FORMATION OF EQUATION OF STATE IN EVOLVING QUANTUM FIELD

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Formulation of a problem (K. Dusling et al., 2011)

- Evolution of quantum field φ in the presence of an external time-dependent source J:

$$\mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 - J\varphi$$
$$J(x) \sim \theta(-x^0)\frac{Q^3}{\sqrt{\lambda}}e^{\epsilon x^0}$$

- Problem under consideration: evolution of energy-momentum tensor $T^{\mu\nu}$, in particular of $\mathrm{tr}\,T^{\mu\nu} = \varepsilon - 3p$ for the simplest case of a spatially homogeneous field

- Main motivation: studying evolution of quantum field system created at the early stages of heavy ion collisions.

Classical evolution

- Equation of motion at $t \geq 0$:

$$\ddot{\phi}_c^0 + \frac{\lambda}{6}(\phi_c^0)^3 = 0$$

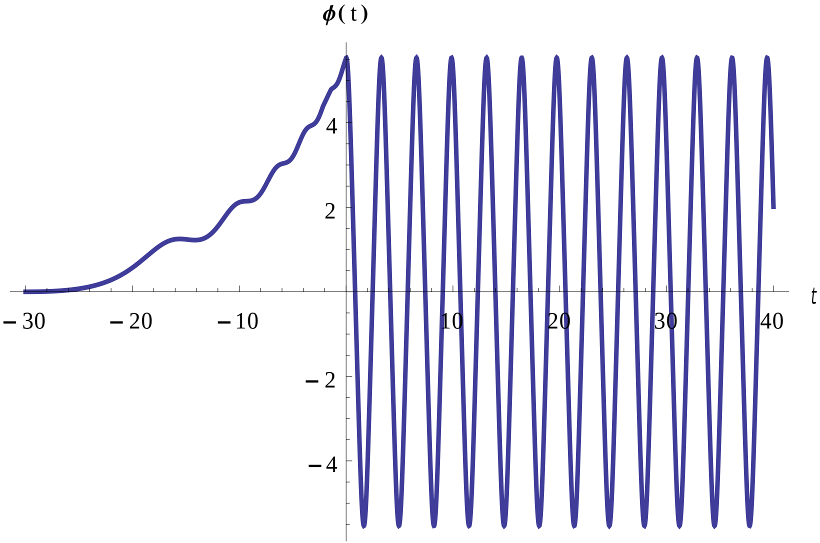
- Solution at $t \geq 0$:

$$\phi_c^0(t) = \phi_{max}\ cn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\ t + C\right)$$

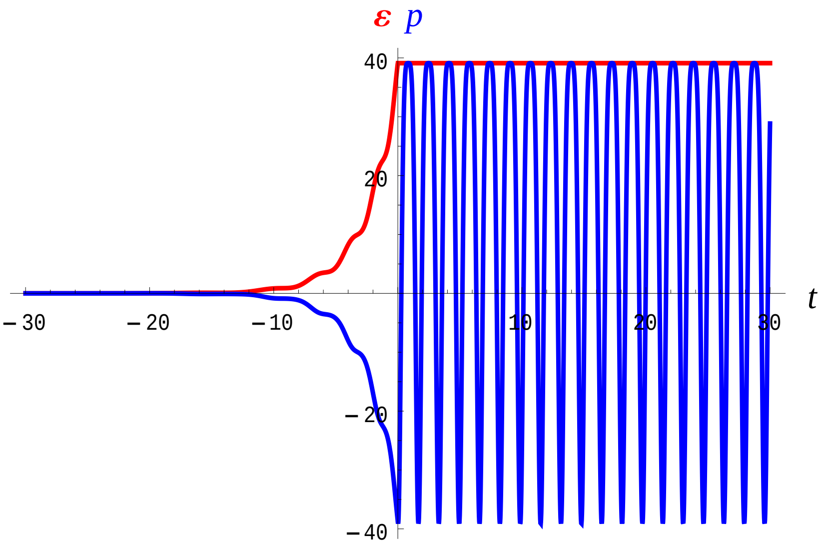
- Energy and pressure:

$$\begin{aligned} \varepsilon &= \frac{(\dot{\phi}_c^0)^2}{2} + \frac{\lambda}{24}(\phi_c^0)^4 = \frac{\lambda}{24}\phi_{max}^4 \\ p &= \frac{(\dot{\phi}_c^0)^2}{2} - \frac{\lambda}{24}(\phi_c^0)^4 \end{aligned}$$

