PROBLEMATIC ASPECTS of KALUZA-KLEIN MODELS

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Evident statement : Any viable physical theory must be in concordance with the observational data.

Gravitational tests: ***the perihelion shift, the deflection of light, the time delay of radar echoes.***

General Relativity satisfies these tests with very high accuracy.

What about multidimensional Kaluza-Klein models ?

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All these gravitational tests are considered in <u>weak-field limit:</u>

1. **gravitational field is weak**
2. **velocities of the gravitating masses are small**

⇓

<u>Parameterized post-Newtonian (PPN) formalism</u> is a useful tool to compare with observations

Four-dimensional static spherically symmetric line element in PPN formalism:

$$ds^2 = \underbrace{\left(1 - \frac{r_g}{r_3} + \beta \frac{r_g^2}{2r_3^2}\right)}_{O(1/c^2)} c^2 dt^2 + \underbrace{\left(-1 - \gamma \frac{r_g}{r_3}\right)}_{O(1/c^4)} \sum_{i=1}^{3} \left(dx^i\right)^2 \qquad O(1/c^2)$$

isotropic coordinates!

$$r_g = \frac{2G_N m}{c^2}$$

h_{00}

PPN parameters

$h_{\alpha\alpha}$

GR: $\gamma = 1, \beta = 1$

$$\gamma = h_{\alpha\alpha} / h_{00}$$

The deflection of light : $\delta\psi = (1+\gamma)\frac{r_g}{\rho}$

The time delay of radar echoes : $\delta t \approx (1+\gamma)\frac{r_g}{c} \ln\left(\frac{4 r_{Earth} r_{planet}}{R_{Sun}^2}\right)$

Perfect *agreement* !

Observed value: $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ **(Cassini spacecraft)**

PPN parameter γ in KK models ?

Calculation of γ in 4-D space-time (Landau&Lifshitz, v.2):

1. Flat background space-time

$$\mathfrak{M}_4 \equiv \mathbb{R}^4 \longrightarrow g_{00} = \eta_{00} = 1, \quad g_{0a} = \eta_{0a} = 0, \quad g_{ab} = \eta_{ab} = -\delta_{ab}$$

2. We perturb this flat background by a point-like gravitating mass

$$\rho = m\,\delta(\vec{r}_3)$$ - rest mass density

$$\longrightarrow \quad g_{ik} \approx \eta_{ik} + h_{ik}\,; \quad h_{ik} = O\left(1/c^2\right); \qquad h_{ik} - ?$$

3. From the Einstein eq. with known r.h.s., we calculate the metric correction terms

$$R_{ik} = \frac{8\pi G_N}{c^4}\left(T_{ik} - \frac{1}{2} g_{ik} T\right) \quad \Rightarrow \quad h_{ik}$$

4. Determination of γ :

$$\gamma = h_{\alpha\alpha} / h_{00}$$

In GR: $\gamma = 1$

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γ in Kaluza-Klein models ?

Background metrics

$$\hat{g}_{MN}(y)dX^M \otimes dX^N = \eta_{\mu\nu}dx^\mu \otimes dx^\nu + \hat{g}^{(d)}_{mn}(y)dy^m \otimes dy^n$$

Background space-time manifold

$$\mathfrak{M}_{\mathcal{D}} = \mathfrak{M}_4 \times \mathfrak{M}_d \qquad \mathcal{D} = 1 + D = 4 + d$$

Minkowski spacetime

internal space

Compact Einstein space (e.g. orbifolds):

$$\hat{R}_{mn}[\hat{g}^{(d)}] = \lambda \hat{g}_{mn}, \quad \hat{R}^m_m[\hat{g}^{(d)}] = \hat{R}^{(d)} = \lambda d, \quad \lambda \equiv const.$$

In general, $\lambda \neq 0$ $\Longrightarrow$ curved internal space !

To create the *curved* background space-time, we should introduce a background matter

$$\kappa \hat{T}'^{M}_{N} = \hat{R}^{M}_{N} - \left(\frac{1}{2}\hat{R}^{(d)} + \kappa \Lambda_{\mathcal{D}} \right) \delta^{M}_{N}, \quad \kappa \equiv \frac{2 S_{D} \tilde{G}_{\mathcal{D}}}{c^{4}},$$

$$\underbrace{\hat{T}'^{\mu}_{\nu} = -\left(\frac{\lambda d}{2\kappa} + \Lambda_{\mathcal{D}} \right) \delta^{\mu}_{\nu},}_{\text{diagonal}} \quad \underbrace{\hat{T}'^{m}_{n} = -\left(\frac{\lambda (d-2)}{2\kappa} + \Lambda_{\mathcal{D}} \right) \delta^{m}_{n}.}_{\text{diagonal}}$$

$$\mu, \nu = 0, 1, 2, 3 \qquad\qquad m, n = 4, 5, \ldots, d$$

In the form of a perfect fluid:

$$\hat{T}'^{M}_{N} = \mathrm{diag}(\hat{\varepsilon}', -\hat{p}'_0, -\hat{p}'_0, -\hat{p}'_0, \underbrace{-\hat{p}'_1, \ldots, -\hat{p}'_1}_{d\ times}),$$

