Hopf Solitons in the Faddeev- Skyrme model

M. Shifman, A. Yung, A.G.

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Motivation : To derive the effective closed string in the low-energy limit with the internal structure, whose stability is provided by the Hopf invariant

The proper model — QED with two charged scalars.

Strings with nontrivial worldsheet theory — perfect probe to investigate the nonperturbative bulk 4D physics

1. Simplified model of the effective closed string — vorton.

2. Hopfion from the twisted semilocal string

3. Stability of the Hopf string

4. Composite Hopf string

5. Conclusion.

Faddeev-Skyrme model. Usually the Skyrme term is added by hands

$$\mathcal{L} = \frac{F^2}{2}\, \partial_\mu \vec{S}\, \partial^\mu \vec{S} - \frac{\lambda}{4} \left(\partial_\mu \vec{S} \times \partial_\nu \vec{S}\right) \cdot \left(\partial^\mu \vec{S} \times \partial^\nu \vec{S}\right) .$$

$$\vec{S}^2 = 1 .$$

Imposing boundary condition we get topology hosting the Hopf fibration

$$\vec{S} \to \{0, 0, 1\} \ \text{at} \ |\vec{x}| \to \infty .$$

$$\pi_3(S_2) = Z ,$$

Solution we shall get: 2d sigma-model instanton lifted to 4d. When Lifted to 3d it corresponds to the particle, to 4d — to the string.

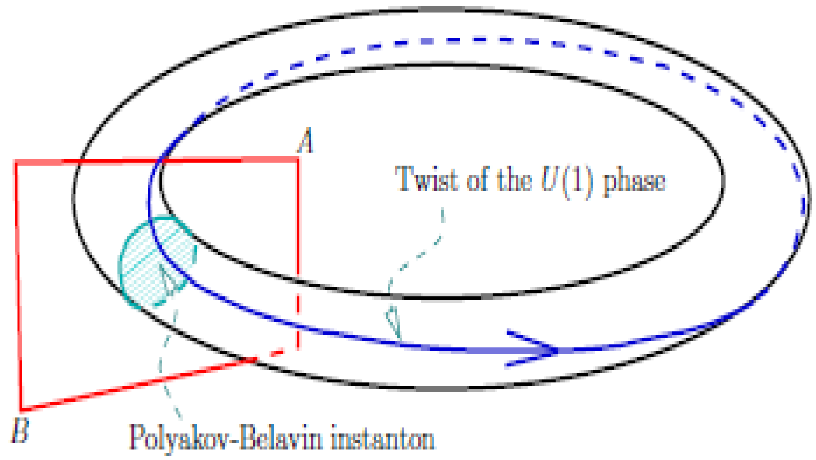


Figure 1: The simplest Hopf soliton, in the adiabatic limit, corresponds to a Belavin-Polyakov "instanton" extended in one extra dimension and bent into a torus, with a 2π twist of the instanton phase modulus.

The simplest model with the closed effective string (version of Witten,s model)

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + |\mathcal{D}_\mu \phi|^2 - \frac{\lambda_\phi}{4} \left(\phi^2 - v_\phi^2\right)^2$$
$$+ |\partial_\mu \chi|^2 - \frac{\lambda_\chi}{4} \left(\chi^2 - v_\chi^2\right)^2 - \beta \phi^2 \chi^2 .$$

Worldsheet action on the effective string

$$S = \int dt\, dz \left\{ \frac{T}{2} \left[(\partial_\mu x_0)^2 + (\partial_\mu y_0)^2 \right] + f^2 (\partial_\mu \alpha)^2 \right\}$$

$$\alpha_k(t, z) = 2\pi z\, k/L\,, \qquad k = 2, 3, \ldots$$

The energy of the string has the evident extremum.

$$E = TL + \frac{(2\pi f)^2}{L}$$

Our model- QED with two scalars and potenial. May by thought of as the bosonic sector of the super QED

$$S_0 = \int d^4x \left\{ -\frac{1}{4g^2} F_{\mu\nu}^2 + \left|\mathcal{D}_\mu \varphi^A\right|^2 - \lambda \left(|\varphi^A|^2 - \xi \right)^2 \right\},$$

$$\beta = \frac{2\lambda}{g^2}\,.$$

Vacuum manifold

$$\left|\varphi^1\right|^2 + \left|\varphi^2\right|^2 = \xi\,.$$

And the mass spectrum

$$m_\gamma = \sqrt{2}g\sqrt{\xi}\,, \qquad m_H = m_\gamma\sqrt{\beta}\,,$$

EoM in the BPS limit

$$F_{12} + g^2\left(|\varphi^A|^2 - \xi\right) = 0\,, \qquad (\mathcal{D}_1 + i\mathcal{D}_2)\,\varphi^{1,2} = 0\,.$$

And BPS violating term in the low-energy action

$$\Delta S = \int d^4x \frac{\beta - 1}{4g^2}\, F_{\mu\nu}^2\,.$$

Total low-energy effective action. Faddeev-Skyrme model

$$S = \int d^4x \left\{ \frac{\xi}{4} \partial_\mu S^a \partial^\mu S^a - \frac{\beta - 1}{4g^2}\, F_{\mu\nu} F^{\mu\nu} + \ldots \right\}\,,$$

$$F_{\mu\nu} = \frac{1}{2}\, \varepsilon_{abc} S^a \partial_\mu S^b \partial_\nu S^c\,.$$