Classical evolution: the field



Classical evolution: energy and pressure



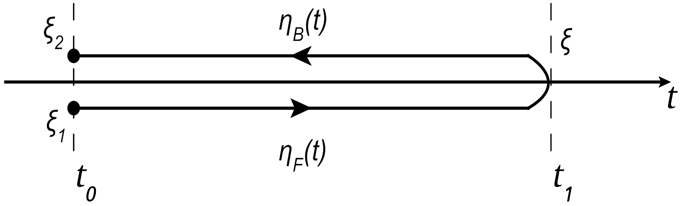
Quantum evolution: general case

- For evolution starting at $t = t_0$ and ending at $t = t_1$

$$< F(\hat{\varphi}) >_{t_1} = \int d\xi \int d\xi_1 \int d\xi_2 < \xi_1 | \hat{\rho}(t_0) | \xi_2 >$$

$$\times F(\xi) \int_{\eta_F(t_0)=\xi_1}^{\eta_F(t_1)=\xi} \mathcal{D}\eta_F \int_{\eta_B(t_0)=\xi_2}^{\eta_B(t_1)=\xi} \mathcal{D}\eta_B \; e^{i(S[\eta_F]-S[\eta_B])}$$

- The fields η_F and η_B live on the Keldysh contour:



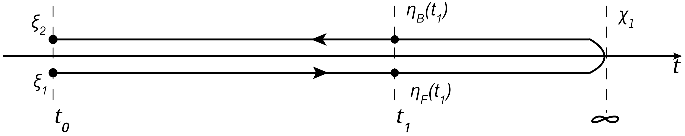
Quantum evolution: general case

- It is convenient to extend time integration to $t = \infty$

$$< F(\hat{\varphi}) >_{t_1} = \int d\chi_1 \int d\xi_1 \int d\xi_2 < \xi_1 | \hat{\rho}(t_0) | \xi_2 >$$

$$\times \int_{\eta_F(t_0)=\xi_1}^{\eta_F(\infty)=\chi_1} \mathcal{D}\eta_F \int_{\eta_B(t_0)=\xi_2}^{\eta_B(\infty)=\chi_1} \mathcal{D}\eta_B \; F\left(\frac{\eta_F(t_1)+\eta_B(t_1)}{2}\right) \; e^{i(S[\eta_F]-S[\eta_B])}$$

- The fields η_F and η_B now live on the extended Keldysh contour:



Quantum evolution: general case

- Keldysh rotation:

$$\begin{aligned} \phi_c &= \frac{\eta_F + \eta_B}{2}, \qquad \phi_q = \eta_F - \eta_B \\ \phi_c(t_0) &= \frac{\xi_1 + \xi_2}{2}, \qquad \phi_c(\infty) = \chi_1 \\ \phi_q(t_0) &= \xi_1 - \xi_2, \qquad \phi_q(\infty) = 0 \end{aligned}$$

- Keldysh action:

$$S[\eta_F] - S[\eta_B] \equiv S_K[\phi_c, \phi_q] =$$

$$\int_{t_0}^{\infty} dt[\dot{\phi}_c \dot{\phi}_q - m^2 \phi_c \phi_q - \frac{\lambda}{4!} \phi_c \phi_q^3 - \frac{\lambda}{6} \phi_c^3 \phi_q - J\phi_q]$$

- Expansion in ϕ_q is a quasiclassical expansion

Quantum evolution at the leading order in ϕ_q

- In the leading order in ϕ_q we have $F[\phi_c(t_1)] = F[\phi_c^0(t_1)]$ so that

$$< F(\hat{\varphi}) >_{t_1} = \int \frac{d\tilde{p}}{2\pi} \int d\xi_1 \int d\xi_2 < \xi_1 | \hat{\rho}(t_0) | \xi_2 > e^{i\tilde{p}(\xi_1 - \xi_2)} F[\phi_c^0(t_1)]$$

