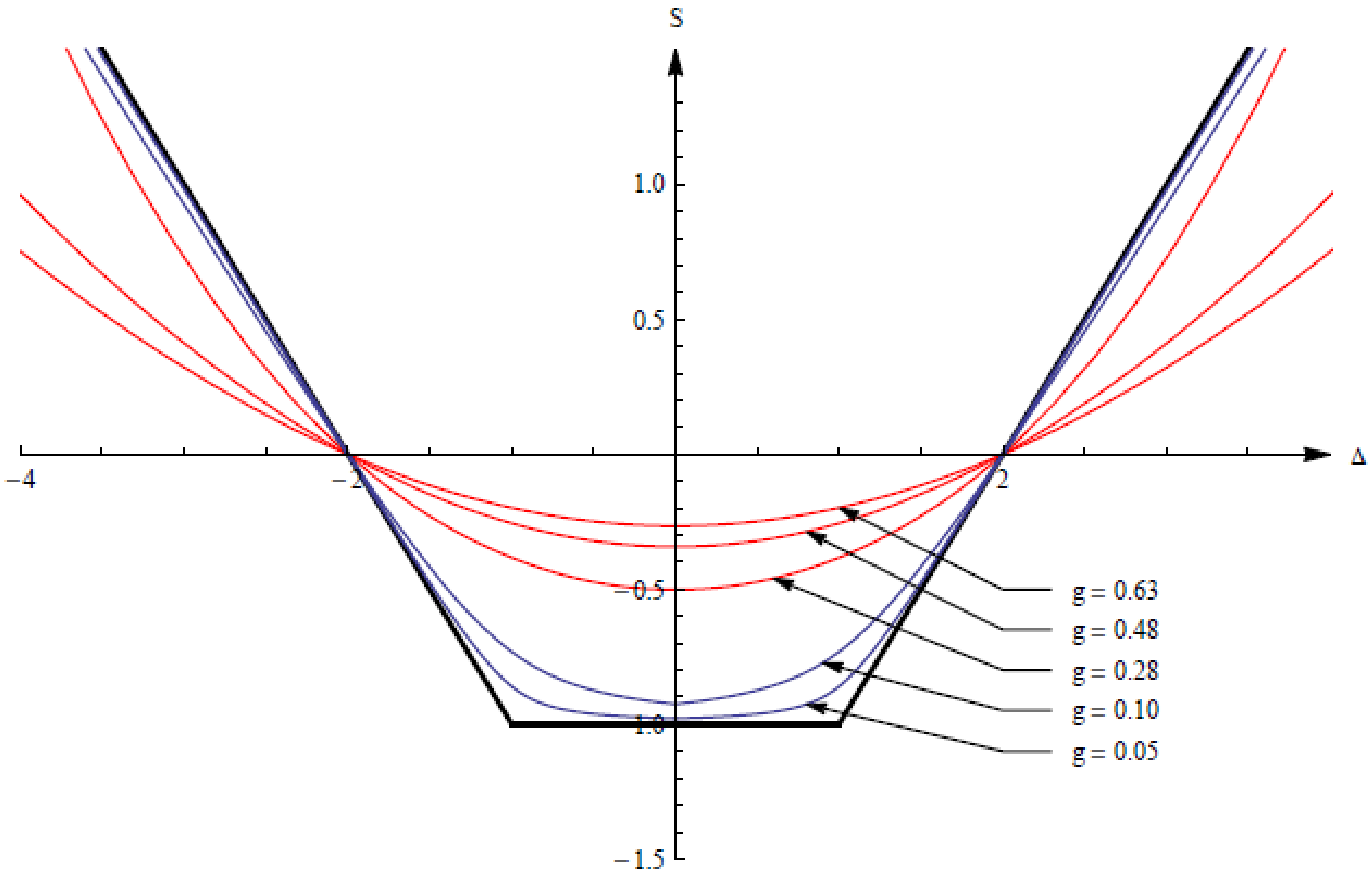
Gubser,Klebanov, Polyakov 98

Gromov,Serban, Shenderovich,Volin '11 Roiban,Tseytlin '11 Mazzuchato,Vallilo '11

Gromov, Valatka'11

Gromov, F.L.-M. Sizov, Valatka '14

S (Δ) for L=2



BFKL pomeron intercept

$$j(\Delta) \equiv 2 + S(\Delta)$$

With our results we can compute the intercept $j(0)$ at strong coupling:

