Dynamical regimes in the model of the Universe with a scalar field nonminimally coupled to gravity

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Abstract

We investigate the cosmological dynamics of non-minimally coupled scalar field system described by $F(\phi)R$ coupling with $F(\phi) = \left(1 - \xi\phi^N\right) R$ $(N \geq 2)$ and the field potential, $V(\phi) = V_0\phi^n$. We use a generic set of dynamical variables to bring out new asymptotic regimes of the underlying dynamics. However, our dynamical variables miss the most important fixed point− the de Sitter solution. We make use of the original form of system of equations to investigate the issues related to this important solution. In particular, we show that the de-Sitter solution which is a dynamical attractor of the system lies in the region of negative effective gravitational constant G_N thereby leading to a ghost dominated universe in future and a transient quintessence(phantom) phase with $G_N > 0$ around the present epoch[1].

1 Introduction

Theories with a scalar field non-minimally coupled to gravity dubbed scalar tensor theories have been studied for decades. The first well-known example of non-minimal coupling *a la* Brans-Dicke theory, was proposed in 1961 with an aim to match Mach principle with General Relativity [1]. In this theory the gravitational constant is replaced by a scalar field ϕ entering into the action in a specific combination with Riemanian curvature as $\phi^2 R$.

Followed by the Brans-Dicke proposal, other forms of scalar-tensor action were investigated, a well known example of a non-minimally coupled system is provided by $F(\phi)R$ coupling with $F = 1 - \xi\phi^2$. The cosmological dynamics of such a theory is rather rich and deserves attention. For a recent development in this direction, it is worth nothing that non-minimally coupled Higgs field due to a large coupling ξ might give rise to a successful inflation [2] which is otherwise impossible.

The non-minimally coupled scalar field system due to novel features are of great interest to dark energy model building [3-11]. For instance, non-minimal coupling might allow phantom crossing and may give rise to cosmological scaling solutions of interest to models of dark energy. Phantom scaling solutions are generic features of a non-minimally coupled system with $F = 1 - \xi\phi^2$ [12].

In recent years, methods of dynamical system theory have been extensively used in cosmology for obtaining a general picture of dynamics for many cosmological models including those with a scalar field and modified gravity. The advantage of this method is in having some kind

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[1] However, as demonstrated by Starobinsky in 1981, the ghost dominated universe, if exists, can not be accessed from the Universe we live in, we shall say more about this important result in the last section.

of "machinery " for deriving asymptotic solution using a simple programmed algorithm. This requires introduction of new set of variables in which the initial system can be rewritten as a system of first-order equations. On the other hand, there is a danger of losing some important solution as well as inability for this scheme to find transient regimes which can also be important. Nevertheless, a classification of stable asymptotic regimes, given by this method may be useful for understanding the underlying dynamics. The success and the limitation of the framework of dynamical systems applied to $f(R)$ theory can be found in [13] and [14].

We shall restrict our discussion to a polynomial functions F of the form $F(\phi) = 1 - \xi\phi^N$ ($N \geq 2$) and power-law potentials for the scalar field, $V(\phi) = V_0\phi^n$ giving rise to generalization of models considered earlier [12, 14-16](see also Ref. [17] on the related theme). The set of variables that we use can help in bringing out some generic features of the underlying dynamics and new asymptotic regimes missed in earlier studies. We should, however, note that the set of variables used in the present paper is not useful for the study of approximate Einstein regime in the system under consideration. For detailed description of this regime, other methods are required [18]. Secondly, our variables miss the existence of de Sitter solution and in order to investigate its existence and stability, we need to go back to initial variables to perform the analysis.

In this paper, we investigate cosmological dynamics of non-minimally coupled scalar field system with specific functional forms of coupling and field potential using a convenient set of dynamical variables. We shall focus on the asymptotic regimes of the solutions of interest and reveal the important features associated with the de Sitter solution.