- Defining $(\xi_1 + \xi_2)/2 = \alpha$ and $\xi_1 - \xi_2 = \beta$ one gets:

$$\begin{aligned} < F(\hat{\varphi}) >_{t_1} &= \int \frac{d\tilde{p}}{2\pi} \int d\alpha f_W(\alpha, \tilde{p}, t_0) F[\phi_c^0(t_1)] \\ f_W(\alpha, \tilde{p}, t_0) &= \int d\beta < \alpha + \frac{\beta}{2} | \hat{\rho}(t_0) | \alpha - \frac{\beta}{2} > e^{i\tilde{p}\beta} \\ \phi_c^0(t_0) &= \alpha, \qquad \dot{\phi}_c^0(t_0) = \tilde{p} \end{aligned}$$

Quantum evolution at LO: analytical solution

- Ansatz for the Wigner function:

$$f_W(\alpha, p, 0) = \frac{1}{\alpha_0 p_0 \pi} e^{-\frac{(\alpha - A)^2}{\alpha_0^2} - \frac{p^2}{p_0^2}}$$

- In the saddle point approximation

$$< \varphi_c > = 2A \sum_{k=0}^{\infty} u_k e^{-\frac{6\pi^2 p_0^2}{\lambda A^4 T^2} k^2} e^{-\frac{\alpha_0^2 \pi^2 \lambda}{6 T^2} k^2 t^2} \cos\left(\frac{2A\pi k}{T} \sqrt{\frac{\lambda}{6}} t\right)$$

Quantum evolution at LO: analytical solution

- In the same approximation the pressure reads

$$\frac{p}{\varepsilon} = -8\left(\frac{2\pi}{T}\right)^2 \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} k\ l\ u_k u_l\ e^{-\frac{6\pi^2 p_0^2}{\lambda A^4 T^2}(k+l)^2} \times e^{-\frac{\alpha_0^2\pi^2\lambda}{6T^2}(k+l)^2 t^2} \cos\left(\frac{2\pi A(k+l)}{T}\sqrt{\frac{\lambda}{6}}t\right) - 1$$

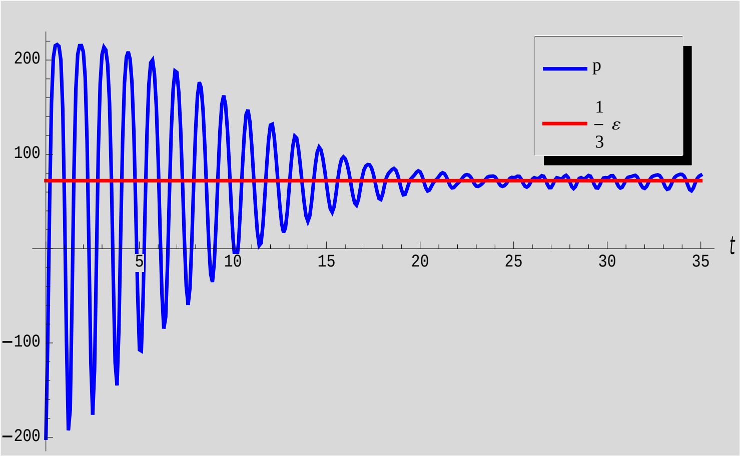
- At $t \to \infty$

$$p(t\to\infty) = \varepsilon\left[-4\cdot 2\left(\frac{2\pi}{T}\right)^2 \sum_{k=-\infty}^{\infty}(-k^2 u_k^2) - 1\right]$$

- Finally,

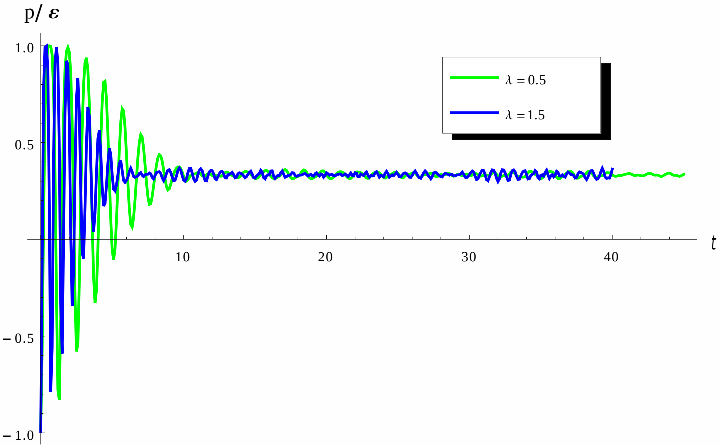
$$2\left(\frac{2\pi}{T}\right)^2 \sum_{k=-\infty}^{\infty}(-k^2 u_k^2) = -\frac{1}{3} \ \Rightarrow\ p(t\to\infty) = \frac{\varepsilon}{3}$$