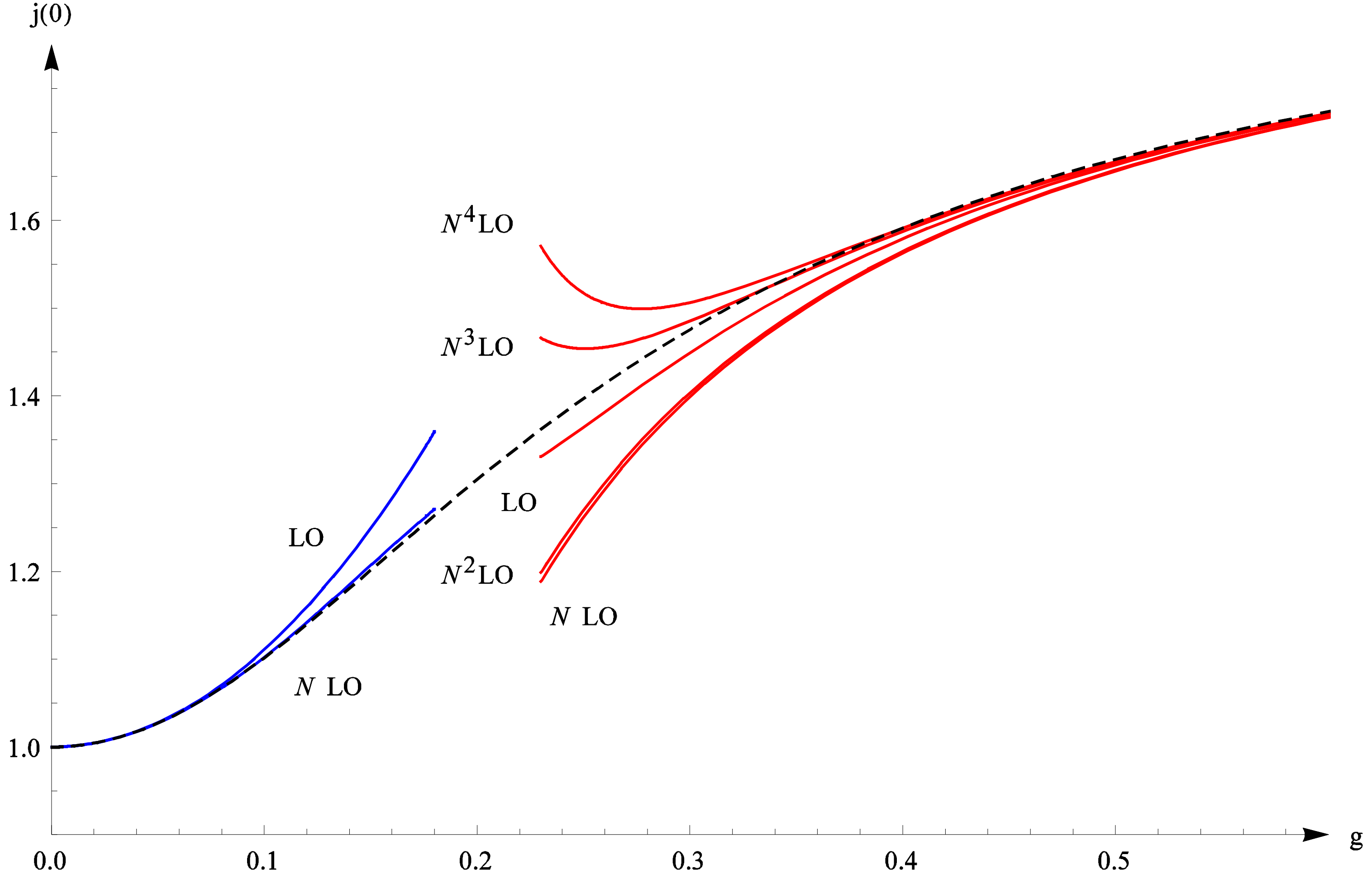
Costa,Goncalves,Penedones 2012

Kotikov,Lipatov 2013

$$j(0) = 2 + S(0) = 2 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4\,\lambda^{3/2}} + (6\zeta_3 + 2)\,\frac{1}{\lambda^2}$$
$$+ \left(18\,\zeta_3 + \frac{361}{64}\right)\frac{1}{\lambda^{5/2}} + \left(39\,\zeta_3 + \frac{447}{32}\right)\frac{1}{\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^{7/2}}\right)$$

