

# Full two-loop electro-weak matching corrections for SM RGE analysis

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# Motivation: (1)

1. After Higgs boson discovery and its mass measurement the last building block for SM RGE analysis become available
2. Attempt to reduce theoretical uncertainty

## $N$ -loop RGE SM analysis

Low energy input  
values from  
experiment

$(n - 1)$ -loop matching  
to obtain input values  
in  $\overline{MS}$ -scheme

$n$ -loop RGE equations

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$n$ -loop RGE equations

do not touch

this work,  $\mathcal{O}(\alpha^2)$

3-loop precision

# SM couplings running and beta-functions

- ▶ Gauge couplings

$$a_i = \frac{g_i^2}{16\pi^2}$$

- ▶ Yukawa couplings

$$a_y = \frac{y^2}{16\pi^2}$$

- ▶ Higgs self-coupling

$$a_\lambda = \frac{\lambda}{16\pi^2}$$

- ▶ Initial conditions

$$a_i^{(0)} = a_i(\mu_0)$$

$$\frac{da_1(\mu^2)}{d\log\mu^2} = \beta_{a_1}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

$$\frac{da_2(\mu^2)}{d\log\mu^2} = \beta_{a_2}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

$$\frac{da_s(\mu^2)}{d\log\mu^2} = \beta_{a_s}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

$$\frac{da_t(\mu^2)}{d\log\mu^2} = \beta_{a_t}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

$$\frac{da_b(\mu^2)}{d\log\mu^2} = \beta_{a_b}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

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**Most relevant**

$$\frac{da_1(\mu^2)}{d\log\mu^2} = \beta_{a_1}(a_1, a_2, a_s, a_t, a_b, a_\tau, a_\lambda)$$

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# State of the art: beta-functions

- ▶ 3-loop QCD [Tarasov,Vladimirov,Zharkov'80] [Larin,Vermaseren'93]
- ▶ 4-loop QCD [Larin,van Ritbergen,Vermaseren'97] [Czakon'05]
- ▶ 2-loop SM [Machacek,Vaughn'83] [Arason,et al.'91] [Luo,Wang,Xiao'02]
- ▶ 3-loop SM,  $g_1 = g_2 = 0$  [Chetyrkin,Zoller'12]
- ▶ 3-loop SM
  - ▶ Gauge couplings [Mihaila,Salomon,Steinhauser'12] [Bednyakov,AP,Velizhanin'12]
  - ▶ Yukawa couplings [Bednyakov,AP,Velizhanin'12]
  - ▶ Higgs self-coupling [Chetyrkin,Zoller'13] [Bednyakov,AP,Velizhanin'13]

# Pole mass, definition

Higgs boson example

$$\Pi = \text{[hatched circle]} = \text{[red circle labeled 1PI]} + \text{[tadpole with H]} + \text{[self-energy with H]} + \text{[red circle labeled 1PI with tadpole]} + \dots$$

$\Pi$  is a sum of all 1PI diagrams, but Higgs reducible - **tadpoles**

$$D(p) = \text{[hatched circle]} + \text{[hatched circle] - [hatched circle]} + \dots = \frac{1}{p^2 - m_0^2 - \Pi(p)}$$

We can solve perturbatively, using ansatz  $p^2 = (1 + X_1 + X_2 + \dots)m_0^2$

# Pole mass, definition

Higgs boson example

$$\Pi = \text{[diagram]} = \text{[1PI diagram]} + \boxed{\text{[Tadpoles diagrams]}} + \dots$$

Tadpoles

$\Pi$  is a sum of all 1PI diagrams, but Higgs reducible - **tadpoles**

$$D(p) = \text{[diagram]} + \text{[diagram]} + \dots = \frac{1}{p^2 - m_0^2 - \Pi(p)}$$

