High energy production amplitudes in $N = 4$ SUSY and unitarity

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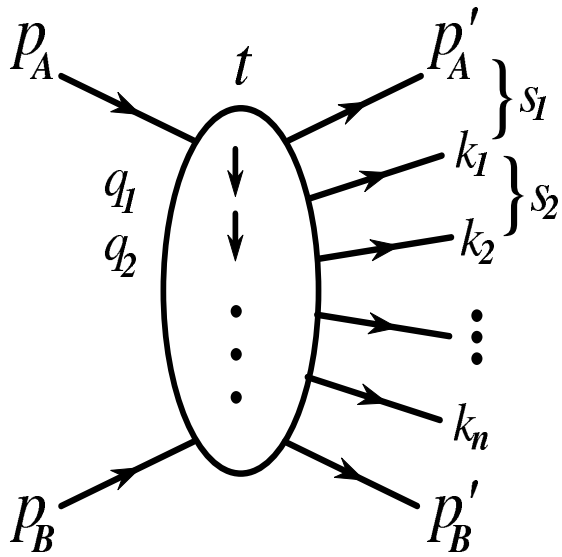
1 Amplitudes in multi-Regge kinematics



$$M^{FKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{q_1^2}\, gT^{d_1}_{c_2 c_1} C_1^{\mu} e^1_{\mu}\, \frac{s_2^{\omega_2}}{q_2^2} ... gT^{d_n}_{c_{n+1} c_n}\, C_n^{\sigma} e^n_{\sigma}\, \frac{s_{n+1}^{\omega_{n+1}}}{q_{n+1}^2}\,,\ \omega_r = \frac{-g^2}{16\pi^3} \int \frac{d^2k\ q_r^2}{k^2\,(q_r-k)^2}\,,$$

$$C_1 = -q_1^{\perp} - q_2^{\perp} + p_A \left(\frac{t_1 - m^2}{p_A k_1} + \frac{p_B k_1}{p_A p_B} \right) - p_B \left(\frac{t_2 - m^2}{p_B k_1} + \frac{p_A k_1}{p_A p_B} \right)$$

2 Fadin-Kuraev-Lipatov equation (1975)

Total cross-section in the Higgs model at high energies in LLA

$$\sigma_t \sim s^{\Delta}\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0 = \frac{4\alpha_s N_c}{\pi}\,\ln 2$$

FKL equation for the Pomeron wave function at $q = t = 0$

$$E f(r) = H f(r)\,,\ H = T(p) + V(r)$$

Kinetic energy related to two Regge trajectories

$$T(p) = \frac{2(|p|^2 + m^2)}{|p|\sqrt{|p|^2 + 4m^2}}\,\ln\frac{\sqrt{|p|^2 + 4m^2} + |p|}{\sqrt{|p|^2 + 4m^2} - |p|}\,,\ |p|^2 = -\frac{1}{r}\,\partial\, r\,\partial$$

Potential energy

$$V(r) = -4\,K_0(r\,m) + \frac{N_c^2 + 1}{N_c^2}\,\hat{P}\,,\ \hat{P}\,\phi(p) = \frac{m^2}{|p|^2 + m^2}\int\frac{d^2p'}{\pi}\,\frac{\phi(p')}{|p'|^2 + m^2}$$

Semiclassical solution (Levin, Lipatov, Siddikov (2014))

3 BFKL equation in LLA (1978)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ :\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0 = \frac{4\alpha_s N_c}{\pi}\,\ln 2$$

Holomorphic separability

$$H_{12} = h_{12}+h_{12}^*\,,\ [h_{12},h_{12}^*] = 0\,,\ E = \epsilon+\epsilon^*\,,\ \epsilon = \psi(m)+\psi(1-m)-2\psi(1)$$

Holomorphic Hamiltonian

$$h_{12} = \frac{1}{p_1}(\ln\rho_{12})p_1 + \frac{1}{p_2}(\ln\rho_{12})p_2 + \ln(p_1p_2) - 2\psi(1)$$

Möbius invariant solution

$$\Psi = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}}\,,\ m = i\nu + \frac{1+n}{2}\,,\ \widetilde{m} = i\nu + \frac{1-n}{2}$$

4 BKP equation in LLA

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h + h^*)\,,\;\; h = \sum_{k=1}^{n} h_{k,k+1}\,,$$