New result
[Gromov,F.L.-M.,Sizov,Valatka 2014]

BFKL pomeron intercept



Conclusions

- Quantum Spectral Curve/ $\mathbf{P}\mu$ system applied to study twist operators in N=4 SYM
- S^2 term in conformal dimension found at any coupling
- Tested at weak and strong coupling
- New strong coupling prediction for Konishi operator
- Two new terms in BFKL intercept at strong coupling
- Other applications: quark-antiquark potential, ABJM theory

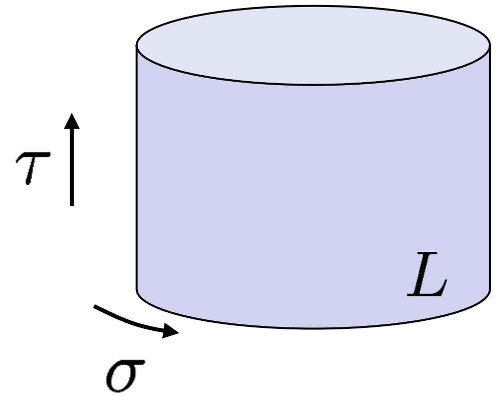
Gromov,Sever 12
Gromov, F.L.-M.,Sizov 13

Cavaglia,Fioravanti,Gromov,Tateo 14
Gromov, Sizov 14

- Many future directions: exact wavefunctions and separated variables, correlators, BFKL, …

Asymptotic energy spectrum

string sigma model =
2d QFT with finite spatial size L

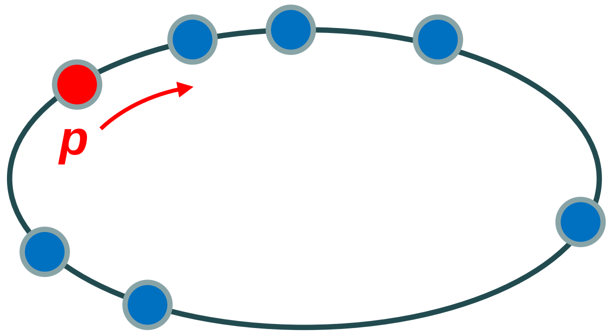


$L \longrightarrow \infty$

Factorized scattering;
quantization of momenta

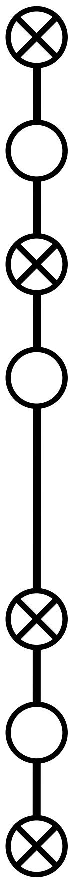
$$e^{ip_i L} = \prod_{j=1}^{M} S(p_i, p_j)$$

– Bethe ansatz equations



$$\Delta = \sum \epsilon(p_j)$$

All-loop asymptotic Bethe ansatz (ABA)



$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{1,k}x^+_{4,j}}{1-1/x_{1,k}x^-_{4,j}},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j}^{K_4} \left(\frac{1-1/x^+_{4,k}x^-_{4,j}}{1-1/x^-_{4,k}x^+_{4,j}}\right) \sigma^2(x_{4,k},x_{4,j})$$
$$\times \prod_{j=1}^{K_1} \frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}} \prod_{j=1}^{K_7} \frac{1-1/x^-_{4,k}x_{7,j}}{1-1/x^+_{4,k}x_{7,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{7,k}x^+_{4,j}}{1-1/x_{7,k}x^-_{4,j}}.$$

At any coupling!