Consider the anzatz

$$
\begin{aligned}
\varphi^1(x) &= \phi_1(r)\, e^{i\theta} \,, \\
\varphi^2(x) &= \phi_2(r)\,, \\
A_i(x) &= -\varepsilon_{ij}\, \frac{x_j}{r^2}\, [1 - f(r)]\,, \\
r &\equiv \sqrt{\vec{x}_\perp^{\,2}}\,,
\end{aligned}
$$

Semilocal string solution

$$
\begin{aligned}
\phi_1(r) &= \sqrt{\xi}\frac{r}{\sqrt{r^2 + |\rho|^2}}\,, \\
\phi_2(r) &= \sqrt{\xi}\frac{\rho}{\sqrt{r^2 + |\rho|^2}}\,, \\
f &= \frac{|\rho|^2}{r^2 + |\rho|^2}\,.
\end{aligned}
$$

Effective action of the length L string

$$\mathcal{E}_{\rm eff} = 2\pi \int dt\, dz \left\{ \xi\, |\partial_k \rho|^2 \; \ln \frac{L}{|\rho|} + \frac{1}{3}\, \frac{(\beta - 1)}{g^2}\, \frac{1}{|\rho|^2} \right\}.$$

Valid if

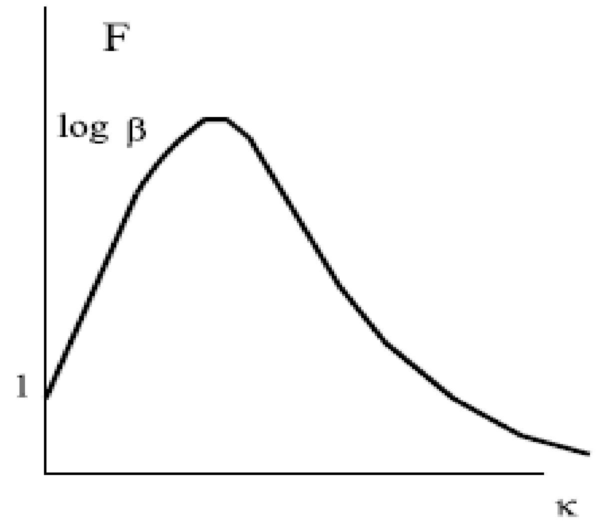
$$|\rho| \gg \frac{1}{m_\gamma}$$

Consider the twisting of the instanton modulus

$$\alpha(z) = \frac{2\pi k z}{L}$$

Total effective energy of the closed string

$$\mathcal{E} = 2\pi \left\{ \xi L + \frac{\xi (2\pi|\rho|k)^2}{L} \ln \frac{L}{|\rho|} + \frac{\beta}{3g^2} \frac{L}{|\rho|^2} F\left(\frac{\beta}{|\rho|^2 m_\gamma^2} \right) \right\}$$



Upon extremization we obtain the energy

$$\mathcal{E} = \frac{4\sqrt{2}\pi^2\,\sqrt{\beta}}{g}\,\sqrt{\xi}\,k\,\sqrt{\log k}\,\sqrt{\frac{1+\frac{2}{3}\kappa F(\kappa)}{\kappa}}.$$

where

$$\kappa_* \sim \frac{1}{\sqrt{\log\beta}}, \qquad |\rho_*| \sim \frac{\sqrt{\beta}}{m_\gamma}\,(\log\beta)^{1/4}, \qquad \frac{L_*}{|\rho_*|} \sim k\sqrt{\log k}$$

$$L \gg \rho \gg 1/m_\gamma.$$

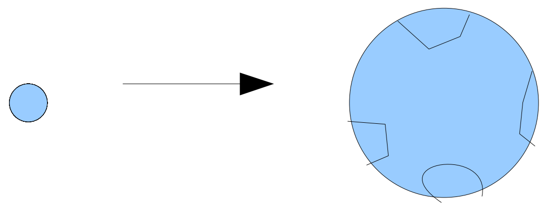
Consider one compact coordinate in D=4

General phenomena — instantons get splitted

Into monopole-like constituents. Wellknown in D=4,

The same phenomena in D=2 Belavin-Polyakov instanton in D=2

CP(N-1) model is splitted into N constituents



Constituents have two charges similar to D=4 case

Solution has the following structure

$$\omega = \frac{z - z_+}{z - z_-} \,,$$

$$Z = \frac{1}{2}\left(z_+ + z_-\right), \qquad \rho = |\rho| e^{i\theta} = \frac{1}{2}\left(z_+ - z_-\right).$$

In four dimension the scalar is dual to the two-form field

$$\partial_\rho \phi = \epsilon_{\rho\alpha\mu\nu} \partial_\alpha B_{\mu\nu} \,.$$

$$d^* H = \left(\delta(z - z_+) - \delta(z - z_-)\right),$$

Upon lifting to 4d splitted Belavin-Polyakov instanton yields
the splitted semilocal string which can be made closed
via several nontrivial windings of constituents

Conclusion

----There are closed Hopf strings whose stability is provided by the Hopf invariant

---- Faddeev-Skyrme model can be considered as the low-energy limit of QED with two scalars

---- Interesting possibility of the composite Hopf closed string

--- Application in the solid state physics (multiband structure, Babaev et al)