Monodromy matrix (L. (1993))

$$\prod_{k=1}^{n} L_k = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},\;\; L_k = \begin{pmatrix} u + \rho_k\,p_k & p_k \\ -\rho_k^2\,p_k & u - \rho_k\,p_k \end{pmatrix}$$

Transfer matrix and integrability (L. (1993))

$$T(u) = A(u) + D(u)\,,\;\; [T(u), T(v)] = [T(u), h] = 0$$

5 Confinement and integrability

Gluon coordinates and momenta at t-channel temperature $T \neq 0$

$$\rho_r = x_r + i y_r \,,\ 0 < y_r < 1/T \,;\ p_r = p_r^x + i p_r^y \,,\ p_r^y = 2\pi k \,,\ k = 0, \pm 1, \pm 2, ...$$

Hamiltonian for Pomeron in the thermostat (de Vega, Lipatov)

$$h = \sum_{s=1,2} \left(\psi\left(1 + i\frac{p_s}{2\pi T}\right) + \psi\left(1 - i\frac{p_s}{2\pi T}\right) - 2\psi(1) + \frac{2}{p_s} \ln(2\sinh(\pi T \rho_{12}))\, p_s \right)$$

Conformal transformation to the zero temperature and integrability

$$\rho_r = \frac{1}{2\pi T} \ln \rho_s'$$

Conformal transformation to a rectangle and anti-Meissner effect

$$\rho_r' = sn(K\rho/a; k) = sn(K'\rho/b; k') \,,\ K = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \,,\ k' = \sqrt{1-k^2}$$

6 Non-Fredholm BFKL equation

Property of the integral kernel for the Fredholm equation

$$\int d^2p d^2p' |K(p^2, p'^2, (p-p')^2)|^2 < \infty$$

Asymptotic freedom for the BFKL kernel at large $p \sim p'$

$$\lim_{p \sim p' \to \infty} |K(p^2, p'^2, (p-p')^2)|^2 \sim f(p/p') \frac{1}{|p|^2 |p'|^2} \frac{1}{\ln^2(\max(p^2, p'^2))}$$

Convergency at large distances due to the anti-Meissner effect

$$|K(\rho, \rho')|^2 \sim \theta(\min(a,b) - |\rho|)\, \theta(\min(a,b) - |\rho'|)$$

Integral divergency at small relative momentum $p - p'$

$$|K(p^2, p'^2, (p-p')^2)|^2 \sim \left(\omega(p^2)\delta^2(p-p')\right)^2 \,, \; E_{m\to\infty} \sim \gamma_K(\alpha) \ln |m|^2$$

7 Spectrum of Pomerons in QCD

Asymptotic freedom

$$\frac{\alpha_s(k^2)}{2\pi} = \frac{1}{\beta_0 \ln \frac{k^2}{\Lambda^2_{QCD}}}\,, \;\; \beta_0 = 11 - \frac{2}{3} n_f$$

Solutions of the BFKL equation with running coupling

$$f_n(k) = \int_{-\infty}^{\infty} d\,\nu \left(\frac{k^2}{\Lambda^2_{QCD}}\right)^{i\,\nu} \left(e^{-2i\psi(1)}\,\frac{\Gamma\left(\frac{1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)}\right)^{\frac{12}{\beta_0\,\omega_n}}$$

Pomeron intercepts and parton distributions at HERA (KLR)

$$\omega_n \approx \frac{0.5}{1+0.95n}\,, \;\; g(x,k^2) = \sum_{n=1}^{\infty} c_n x^{-\omega_n} f_n(k)$$

Essential transverse momenta and physics BSM (KLR)

$$\bar{k}_n \sim \Lambda_{QCD}\, e^{4n}$$

8 Pomeron in $N = 4$ SUSY

Eigenvalue of BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,\;\; \gamma = i\nu + (1+\omega)/2$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.L. (2002)

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big)$$

9 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - \Delta\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling expansion of Δ (KLOV, BPST, KL)

$$\Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2} + \ldots\,,\ \lambda = \frac{\alpha N_c}{2\pi}$$

Exact expression for the slope of γ at $j = 2$ (KLOV, V., Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

10 Maximal helicity violation

BDS amplitudes in $N = 4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,\ldots,a_n} = \sum_{\{i_1,\ldots,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} \ldots T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, \ldots, p_{i_n})\,, \;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right) ,$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon} \,,\; C^{(1)} = 0\,,\; C^{(2)} = -\zeta_2^2/2\,,\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)} \,,\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + \ldots$$