Quantum evolution at LO: pressure relaxation



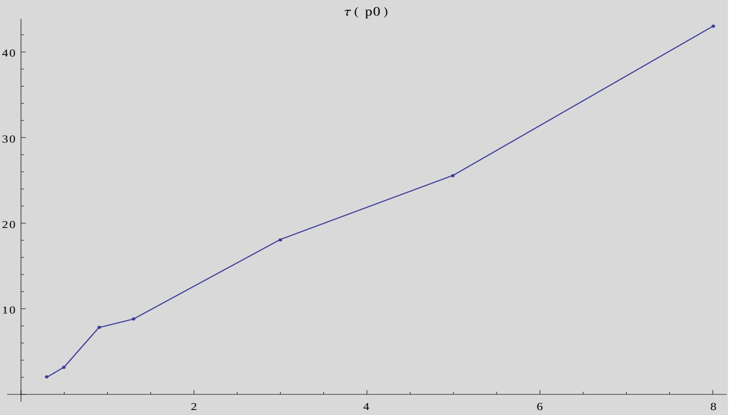
Quantum evolution at LO: pressure relaxation

Pressure relaxation as a function of coupling:



Quantum evolution at LO: pressure relaxation

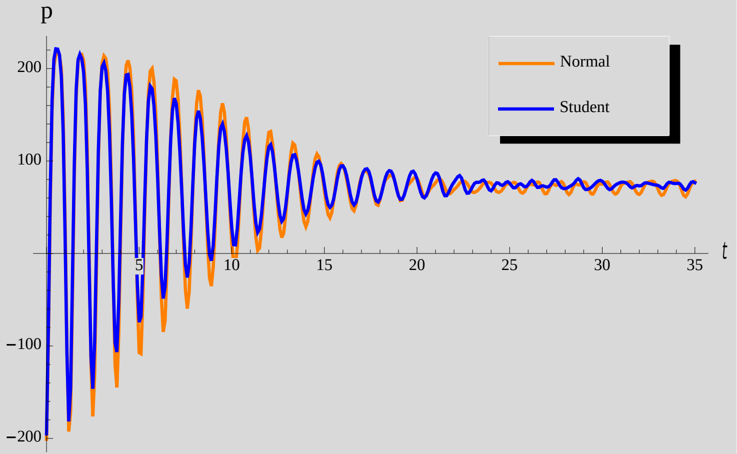
- Let us define relaxation time τ as a time at which $\left|\frac{\varepsilon - 3p}{\varepsilon}\right| = 0.1$



- $\tau(p_0)$ is approximately linear in p_0 in agreement with analytical expression

Quantum evolution at LO: pressure relaxation

Pressure relaxation for different initial distributions:



Quantum evolution at NLO

- NLO corrections are coming from expansion in ϕ_q:

$$e^{iS_K} = e^{i\tilde{S}_K} e^{-i\frac{\lambda}{4!}\int\limits_{t_0}^{\infty} dt' \phi_c \phi_q^3} \approx e^{i\tilde{S}_K}[1 - i \int\limits_{t_0}^{\infty} dt' \frac{\lambda}{4!} \phi_c(t')\phi_q^3(t') + O(\phi_q^6)]$$

- The corresponding NLO contribution reads

$$< \Delta F(\hat{\varphi}) >_{t_1} = -i\frac{\lambda}{4!} \int d\chi_1 \int d\xi_1 \int d\xi_2 < \xi_1|\hat{\rho}(t_0)|\xi_2 > \times$$

$$\int\limits_{\phi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\phi_c(\infty)=\chi_1} \mathcal{D}\phi_c \; F[\phi_c(t_1)] \int\limits_{t_0}^{\infty} dt' \phi_c(t') \int\limits_{\phi_q(t_0)=\xi_1-\xi_2}^{\phi_q(\infty)=0} \mathcal{D}\phi_q \; \phi_q^3(t') e^{i\tilde{S}_K[\phi_c,\phi_q]}$$