But only for $L \to \infty$

$$E = \sum_j \epsilon(u_{4,j})$$

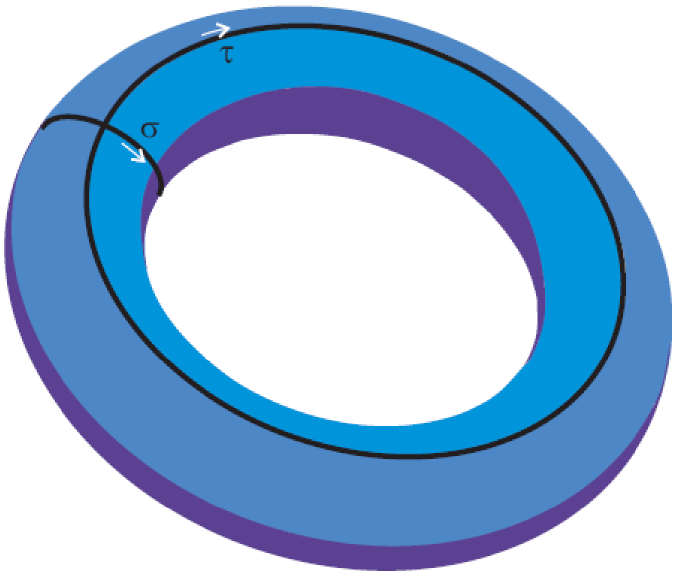
Beisert, Staudacher 2005, 2006
Beisert,Hernandez,Lopez
Beisert,Eden,Staudacher

L = # of fields in operator = spatial size in 2d

What is the spectrum at finite L ???

Thermodynamic Bethe ansatz (TBA)

Zamolodchikov; Yang,Yang



$$Z(L,R) = Z(R,L)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\sum e^{-E_n(L)R} \qquad\qquad \sum e^{-E_n(R)L}$$

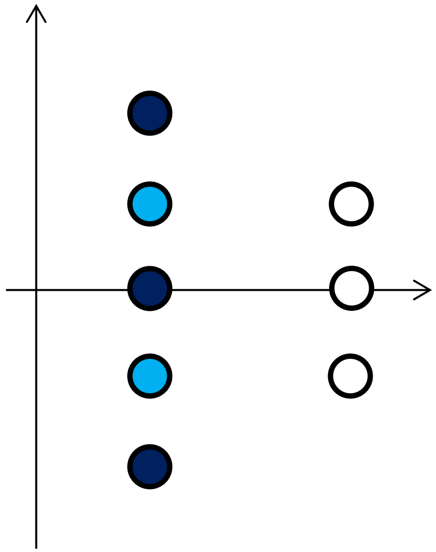
$$\downarrow$$

$$e^{-E_0(L)R}$$

From **asymptotic** spectrum $(R \to \infty)$ we get ground state energy at **finite** volume!

$$E_0(L) = -\lim_{R\to\infty} \frac{\log \sum e^{-E_n(R)L}}{R}$$

TBA equations



$R \to \infty$: Bethe roots form complexes

$Y_n(u)$ are expressed in terms of densities $\rho(u)$

From (mirror) asymptotic Bethe equations

Gromov, Kazakov, Kozak, Vieira 2009
Arutyunov, Frolov 2009
Bombardelli, Fiorovanti, Tateo 2009