11 Elastic BDS amplitude

Regge form of the amplitude at large s/t

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

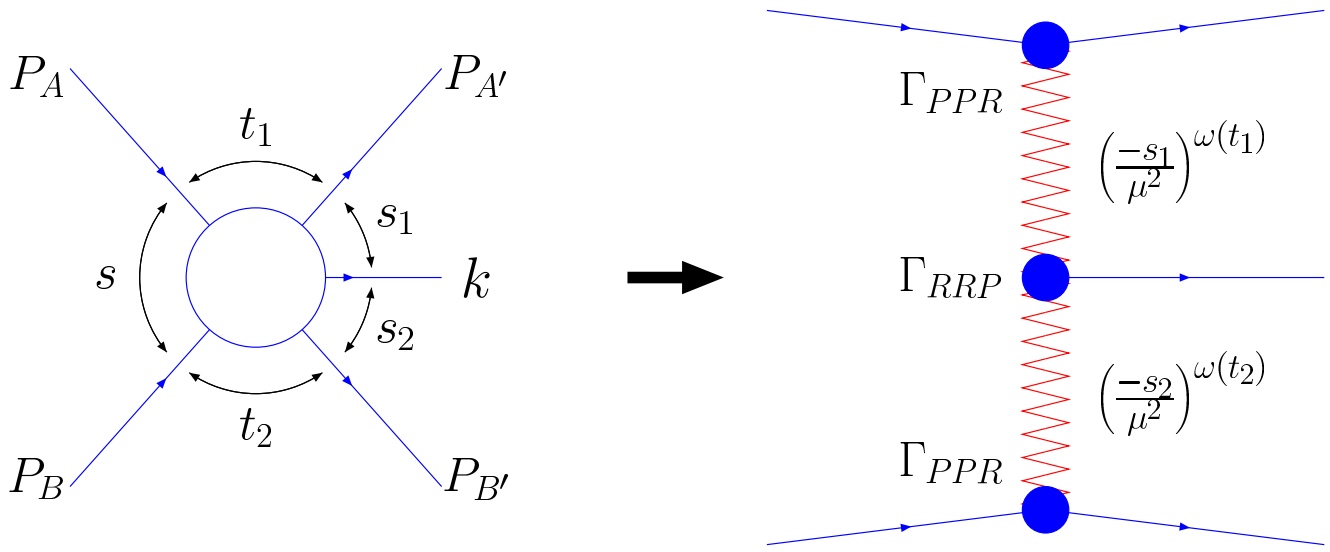
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2$$

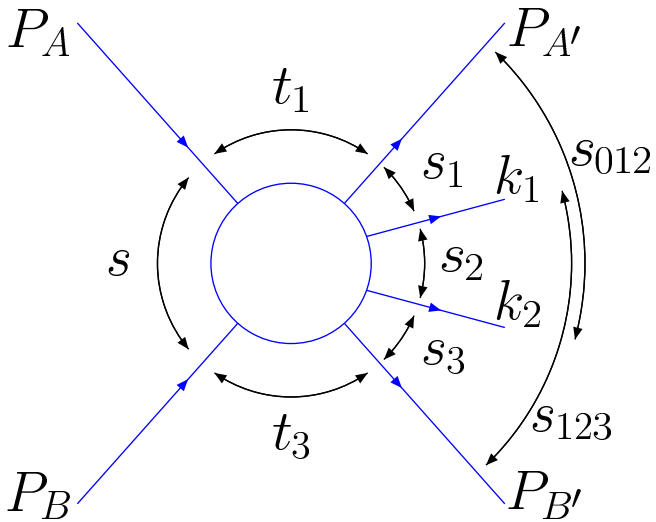
$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

12 One particle production (BLV)



$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2)-\int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2}-$$
$$\frac{\gamma_K(a)}{16}\left(\ln^2\frac{-\kappa}{\mu^2}-\ln^2\frac{-t_1}{-t_2}-\zeta_2\right)-\frac{1}{2}\int_0^a\frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

13 Regge factorization violation (BLV)



$$M_{2\to 4}|_{s_2>0;\, s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln\frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}\Gamma(t_3)$$