- NLO corrections as derivatives in $J(t)$:

$$< \Delta F(\hat{\varphi}) >_{t_1} = -\frac{\lambda}{4!} \int d\chi_1 \int d\xi_1 \int d\xi_2 < \xi_1 | \hat{\rho}(t_0) | \xi_2 > \times$$

$$\int_{t_0}^{\infty} dt' \frac{\delta^3}{\delta J^3(t')} \int_{\phi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\phi_c(\infty)=\chi_1} \mathcal{D}\phi_c \; F[\phi_c(t_1)]\phi_c(t') \int_{\phi_q(t_0)=\xi_1-\xi_2}^{\phi_q(\infty)=0} \mathcal{D}\phi_q \; e^{i\tilde{S}_K[\phi_c,\phi_q]} =$$

$$-\frac{\lambda}{4!} \int \frac{dp}{2\pi} \int d\xi_{1,2} < \xi_1 | \hat{\rho}(t_0) | \xi_2 > e^{ip(\xi_1-\xi_2)} \int_{t_0}^{\infty} dt' \frac{\delta^3}{\delta J^3(t')} F[\phi_c^0(t_1)]\phi_c^0(t')$$

Quantum evolution at NLO

- The quantity to compute is

$$I(t_1) = \int_{t_0}^{\infty} dt' \frac{\delta^3}{\delta J^3(t')} F[\phi_c^0(t_1)]\phi_c^0(t') = \int_{t_0}^{t_1} dt' \phi_c^0(t') \frac{\delta^3}{\delta J^3(t')} F[\phi_c^0(t_1)]$$

- We have

$$\frac{\delta^3 F[\phi(t_1)]}{\delta J^3(t')} = \frac{dF}{d\phi} \cdot \frac{\delta^3 \phi(t_1)}{\delta J^3(t')} + 3\frac{d^2 F}{d\phi^2} \cdot \frac{\delta^2 \phi(t_1)}{\delta J^2(t')} \cdot \frac{\delta \phi(t_1)}{\delta J(t')} + \frac{d^3 F}{d\phi^3} \left(\frac{\delta \phi(t_1)}{\delta J(t')} \right)$$

- Let

$$\frac{\delta \phi(t_1)}{\delta J(t')} = \Phi_1(t_1, t'),\ \frac{\delta^2 \phi(t_1)}{\delta J^2(t')} = \Phi_2(t_1, t'),\ \frac{\delta^3 \phi(t_1)}{\delta J^3(t')} = \Phi_3(t_1, t'),$$

$$\hat{L}_t = \partial_t^2 + \frac{\lambda}{2}\phi^2(t)$$

Quantum evolution at NLO

- Evolution equations for field variations:

$$\begin{aligned}
\hat{L}_{t_1}\Phi_1(t_1,t') &= -\delta(t_1-t'),\\
\hat{L}_{t_1}\Phi_2(t_1,t') &= -\lambda\phi(t_1)\Phi_1^2(t_1,t'),\\
\hat{L}_{t_1}\Phi_3(t_1,t') &= -\lambda\Phi_1^3(t_1,t')-3\lambda\phi(t_1)\Phi_1(t_1,t')\Phi_2(t_1,t')
\end{aligned}$$

- Φ_1 can be computed analytically:

$$\Phi_1(t,t') = G(t,t') = \theta(t-t')[f_2(t')f_1(t)-f_1(t')f_2(t)]$$

Quantum evolution at NLO

- $$f_1(t) = \dot{\phi}_c^0(t) = -\sqrt{\frac{\lambda}{6}\phi_{max}^2}\; sn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\; t + C\right) \times dn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\; t + C\right)$$

- $$f_2(t) = -\sqrt{\frac{6}{\lambda}}\frac{1}{\phi_{max}^2}\left(t\; sn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\; t + C\right) \times dn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\; t + C\right) - \frac{cn\left(\frac{1}{2}; \sqrt{\frac{\lambda}{6}}\phi_{max}\; t + C\right)}{\sqrt{\frac{\lambda}{6}}\phi_{max}}\right)$$

Conclusions

- The problem of evolution of quantum field created by an external source typical, e.g., for studying early stages of heavy ion collisions is rigorously formulated using Keldysh formalism.
- Detailed study of the dependence of the process of formation of equation of state on the Wigner function describing the initial field distribution was performed.
- Systematic procedure of computing NLO is set up. Sample analytical calculations were performed.