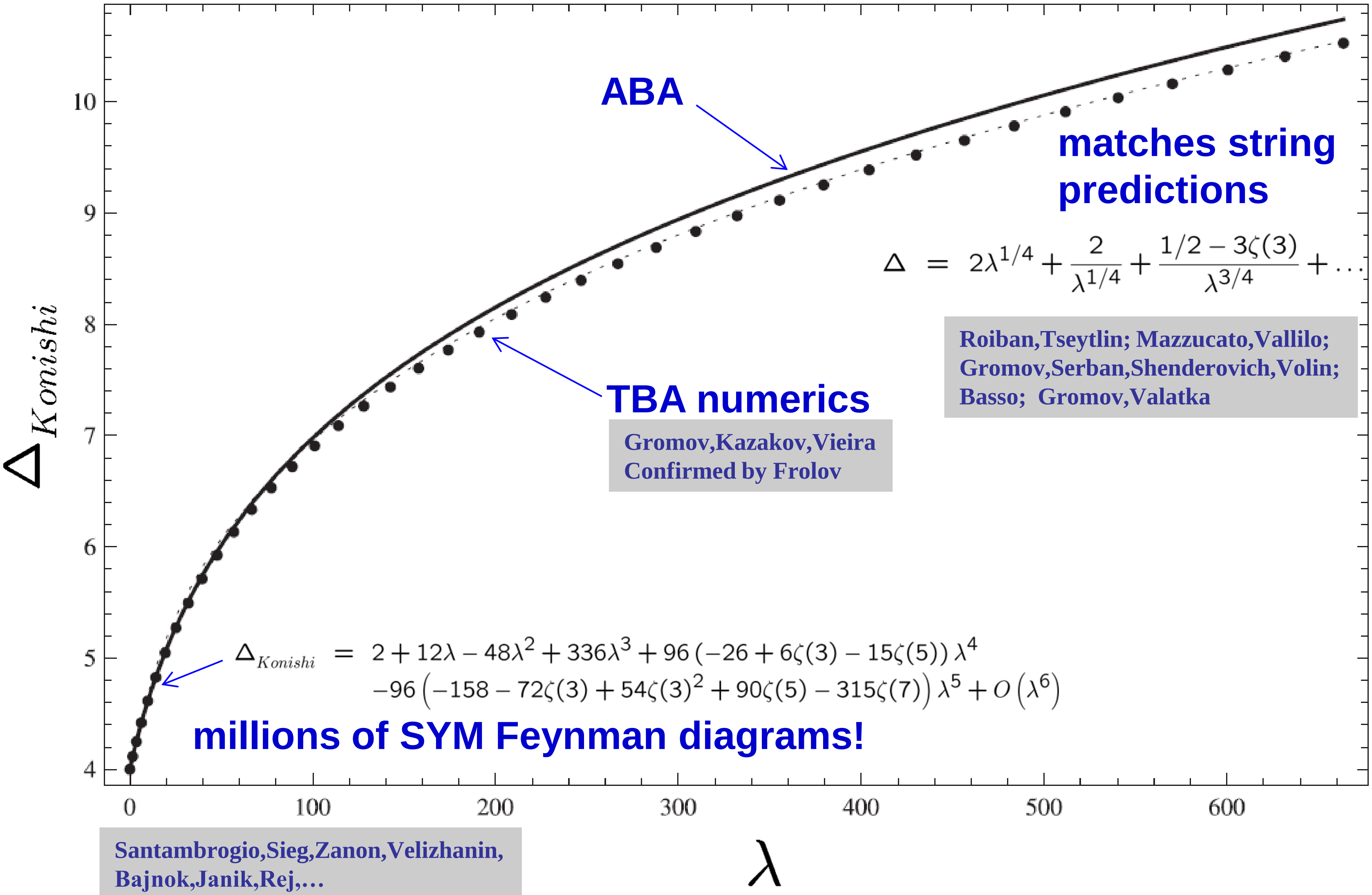
$\Longrightarrow$ TBA equations for Y-functions

$$\log Y_n(u) = \Phi_n(u) + \int dv\; K_{n,m}(u,v) \log(1+Y_m(v))$$

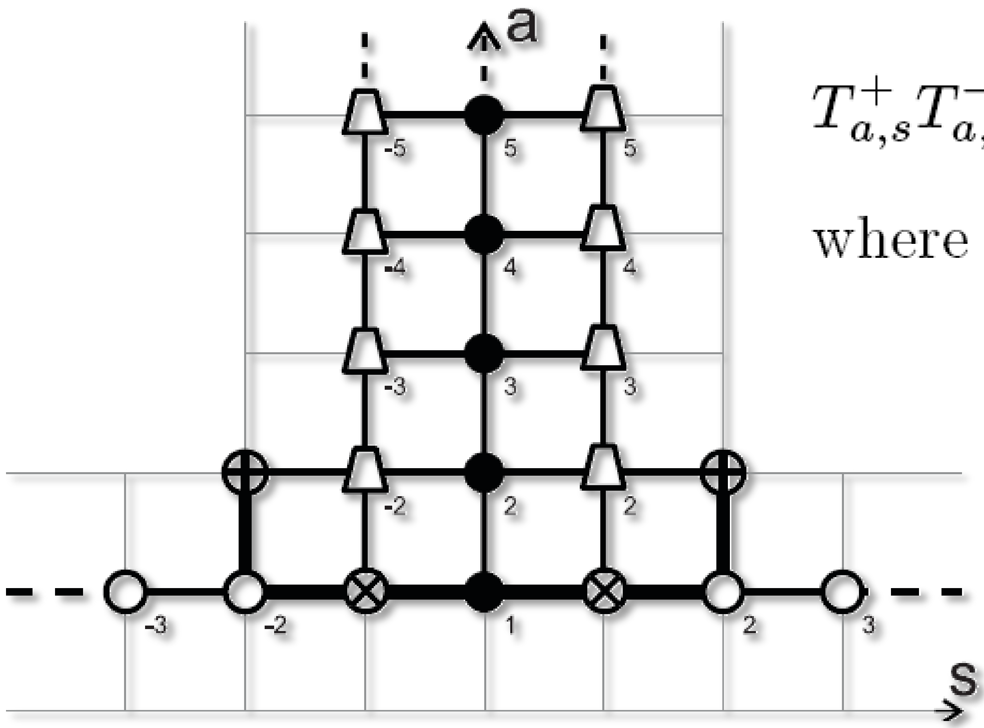
For excited states – contour deformation trick