Energy density: $\hat{\varepsilon}' \equiv -\left(\frac{\lambda d}{2\kappa} + \Lambda_{\mathcal{D}}\right), \quad \hat{p}'_0 = \omega_0 \hat{\varepsilon}', \quad \hat{p}'_1 = \omega_1 \hat{\varepsilon}'.$

EoS: $\underbrace{\omega_0 = -1}_{\textbf{fixed}}, \quad \underbrace{\omega_1 = \frac{(2-d)\lambda - 2\kappa\Lambda_{\mathcal{D}}}{\lambda d + 2\kappa\Lambda_{\mathcal{D}}}}_{\textbf{arbitrary}}.$

e.g. monopole form-fields, Casimir effect

$\omega_1 = 1$

$\omega_1 = 4/d$

auxiliary relation: $\hat{\varepsilon}' = -\lambda/[\kappa(1+\omega_1)]$

We perturb this background (metrics and matter) by a gravitating mass with the EMT:

$$\tilde{T}^{M\nu} = \tilde{\rho}^{(\mathcal{D})} c^2 \frac{ds}{dx^0} u^M u^\nu, \quad u^M = \frac{dX^M}{ds},$$

$$\tilde{T}^{mn} = -\tilde{p} g^{mn} + \tilde{\rho}^{(\mathcal{D})} c^2 \frac{ds}{dx^0} u^m u^n, \quad \tilde{p} = \Omega \tilde{\rho}^{(\mathcal{D})} c^2 \frac{ds}{dx^0}.$$

$$\tilde{\rho}^{(\mathcal{D})} = [|g|]^{-1/2} m \delta(\tilde{X}).$$

For ordinary astrophysical objects (e.g. our Sun) $p \ll \varepsilon$

This gravitating body is pressureless in the external space but it has arbitrary EoS Ω **in the internal space**

We don't know pressure in the internal space

The perturbed metrics in the weak-field limit:

$$g_{MN} \approx \hat{g}_{MN} + h_{MN}, \qquad h_{MN} \sim O(1/c^2)$$

?

We can find these correction terms from the Einstein eq.

$$R_{MN} = \kappa \underbrace{\left[T_{MN} - \frac{1}{d+2} T g_{MN} - \frac{2}{d+2} \Lambda_{\mathcal{D}} g_{MN} \right]}_{O(\mathrm{c}^2)}$$

$$\kappa \sim 1/\mathrm{c}^4$$

Total EMT: $T_{MN} = T'_{MN} + \tilde{T}_{MN}$

Perturbed background matter:

$$T'_{\mu\nu} \approx (\hat{\varepsilon}' + \varepsilon'_1) g_{\mu\nu}, \quad T'_{mn} \approx -\omega_1 (\hat{\varepsilon}' + \varepsilon'_1) g_{mn}.$$

?

Perturbation (gravitating mass)

The pressure of the perturbing gravitating body is isotropic in each factor manifolds. Such perturbation does not change the topologies of the factor manifolds and it preserves also the block-diagonal structure of the metric tensor.
In the case of a steady-state model (our case) the non-diagonal perturbations $h_{0\tilde{M}}$ **are also absent.**

The metric correction terms are conformal to the background metrics and can be written in the block-diagonal form:

radion !

$$\left[h_{MN}(X)\right] = \left[\xi_1 \eta_{00}\right] \oplus \left[\xi_2 \eta_{\tilde{\mu}\tilde{\nu}}\right] \oplus \left[\xi_3 \hat{g}^{(d)}_{mn}\right]$$

$$\xi_{1,2,3} = \xi_{1,2,3}(X) \sim O(1/c^2) \quad -?$$

D-dimensional spatial coordinates

Einstein equations

D-dimensional Laplace operator

$(\hat{g}_{\tilde{M}\tilde{N}})$

$$\Delta_D \xi_1 = 2\frac{1+d(\Omega+1)}{d+2}\kappa\tilde{\varepsilon} + 2\frac{d(1+\omega_1)-2}{d+2}\kappa\varepsilon_1', \quad (1)$$

$$\Delta_D \xi_2 = -2\frac{1-\Omega d}{d+2}\kappa\tilde{\varepsilon} + 2\frac{d(1+\omega_1)-2}{d+2}\kappa\varepsilon_1', \quad (2)$$

$$\Delta_D \xi_3 = -2\frac{1+2\Omega}{d+2}\kappa\tilde{\varepsilon} + 2\lambda\xi_3 - \frac{4(\omega_1+2)}{d+2}\kappa\varepsilon_1', \quad (3)$$