14 Mandelstam cuts in j_2-plane

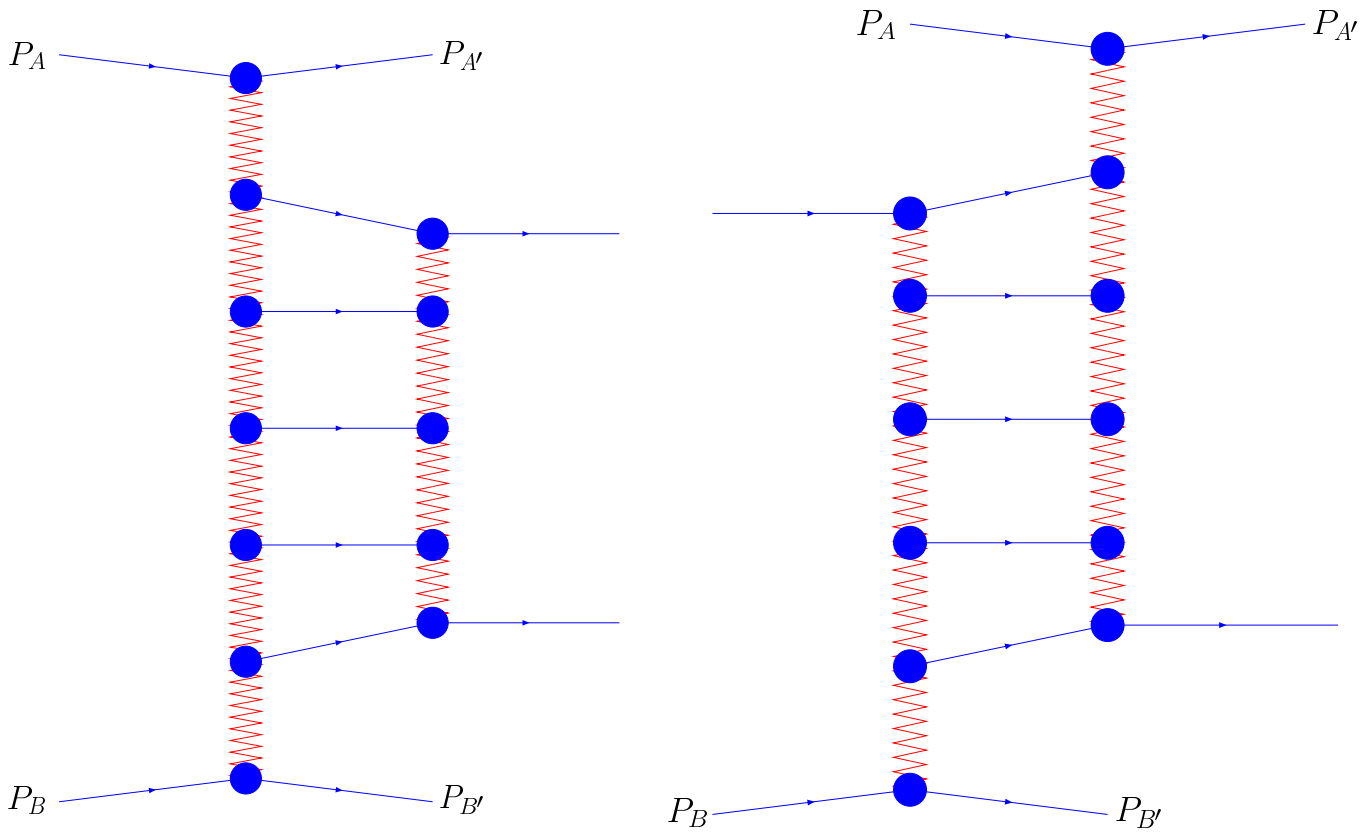


Figure 1: BFKL ladders in $M_{2\to 4}$ and $M_{3\to 3}$

15 Analyticity of production amplitude

Steinmann constraints for overlapping channels

$$\Delta_{s_r}\Delta_{s_{r+1}}M_{2\to 2+n}=0$$

Regge representation for planar amplitude $M_{2\to 3}$

$$\frac{M_{2\to 3}}{\Gamma(t_1)\Gamma(t_2)|\Gamma_a|}=c_R^a(-\widetilde{s})^{j_2}(-s_1)^{j_1-j_2}+c_L^a(-\widetilde{s})^{j_1}(-s_2)^{j_2-j_1}\ ,\ \widetilde{s}=s|k_\perp^a|^2$$