Dorey,Tateo

Konishi operator: $\mathrm{Tr}[\Phi_1, \Phi_2]^2$



Hirota equation



$$T_{a,s}^{+}T_{a,s}^{-} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1} \, ,$$

$$\text{where } \; Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}} \quad .$$

$$T_{a,s}^{\pm} \equiv T_{a,s}(u \pm \tfrac{i}{2})$$

Gromov,.Kazakov,Vieira 2009

- Large L solutions ⟹ ABA
- Efficiently compute weak-coupling corrections

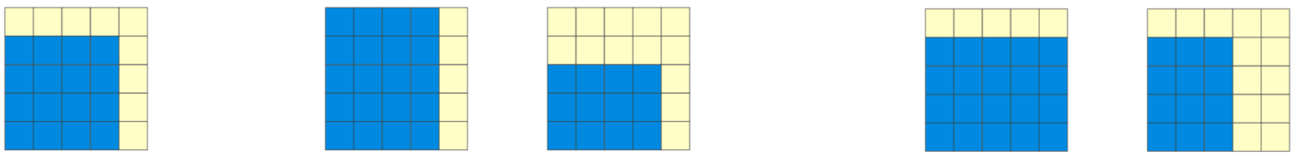
Allowed to reduce TBA to a finite set of integral equations!

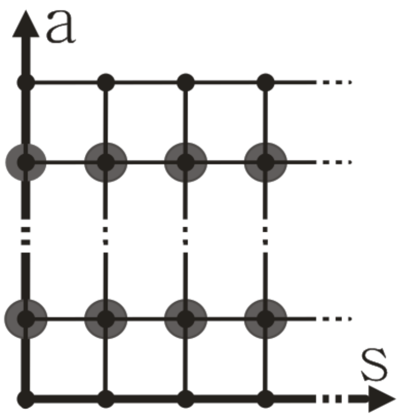
Gromov,Kazakov, Leurent,Volin, 2011

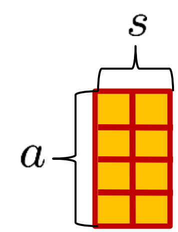
Strong coupling

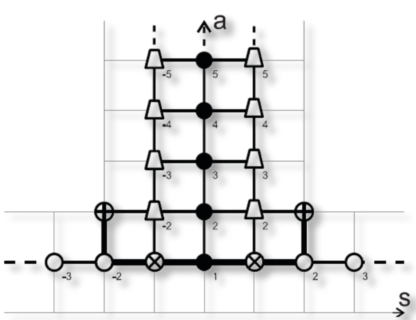
No shifts $\pm\frac{i}{2}$ in l.h.s.

$$T^2_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$





$T_{a,s}$ = characters of GL(n) for reps 



Full T-hook: characters of PSU(2,2|4)

Matches 1-loop spectral curve results!

Gromov 2009
Gromov,Kazakov,Tsuboi 2010

Other integrable cases of AdS/CFT

Deformations of N=4 SYM

- N=1 or no SUSY
- Dual to deformed versions of AdS5 x S5
- appears to be integrable (ABA, TBA, …)
- Y-system is the same!