We can solve perturbatively, using ansatz  $p^2 = (1 + X_1 + X_2 + \dots)m_0^2$



# Pole mass, definition for top quark

- ▶ Fermion propagator

$$\not{S}^{-1}(p) = \not{p} - m_0 - \Sigma(\not{p})$$

- ▶ Decompose self-energy, with  $P_{L,R} = (1 \mp \gamma_5)/2$

$$\not{\Sigma}(p) = \not{p} P_L A_L(p^2) + \not{p} P_R A_R(p^2) + m_0 B(p^2)$$

- ▶ equation for  $p$  to find pole mass  $\not{S}^{-1}(p) = 0$

$$p^2[1 - A_L(p^2)][1 - A_R(p^2)] - m_0^2[1 + B(p^2)]^2 = 0$$

- ▶ Perturbative solution, using ansatz  $p = (1 + X_1 + X_2 + \dots)m_0$

$$X_1 = B_1 + \frac{1}{2}A_{L,1} + \frac{1}{2}A_{R,1}$$

$$X_2 = B_2 + \frac{1}{2}A_{L,2} + \frac{1}{2}A_{R,2} + X_1(A_{L,1} + A_{R,1} + A'_{L,1} + A'_{R,1} + 2B'_1) - \frac{1}{2}X_1^2 - \frac{1}{2}A_{L,1}A_{R,1} + \frac{1}{2}B_1^2$$

# Pole mass, relation with running mass

- ▶  $X_1, X_2$  - infrared safe and gauge invariant quantities, after tadpoles inclusion
- ▶ performing renormalization of  $m_0$  in  $\overline{MS}$  scheme we obtain relation between  $\overline{MS}$  mass  $m(\mu)$  and pole mass  $M$

# State of the art - threshold corrections

	LO	NLO	NNLO	NNNLO
$g_1$	$2\sqrt{(M_Z^2 - M_W^2)}/v$	full	(1)- $\mathcal{O}(\alpha^2)$	-
$g_2$	$2M_W/v$	full	(1)- $\mathcal{O}(\alpha^2)$	-
$y_t$	$\sqrt{2}M_t/v$	full	(2)- $\mathcal{O}(\alpha\alpha_s)$ (1)- $\mathcal{O}(\alpha^2)$	(3)- $\mathcal{O}(\alpha_s^3)$
$y_b$	$\sqrt{2}M_b/v$	full	(4)- $\mathcal{O}(\alpha^2)$ , $g_{1,2} = 0$	-
$\lambda$	$M_H^2/2v^2$	full	(2)- $\mathcal{O}(\alpha\alpha_s)$ (1)- $\mathcal{O}(\alpha^2)$	-
$\mu^2$	$M_H^2$	full	(1)- $\mathcal{O}(\alpha^2)$	-

(1) [Buttazzo, Degrandi, Giardino, Giudice, Sala, Salvio, Strumia]

numerical

(2) [Bezrukov, Kalmykov, Kniehl, Shaposhnikov]

analytical

(3) [Melnikov, Ritbergen]

analytical

(4) [Kniehl, Veretin]

analytical

## Motivation: (2)

1. Calculate  $\mathcal{O}(\alpha)$ ,  $\mathcal{O}(\alpha\alpha_s)$ ,  $\mathcal{O}(\alpha^2)$  corrections using unified setup
2. Calculate running masses for  $b, t$ -quarks,  $W, Z, H$ -bosons
3.  $b, t$ -yukawa couplings,  $\lambda$  and  $g_1, g_2$
4. Compare with available exact  $\mathcal{O}(\alpha)$ ,  $\mathcal{O}(\alpha\alpha_s)$  results
5. Extract couplings from running masses using universal procedure

Loop integrals are calculated numerically:

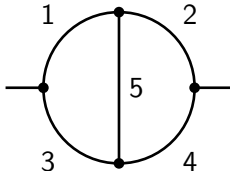
- ▶ Calculation in  $R_\xi$ -gauge with 4 gauge fixing parameters  $\xi_g, \xi_W, \xi_A, \xi_Z$
- ▶ Expansion up to second order in  $a_i = 1 - \xi_i$  around Feynman gauge

# Massive self-energies in SM

- ▶ We use **DIANA** for diagrams generation up to two-loops, including tadpoles
- ▶ Tarasov dimensional recurrence relations for integrals reduction, implemented in **TARCER** package for arbitrary  $q^2$  case, but we need on-shell case  $q^2 = m^2$
- ▶ Additional cases for on-shell diagrams and reduction of special type of sun-set diagrams
- ▶ Master integrals numerical evaluation using **TSIL** package
- ▶ Packaging analytical results in terms of **TSIL** master integrals for easy numerical evaluation