$$\tilde{\varepsilon} \equiv \rho^{(\mathcal{D})}c^2$$

$$\underline{\xi_1, \xi_2, \xi_3, \varepsilon_1' - ?}$$

Gauge condition (Landau&Lifshitz, V.2):

$$\hat{\nabla}_L h_N^L - \frac{1}{2}\partial_N h_L^L = 0, \quad h_N^M \equiv \hat{g}^{MS} h_{NS},$$

$$\Downarrow$$

$$\partial_0 \xi_1 - (1/2)\partial_0\left(\xi_1 + 3\xi_2 + d\xi_3\right) = 0, \quad (4)$$

← Satisfied automatically

$$\partial_{\tilde{\nu}} \xi_2 - (1/2)\partial_{\tilde{\nu}}\left(\xi_1 + 3\xi_2 + d\xi_3\right) = 0, \quad (5)$$

$$\partial_n \xi_3 - (1/2)\partial_n\left(\xi_1 + 3\xi_2 + d\xi_3\right) = 0. \quad (6)$$

$$\xi_1 + \xi_2 + d\xi_3 = C(y)\big|_{|r|\to+\infty} \to 0 \quad \Rightarrow \quad \boxed{\xi_3 = -\frac{1}{d}\left(\xi_1 + \xi_2\right)}$$

$\Delta_D \xi_3$ **+ (1)-(3)** $\Longrightarrow$ $\boxed{\varepsilon_1' = (\lambda d/2\kappa)\xi_3}$ (7)

(3)+(7) ➡

$$\boxed{\Delta_D \xi_3 = 2\left[\lambda \frac{2-d(1+\omega_1)}{d+2}\xi_3 - \frac{1+2\Omega}{d+2}\kappa\rho^{(\mathcal{D})}c^2\right]} \quad (8)$$

Taking into account the relation $\xi_3 = -(\xi_1+\xi_2)/d$ **and Eqs. (1),(2),(8) we get:**

$$\boxed{\xi_1 = -\frac{d}{2}\xi_3 + f, \quad \xi_2 = -\frac{d}{2}\xi_3 - f} \quad (9)$$

h_{00}

$-h_{\tilde{\mu}\tilde{\mu}}$

where

$$\boxed{\Delta_D f = \kappa\rho^{(\mathcal{D})}c^2} \quad (10)$$

$$\gamma = h_{\tilde{\mu}\tilde{\mu}} / h_{00} = -\xi_2 / \xi_1 = ?$$

$\xi_3, f - ?$

Crucial point:

(6) $\longrightarrow$ $$\underbrace{\lambda\left[2-d(1+\omega_1)\right]}_{>0}\xi_3 - 2\Omega\kappa\rho^{(\mathcal{D})}c^2 = C_2(\vec{r})/2$$

3D

Internal space stabilization

$\xi_3, \rho^{(\mathcal{D})}$ are functions of $\vec{r}$!

f, ξ_1, ξ_2 are functions of $\vec{r}$

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1.Gravitating mass is uniformly smeared over the internal space.

2. KK modes corresponding to the metric fluctuations are absent.

Therefore, in Eqs. (8) and (10) we need to make the substitution

$$\Delta_D \longrightarrow \Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \rho^{(\mathcal{D})} = \rho^{(3)}(\vec{r})/V_{\text{int}}$$

internal space volume

(8),(10) $\Rightarrow$

$$4\pi G_N = \frac{S_D \tilde{G}_{\mathcal{D}}}{V_{\text{int}}}$$

$$\left\{ \begin{aligned} \Delta_3 f &= \frac{8\pi G_N}{c^2}\rho^{(3)}, \quad \rho^{(3)} = m\delta(\vec{\mathrm{r}}) \\ \Delta_3 \xi_3 - \mu^2 \xi_3 &= -\frac{2(1+2\Omega)}{d+2}\frac{8\pi G_N}{c^2}\rho^{(3)}, \end{aligned} \right. \qquad (*)$$

$$\boxed{\frac{2\lambda[2-d(1+\omega_1)]}{d+2}}$$

Yukawa squared mass

Scalar curvature λ **depends on the size of the internal space.**

E.g., for d-sphere of the radius a **:** $\lambda = -(\mathrm{d}-1)/a^2$

Boundary condition: $\xi_3 \to 0$ **for** $|r| \to +\infty$

$$\Downarrow$$

Positivity of the Yukawa mass squared:

$$\mu^2 > 0 \quad \Rightarrow \quad \boxed{\lambda\left[2 - d(1+\omega_1)\right] > 0} \Rightarrow \begin{cases} \omega_1 > (2/d) - 1, & \lambda < 0, \\ \omega_1 < (2/d) - 1, & \lambda > 0. \end{cases}$$