Beisert,Roiban 2005
Gromov,F.L-M 2010
Arutyunov,Leeuw,Tongeren 2010,2012
Ahn, Bajnok,Bombardelli, Nepomecie 2011

Example: weak-coupling checks of Y-system vs direct perturbative results at 11 loops!

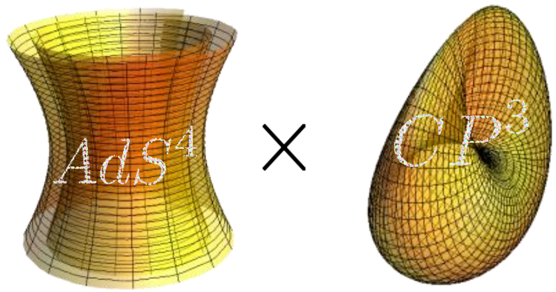
ABJM duality

Aharony,Bergman,Jafferis,Maldacena 2008

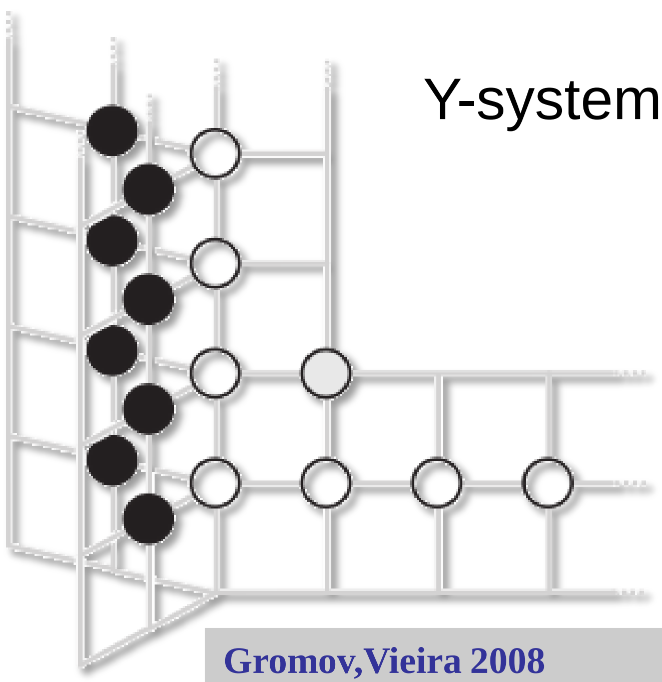
$\mathcal{N} = 6$ 3d superconformal
SU(n) x SU(N) Chern-Simons

⟺

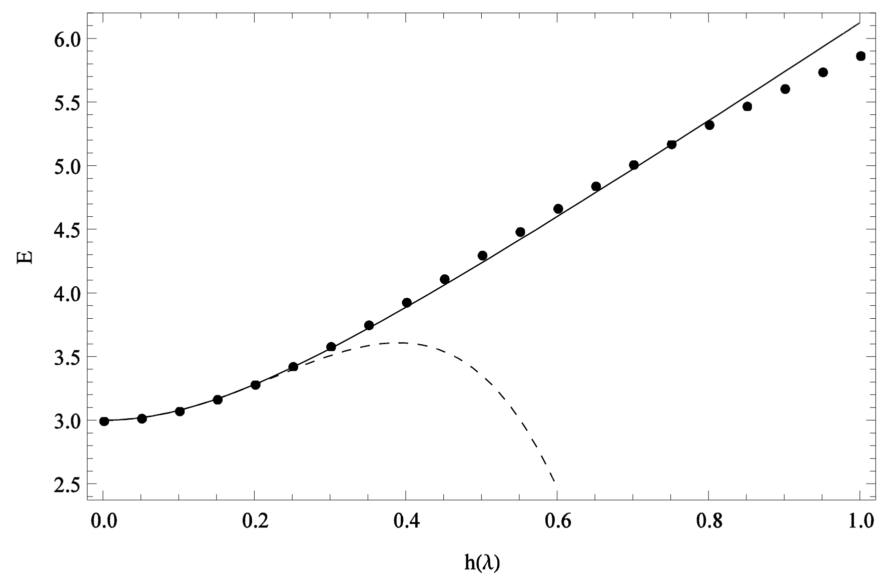
superstrings in
$AdS^4 \times CP^3$



Y-system and TBA are known



Gromov,Vieira 2008
Gromov,Kazakov,Vieira 2009
Bombardelli,Fioravanti,Tateo 2009
Gromov,F.L-M 2009