Regge ansatz for planar amplitude $M_{2\to 4}$

$$\frac{M_{2\to 4}}{\Gamma(t_1)\Gamma(t_2)|\Gamma_a||\Gamma_b|}=c_R^a c_L^b(-\widetilde{s})^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}$$

$$+c_R^a c_R^b(-\widetilde{s})^{j_3}(-\widetilde{s}_{012})^{j_2-j_3}(-s_1)^{j_1-j_2}+c_L^a c_L^b(-\widetilde{s})^{j_1}(-\widetilde{s}_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

$$+c_L^a c_R^b\left(k(-\widetilde{s})^{j_3}(-\widetilde{s}_{012})^{j_1-j_3}(-s_2)^{j_2-j_1}+l(-\widetilde{s})^{j_1}(-\widetilde{s}_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}\right),$$

$$c_R^a=\frac{\sin\pi\omega_{1a}}{\sin\pi\omega_{12}}\ ,\ c_L^a=\frac{\sin\pi\omega_{2a}}{\sin\pi\omega_{21}}\ ,\ k=\frac{\sin\pi\omega_1}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{23}}{\sin\pi\omega_{13}}\ ,\ l=\frac{\sin\pi\omega_3}{\sin\pi\omega_2}\,\frac{\sin\pi\omega_{21}}{\sin\pi\omega_{31}}$$

16 Regge factorization and crossing

Twist operators in the t_r-channels

$$\tau_r = \pm 1$$

Generating function for amplitudes in crossing channels

$$A^{\tau_1 \ldots \tau_n}_{2\to n+1} = A + \sum_{r=1}^{n} \tau_r \, A_r + \sum_{r_1 < r_2} \tau_{r_1} \tau_{r_2} \, A_{r_1 r_2} + \ldots + \tau_1 \tau_2 \ldots \tau_n \, A_{1 \ldots n}$$

Regge factorization (Weis (1971))

$$\frac{A^{\tau_1 \ldots \tau_n}_{2\to n+1}}{s\Gamma(t_1)\Gamma(t_2)|\Gamma_a||\Gamma_b|\ldots} = |s_1|^{\omega_1} \xi_1 V^{1,2} |s_2|^{\omega_2} \xi_2 V^{2,3} \ldots V^{n-1,n} |s_n|^{\omega_n} \xi_n$$

Signature factors

$$\xi_r = e^{-i\pi\omega_r} - \tau_r \, , \; \xi_{12} = e^{-i\pi\omega_{12}} + \tau_1 \tau_2 \, , \; V^{1,2} = \frac{\xi_{12}}{\xi_1} c^a_R + \frac{\xi_{21}}{\xi_2} c^a_L$$

17 Shortage of Regge ansatz and cuts

Regge pole expression for the 6-point amplitude

$$\frac{A^{\tau_1\ldots\tau_n}_{2\to n+1}}{s|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\Gamma(t_1)\Gamma(t_2)|\Gamma_a||\Gamma_b|} = \ldots+\left(\tau_1\tau_3\, e^{-i\pi\omega_2} + \tau_1\tau_2\tau_3\right) A+\ldots$$

Singular Regge contribution

$$A = \frac{2\cos\pi\omega_2\ \sin\pi\omega_a\ \sin\pi\omega_b}{i\sin\pi\omega_2} + i\sin\pi(\omega_a+\omega_b) + \cos\pi\omega_{ab}$$

Remainder function for $A_{2\to 4}$

$$A = R(u_1,u_2,u_3)\, A^{BDS}_{2\to 4}\ ,\ u_1 = \frac{ss_2}{s_{012}s_{123}}\ ,\ u_2 = \frac{s_1t_3}{s_{012}t_2}\ ,\ u_3 = \frac{s_3t_1}{s_{123}t_2}\ ,\ |w|^2 = \frac{u_2}{u_3}$$

Regge pole and cut contributions for $R_{2\to 4}$ (L.L. (2009)

$$R\, e^{i\pi\delta} = cos\,\pi\omega_{ab} + i\int \frac{d\omega}{2\pi i} s_2^{\omega}\, f_\omega\ ,\ \delta = \frac{\gamma_K}{8}\ \ln\frac{|w|^2}{|1+w|^4}\ ,\ \omega_{ab} = \frac{\gamma_K}{8}\ \ln|w|^2$$