(**)

Solution of (*):

$$\varphi_N = -G_N m/|r|$$ Newtonian potential

$$f = \frac{2\varphi_N}{c^2}, \quad \xi_3 = -\frac{4\varphi_N}{(2+d)c^2}(1+2\Omega)\exp(-\mu|r|),$$

Conformal excitation of the int. space: radion!

$$h_{\tilde{\mu}\tilde{\mu}} = -\xi_2 = \frac{2\varphi_N}{c^2}\left[1 - \frac{d}{2+d}(1+2\Omega)\exp(-\mu|\vec{r}|)\right]$$

$$\frac{2\varphi}{c^2} \equiv h_{00} = \xi_1 = \frac{2\varphi_N}{c^2}\left[1 + \frac{d}{2+d}(1+2\Omega)\exp(-\mu|\vec{r}|)\right]$$

Standard Newton

Fifth force – admixture of the radion

In general, PPN parameter $\gamma = h_{\tilde{\mu}\tilde{\mu}} / h_{00} \neq 1$!

Problem with observations

The Yukawa *coupling constant* g between any massive particle and radion:

$$g^2 \sim \frac{d}{2+d}(1+2\Omega)G_N \sim O(G_N)$$

Black strings/branes exceptional case: $\Omega = -1/2 \quad \Rightarrow \quad g = 0$ Coupling Is absent !

I. Ricci-flat internal space: $\lambda = 0 \quad \Rightarrow \quad \mu = 0$

infinite range of fifth force

$\gamma = 1$ only if $\Omega = -1/2$

the same accuracy as in GR

This result does not depend on the size of the internal space!

II. Curved internal space: $\lambda \neq 0,\ \mu \neq 0$

There are two possibilities to be in agreement with observations.

A. Large mass of radion: $\mu \to \infty$

$$\gamma \to 1 \quad \text{for} \quad \mu \to \infty$$

Usually $\mu \sim V_{\text{int}}^{-1/d} \sim 1/a$

B. Zero coupling case: $\Omega = -1/2$ $\longleftarrow$ **black strings**

$$\gamma = 1 \quad \text{for any value of} \quad \mu$$

In this case, the result does not depend on the size of the internal space!

Internal space stabilization

Let the scale factor of the internal space be a function of time:

$$\hat{g}^{(d)}_{mn}(y) \quad \rightarrow \quad e^{2\beta(t)}\hat{g}^{(d)}_{mn}(y), \quad t \equiv x^0.$$

radion/
gravexciton

The conservation law $T'^{M}_{N;M} = 0$

(for considered above EMT of the perfect fluid with $\omega_0 = -1$)

$$\varepsilon'(t) = \varepsilon'_c e^{-\beta(t)(1+\omega_1)d}$$

The stabilization is possible if the effective potential has a minimum at $t = 0$ **(present time).**

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$$U_{\text{eff}}(\beta) = e^{-d\beta}\left[\frac{\hat{R}^{(d)}}{2}e^{-2\beta} + \kappa\Lambda_{\mathcal{D}} + \kappa\varepsilon'_c e^{-\beta(1+\omega_1)d}\right]$$

1. External space is flat $\Longrightarrow$ $\Lambda_{\text{eff}}^{(4)} = U_{\text{eff}}(\beta = 0) = 0$

$\Longrightarrow$ $\frac{1}{2}\hat{R}^{(d)} = \frac{\lambda d}{2} = -\kappa\left(\Lambda_{\mathcal{D}} + \varepsilon_c'\right)$ $\Longleftrightarrow$ **fine tuning on p.9** $\Longrightarrow$ $\varepsilon_c' = \hat{\varepsilon}'$

2. Necessary condition for an extremum: $\partial U_{\text{eff}}/\partial\beta\big|_{\beta=0} = 0$

$\Longrightarrow$ $\lambda = -(1+\omega_1)\kappa\hat{\varepsilon}'$ $\Longleftrightarrow$ **auxiliary relation on p.9**

3. Sufficient condition of a minimum: $\partial^2 U_{\text{eff}}/\partial\beta^2\big|_{\beta=0} > 0$

$\Longrightarrow$ $\lambda\left[2 - d(1+\omega_1)\right] > 0$ $\Longleftrightarrow$ **Eq. (**) on p. 18**

This inequality coincides with condition (**) of the positiveness of Yukawa mass squared!

SUMMARY

In considered Kaluza-Klein models:

1.Gravitating masses are uniformly smeared over the internal spaces.

2. KK modes corresponding to the metric fluctuations are absent.

↑

It looks artificial from the point of statistical physics and quantum mechanics.

3. The agreement with the observed PPN parameter γ takes place either in the case of large mass of radion $\mu \to \infty$ or for zero coupling of radion $\Omega = -1/2$ (i.e. for black strings).