TBA numerics, F.L-M. 2011

Conclusions

- For the first time: exact results (in planar limit) for nontrivial 4d gauge theory in non-BPS sectors
- Full spectrum at any coupling from a system of functional (Y-system, Hirota) or integral (TBA) equations
- Confirmed by all known tests
- Extensions to other AdS/CFT dualities
- Many other directions : 3-point functions, scattering amplitudes, …

Classical integrability of the string

Coset sigma model $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(2,2)\times \mathrm{Sp}(4)}$

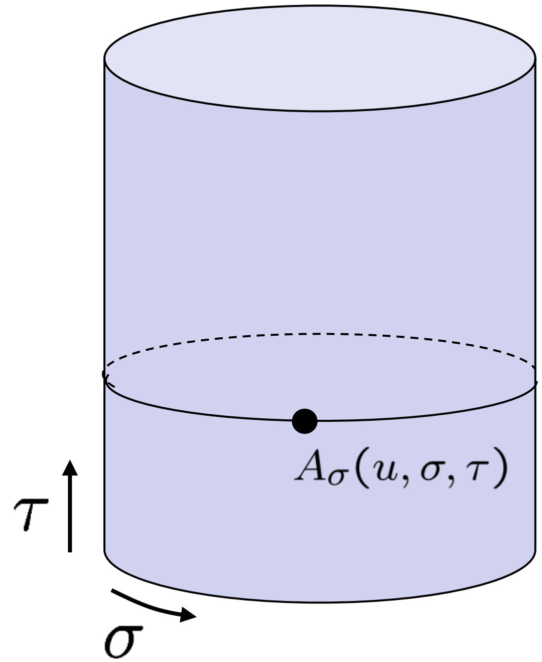
$A(u)$ – flat connection (on e.o.m.)

$u \in \mathbb{C}$

$$\Omega(u,\tau) = \mathrm{Pexp}\oint A_\sigma(u)d\sigma$$

eigenvalues $\lambda(u)$ are conserved

(spectral curve)



Infinitely many integrals of motion!

Bena,Polchinski,Roiban
Kazakov,Marshakov,Minahan,Zarembo

Mixing and integrability in N=4 SYM

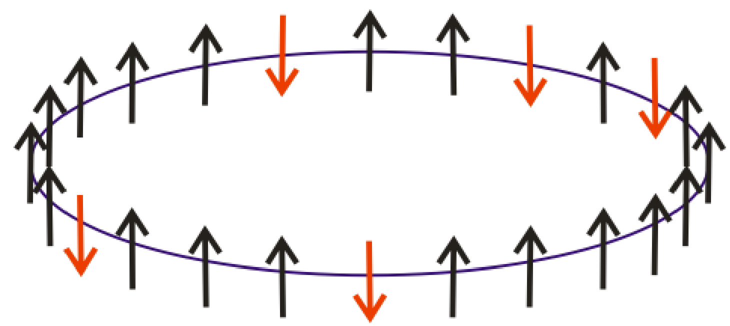
$$\mathcal{O}_i^{\rm ren} = Z_{ij}(\Lambda)\mathcal{O}_j^{\rm bare} \quad \Rightarrow \quad \Delta = \Delta^{(0)} + \gamma$$

Our goal: compute the spectrum of dimensions $\Delta_i(\lambda)$

γ_i are eigenvalues of the mixing matrix: $\Gamma = Z^{-1}\dfrac{dZ}{d\log\Lambda}$

$$\mathcal{O}_i(x) = \mathrm{tr}\left(\Phi_1\Phi_2\Phi_1\ldots\Phi_1\Phi_1\Phi_2\Phi_2\Phi_1\right)(x)$$

1-loop mixing matrix =
integrable spin chain Hamiltonian!



L = number of fields

Solved by Bethe ansatz:

Minahan, Zarembo 2002

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = -\prod_{k=1}^{M}\frac{u_j - u_k + i}{u_j - u_k - i}$$

$$\gamma = \sum_{k=1}^{M}\frac{\lambda}{u_k^2 + 1/4}$$