# Recurrence relations

Generic topology for two-loop self-energy integral

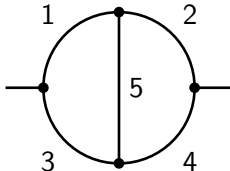


Recurrence relation for arbitrary  $q^2$  [Tarasov'97]

$$\begin{aligned}
 2\nu_1 \Delta \mathbf{1}^+ F_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{(d)} = & \\
 \left\{ (d - 2\nu_1 - \nu_3 - \nu_5) \Delta_1 + \Delta_{345} [\nu_5 \mathbf{5}^+ (\mathbf{2}^- - \mathbf{1}^-) - \nu_3 \mathbf{3}^+ \mathbf{1}^-] \right. & \\
 + \Delta_2 [\nu_1 \mathbf{1}^+ (\mathbf{5}^- - \mathbf{2}^-) + \nu_3 \mathbf{3}^+ (\mathbf{5}^- - \mathbf{4}^-) + \nu_5 - \nu_1] & \\
 \left. + \Delta_6 [\nu_1 \mathbf{1}^+ \mathbf{3}^- + \nu_5 \mathbf{5}^+ (\mathbf{3}^- - \mathbf{4}^-) + \nu_3 - \nu_1] \right\} F_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{(d)} &
 \end{aligned}$$

# Recurrence relations

Generic topology for two-loop self-energy integral

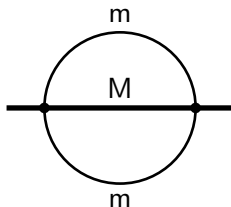


On-shell diagrams in SM lead to  $\Delta = \Delta_2 = \Delta_6 = 0, \Delta_1 = \Delta_{345}$

$$F_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{(d)} = -\frac{[\nu_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{1}^-) - \nu_3\mathbf{3}^+\mathbf{1}^-]}{(d - 2\nu_1 - \nu_3 - \nu_5)}$$

Implement such type of relations in TAR CER with correct conditions

# Reduction of special case of sun-set

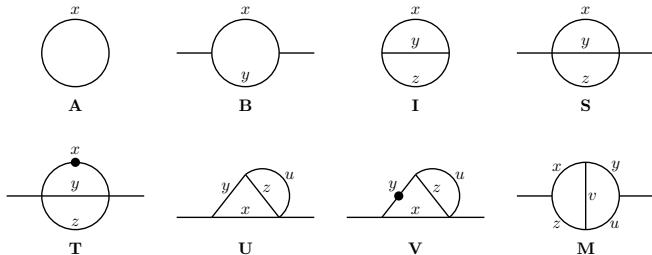


- ▶ For  $q^2 = M^2$  reduction is not possible with original Tarasov algorithm [Tarasov'97]
- ▶ But for special case  $J_{mmM}(n_1, n_2, n_3)$  is available in [Onishchenko, Veretin'02]



# Master integrals evaluation

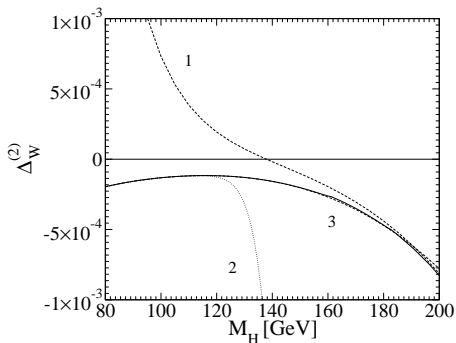
- ▶ Evaluate master integrals numerically
- ▶ We use **TSIL** package [Martin,Robertson'05]
- ▶ Prepare result free from spurious poles i.e. no terms like  $J/(d-4)$
- ▶ Only up to  $\epsilon^0$  part
- ▶ No spurious IR-divergencies, i.e. topology **V** with massless line  $y$



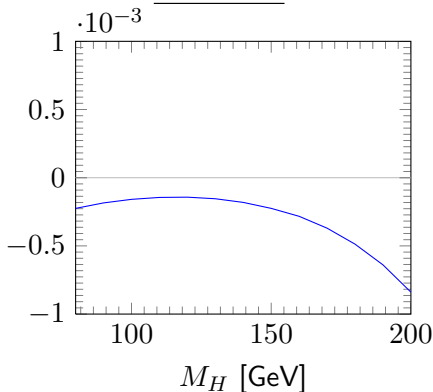
# Validation: $M_W, n_f = 0, \mu = M_Z$

$$m_W(\mu)^2 = M_W^2(1 + X_1 + X_2)$$

[arXiv:hep-ph/0209084]



Our result:

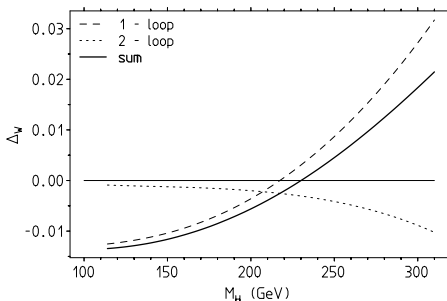


Bosonic part of full 2-loop EW correction

# Validation: $M_W, n_f = 0, \mu = M_W$

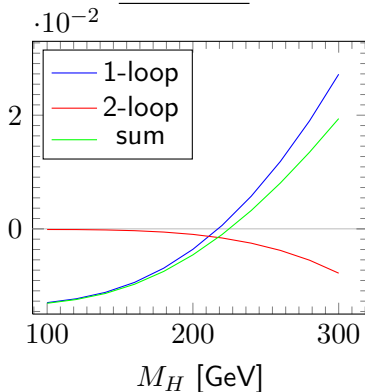
$$m_W(\mu)^2 = M_W^2(1 + X_1 + X_2)$$

[arXiv:hep-ph/0105304]



$$\Delta_W = m_W^2(M_W)/M_W^2 - 1$$

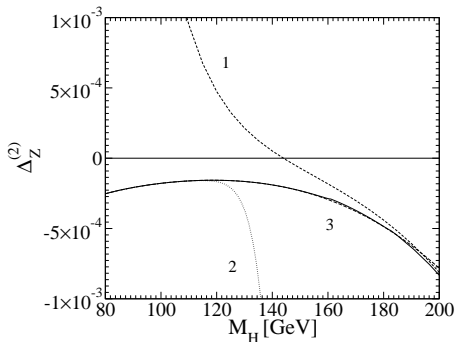
Our result:



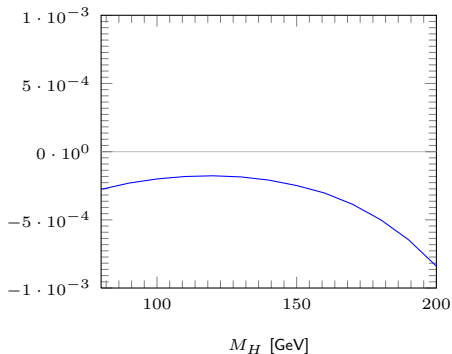
Bosonic part of full 2-loop EW correction

# Validation: $M_Z, n_f = 0, \mu = M_Z$

[arXiv:hep-ph/0209084]



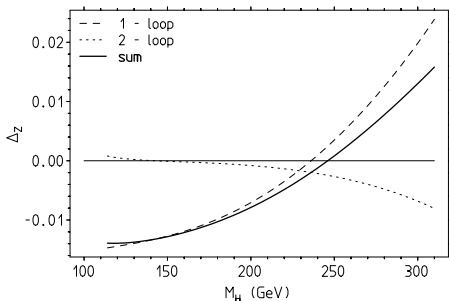
Our result:



Bosonic part of full 2-loop EW correction

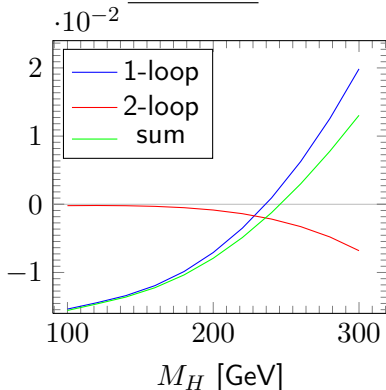
# Validation: $M_Z, n_f = 0, \mu = M_Z$

[arXiv:hep-ph/0105304]



$$\Delta_Z = m_Z^2(M_Z)/M_Z^2 - 1$$

Our result:



Bosonic part of full 2-loop EW correction

# Running top quark mass

Comparison with gaugeless limit

Contributions to  $m_t(M_t) - M_t$  in GeV

$M_H$ [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$ -g.l.	$\mathcal{O}(\alpha^2)$	total
124	-10.38	12.11	-0.39	-0.51	-0.96	0.38
125	-10.38	11.91	-0.39	-0.49	-0.95	0.19
126	-10.38	11.71	-0.38	-0.48	-0.92	0.03