18 Amplitude $A_{2\to 4}$ in $N=4$ SUSY

Remainder factor in next-to-leading LLA (F.,L. (2011))

$$R\, e^{i\pi\delta} = cos\, \pi\omega_{ab} + i\, \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n e^{i\phi n} \int_{-\infty}^{\infty} \frac{|w|^{2i\nu} d\nu}{\nu^2 + \frac{n^2}{4}}\, \Phi(\nu, n) \left(\frac{-1}{\sqrt{u_2 u_3}} \right)^{\omega(\nu,n)} ,$$

$$\cos\phi = \frac{1 - u_1 - u_2 - u_3}{2\sqrt{u_2 u_3}} , \;\; \Phi(\nu, n) = 1 - a \left(\frac{E^2_{\nu n}}{2} + \frac{3}{8}\, n^2 / (\nu^2 + \frac{n^2}{4})^2 + \zeta(2) \right) ,$$

Spectrum of the eigenvalues of the BFKL kernel

$$\omega(\nu, n) = -aE_{\nu,n} - a^2 (\epsilon^{FL}_{\nu n} + 3\zeta(3)) , \;\; E_{\nu n} = -\frac{|n|/2}{\nu^2 + \frac{n^2}{4}} + 2\Re\psi(1 + i\nu + \frac{|n|}{2}) - 2\psi(1) ,$$

Eigenvalue in the next-to-leading order (F.,L. (2011)

$$\epsilon^{FL}_{\nu n} = -\frac{\Re}{2} \left(\psi''(1 + i\nu + \frac{|n|}{2}) - \frac{2i\nu\psi'(1 + i\nu + \frac{|n|}{2})}{\nu^2 + \frac{n^2}{4}} \right) - \zeta(2)\, E_{\nu n} - \frac{1}{4}\, \frac{|n| \left(\nu^2 - \frac{n^2}{4} \right)}{\left(\nu^2 + \frac{n^2}{4} \right)^3}$$

19 Mandelstam cuts with n reggeons

Mandelstam cut contribution

$$A_{2\to 2+r} = \int \prod_{i=1}^{n-1} d^2 k_i^\perp \Phi_1(k_1^\perp, ..., k_{n-1}^\perp) \Phi_2(k_1^\perp, ..., k_{n-1}^\perp) \prod_{t=1}^{n} \frac{s^{j(-|k_i^\perp|^2)}}{|k_i^\perp|^2}$$

Impact factor Sudakov representation

$$\Phi_1(k_1^\perp, ..., k_{n-1}^\perp) = \int \prod_{i=1}^{n-1} d\alpha_i f(k_1^\perp, \alpha_1, ..., k_{n-1}^\perp, \alpha_{n-1})$$

Conditions of a nonzero contribution

$$r = 2n-2\,,\ \ s_1, s_2, ..., s_{n-1} < 0\,,\ \ s_n > 0$$

Mandelstam cuts with the reggeon interaction

$$A_{2\to 2+r} = \int \prod_{i=1}^{n-1} d^2 k_i^\perp d^2 k_i^{\perp\prime} \Phi_1(k_1^\perp, ...) \Phi_2(k_1^{\perp\prime}, ...) \int d\omega s^\omega G_\omega(k_1^\perp, ...k_n^{\perp\prime})$$

20 Integrability of n-reggeon dynamics

Holomorphic separability of the reggeon Hamiltonian

$$H = h + h^* \,, \; [h, h^*] = 0$$

Holomorphic hamiltonian at large N_c

$$h = \ln(Z_1^2 \partial_1) - 2\psi(1) + \ln \partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}$$

Pair hamiltonian

$$h_{1,2} = \ln(Z_{12}^2 \partial_1) + \ln(Z_{12}^2 \partial_2) - 2 \ln Z_{12} - 2\psi(1)$$

Integrals of motion for an open spin chain (L. (2009))

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k \,, \; [D(u), h] = 0$$

21 Discussion

1. Gluon reggeization
2. FKL equation for the Hiiggs model
3. Integrability of the BKP equations on the rectangle
4. BDS ansatz for N=4 SUSY and its breakdown
5. Remainder functions and Mandelstam cuts
6. Analyticity properties of production amplitudes
7. Next-to-leading LLA corrections to $2 \to 4$ transition
8. Integrability of the BFKL dynamics for the octet states