- ▶ QCD is a 3-loop pure QCD result [Melnikov, van Ritbergen'00]
- ▶ g.l.-means result in gaugeless limit, i.e.  $g_1 = g_2 = 0$  [Kniehl, Veretin'14]
- ▶ gauge less limit is not so good approximation as in  $\mathcal{O}(\alpha)$  result:

	g.l.	full
$[m_t(M_t) - M_t]_{\mathcal{O}(\alpha)}$	12.89	11.79

Large cancellation between terms in region close to measured  $M_H$

# $\overline{MS}$ Couplings from running masses

Vacuum expectation value  $v$  is connected with Fermi coupling constant:

$$G_F = 1/\sqrt{2}v^2$$

Similar relation holds for  $\overline{MS}$  quantities, and running Fermi constant is:

$$G_F = (1 + \Delta r^{\overline{MS}})G_F^{\overline{MS}}(\mu)$$

$$\begin{aligned}y^2(\mu) &= 2\sqrt{2}m^2(\mu)G_F^{\overline{MS}}(\mu) \\ \lambda(\mu) &= \sqrt{2}m_H^2(\mu)G_F^{\overline{MS}}(\mu) \\ g_1^2(\mu) &= 4\sqrt{2}(m_Z^2(\mu) - m_W^2(\mu))G_F^{\overline{MS}}(\mu) \\ g_2^2(\mu) &= 4\sqrt{2}m_W^2(\mu)G_F^{\overline{MS}}(\mu)\end{aligned}$$

Tadpole free mass:

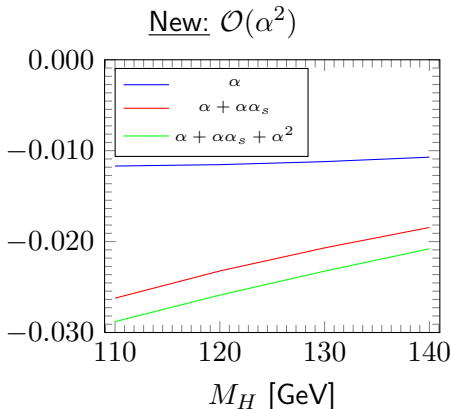
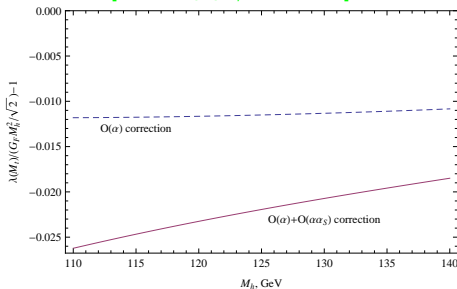
$$m_Y(\mu) = \frac{y(\mu)}{\sqrt{2\sqrt{2}G_F}}$$

# Higgs self-coupling $\lambda(\mu)$

Comparison with analytically known results

$$\frac{\sqrt{2}\lambda(M_t)}{G_F M_H^2} = 1 + X_\alpha + X_{\alpha\alpha_s} + X_{\alpha^2}$$

[arXiv:hep-ph/1205.2893]

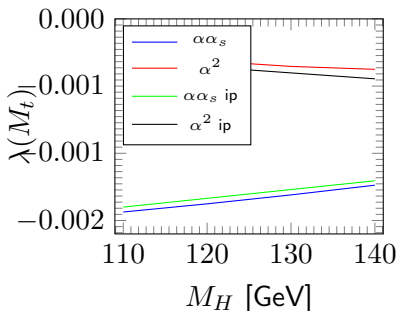




# Higgs self-coupling $\lambda(\mu)$

Comparison with  $\mathcal{O}(\alpha^2)$  interpolation formula at  $\mu = M_t$

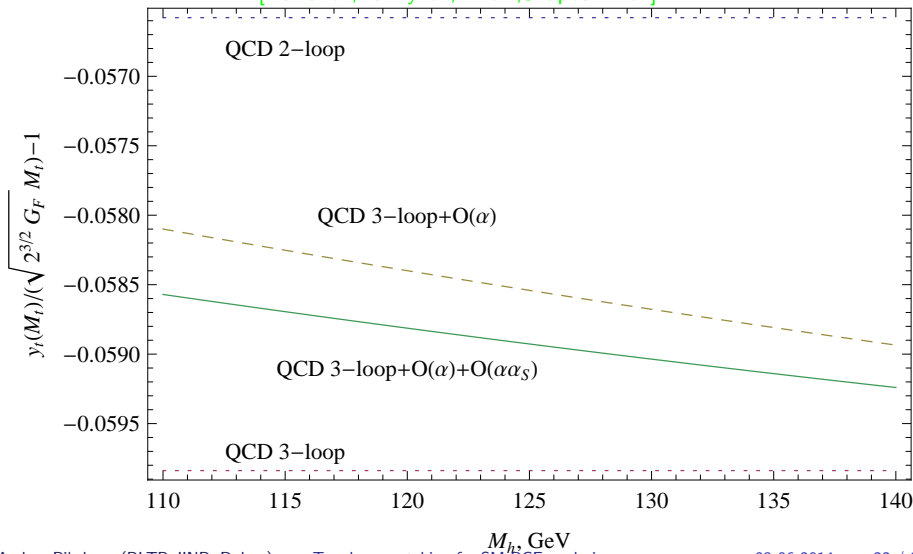
[arXiv:hep-ph/1307.3536]



$$\lambda_{\alpha\alpha_s}^{(2)}(M_t) = \frac{g_3^2}{(4\pi)^4} \left[ -23.89 + 0.12 \left( \frac{M_h}{\text{GeV}} - 125 \right) - 0.64 \left( \frac{M_t}{\text{GeV}} - 173 \right) \right]$$
$$\lambda_{\alpha^2}^{(2)}(M_t) = \frac{1}{(4\pi)^4} \left[ -9.45 - 0.11 \left( \frac{M_h}{\text{GeV}} - 125 \right) - 0.21 \left( \frac{M_t}{\text{GeV}} - 173 \right) \right]$$

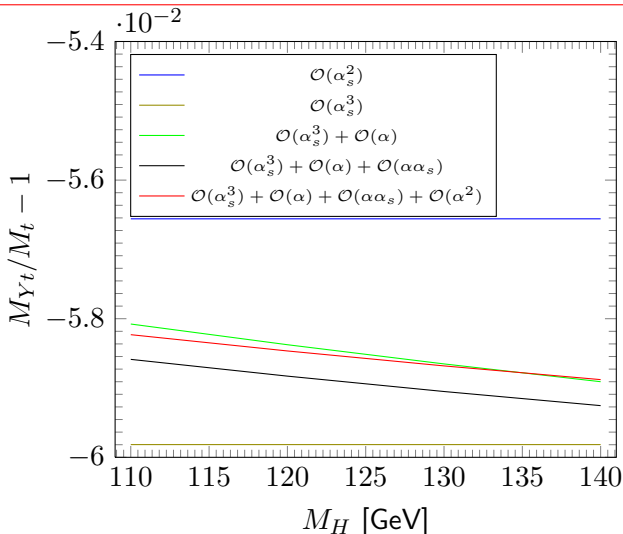
# Yukawa top $y_t$

[Bezrukov, Kalmykov, Kniehl, Shaposhnikov]



# Yukawa top $y_t$

As for  $m_t(\mu)$  large cancellation in full 2-loop EW result



# Realization

- ▶ Publicly available code for 2-loop matching and 3-loop running in full SM
- ▶ Easy to reproduce presented results
- ▶ At the development stage but available from [github.com/apik/mr](https://github.com/apik/mr)

Each threshold correction is present in the form:

$$m = M(1 + \alpha(\mu)m_{10} + \alpha_s(\mu)m_{01} + \alpha(\mu)\alpha_s(\mu)m_{11} + \alpha(\mu)^2m_{20} + \alpha_s(\mu)^2m_{02})$$

$m_{ij}$  are available in package  
 $\alpha(\mu), \alpha_s(\mu) - \overline{MS}$  couplings

# Example usage

$\mathcal{O}(\alpha\alpha_s)$  correction to the top mass

```
1 //                mb          mW          mZ          mH          mt
2 SMinput si(4.40, 80.385, 91.1876, 125.66, 173.5);
3 // Matching scale is Mt
4 tt dMt = tt(si, si.MMt());
5 dm11 = dMt.Mt()*alphaMt/4./Pi*alphaSMt/4./Pi
6          *dMt.m11();
```

For successful matching at scale  $\mu$  we need:

1. Low energy input  $M_b, M_W, M_Z, M_H, M_t$
2.  $\overline{MS}$  couplings  $\alpha(\mu), \alpha_s(\mu)$  at the matching scale  $\mu$
3. For couplings calculation we also need  $G_F$

# Conclusion

1. We have calculated all threshold corrections needed for SM RGE analysis input with  $\mathcal{O}(\alpha^2)$  precision
2. Large cancellation between terms in full EW result, inclusion of only  $\mathcal{O}(\alpha\alpha_s)$  is not enough
3.  $\mathcal{O}(\alpha^2)$  corrections in gaugeless limit are not so good approximation as in 1-loop case
4. There is some difference with  $\mathcal{O}(\alpha^2)$  corrections already known in literature
5. More detail comparison is needed, including different matching scales, not only  $\mu = M_t$
6. Software package made publicly available and ready to use