2 Equations of motion

Let us consider the scalar field system with non-minimal coupling in the form,

$$S = \frac{1}{2}\int \sqrt{-g}d^4x\Big[m_{Pl}^2 R - (g^{\mu\nu}\phi_\mu\phi_\nu + \xi RB(\phi) + 2V(\phi))\Big] + S_M, \tag{1}$$

where $m_{Pl}^2 = 1/8\pi G = 1/\kappa$, and ξ is the dimensionless parameter and S_M designates matter action.

In a homogenous isotropic Friedmann-Robertson-Walker universe with spatially flat metric,

$$ds^2 = -dt^2 + a^2(t)dl^2, \tag{2}$$

the equations of motion which are obtained by varying the action with(1) have the form,

$$H^2 = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + 3\xi(H\dot{\phi}B'(\phi) + H^2B(\phi)) + \rho\right), \tag{3}$$

$$R = \kappa\left(-\dot{\phi}^2 + 4V(\phi) + 3\xi(3H\dot{\phi}B'(\phi) + \tfrac{R}{3}B(\phi) + \dot{\phi}^2B''(\phi) + \ddot{\phi}B'(\phi)) + \rho(1-3\omega)\right), \tag{4}$$

$$\ddot{\phi} + 3H\dot{\phi} + \tfrac{1}{2}\xi RB'(\phi) + V'(\phi) = 0. \tag{5}$$

where ρ and p are the energy density and pressure of the ordinary matter with $p = \omega\rho$.

From the standard form of equations $R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G_N(T_{ij,\phi} + T_{ij,m}) = \kappa T_{ij}^{eff}$, we read off the expression for G_N as the effective Newtonian gravitational constant $G_N = \frac{\kappa}{8\pi(1-\kappa\xi B(\phi))}$, and we shall use this definition of G_N hereafter. Ricci Scalar in chosen metric is given by, $R = 6(2H^2 + \dot{H})$. For convenience, we shall use the system of unites with $\kappa = 6$.

We introduce the following dimensionless variables,

$$x = \tfrac{\dot{\phi}^2}{H^2(1-6\xi B(\phi))},\quad y = \tfrac{2V(\phi)}{H^2(1-6\xi B(\phi))},\quad z = \tfrac{6\xi\dot{\phi}B'(\phi)}{H(1-6\xi B(\phi))},\quad \Omega = \tfrac{2\rho}{H^2(1-6\xi B(\phi))}, \tag{6}$$

and the dimensionless parameters that depend on the specific form of functions $B(\phi)$, $V(\phi)$,

$$A = \frac{B'(\phi)\phi}{(1-6\xi B(\phi))}, \quad b = \frac{B''(\phi)\phi}{B'(\phi)}, \quad c = \frac{V'(\phi)\phi}{V(\phi)}. \tag{7}$$

Here $'$ denotes derivative with respect to ϕ.

We will consider only function in the form $B(\phi) = \phi^N$ and $V(\phi) = V_0\phi^n$. In this case the parameters $b = \frac{N(N-1)\phi^{N-1}}{N\phi^{N-1}} = N-1$ and $c = \frac{V_0 n\phi^n}{V_0\phi^n} = n$ are constants and don't depend on time.

Taking derivative with respect to $\ln a$ of the introduced variables x, y, z and the parameter A (Ω is excluded with the identity $\Omega = 1 - x - y - z$, which is a consequence of (3)) we get the system of equations

$$\begin{aligned}
x' = \frac{dx}{d\ln a} &= 12x\left(\frac{1}{2} - \frac{z}{18(4x+z^2)}\left(-6x + 12y + \frac{z^2 b}{2\xi A} + \frac{yc}{\xi A} + 3(1-x-y-z)(1-3\omega)\right)\right) - \\
&-2x\left(\frac{2x}{3(4x+z^2)}\left(-6x + 12y + \frac{z^2}{4\xi A}\left(2b - \frac{yc}{x}\right) + 3(1-x-y-z)(1-3\omega)\right) - 2\right) + xz, \\
y' &= \frac{yz}{6\xi}\frac{c}{A} - 2y\left(\frac{2x}{3(4x+z^2)}\left(-6x + 12y + \frac{z^2}{4\xi A}\left(2b - \frac{yc}{x}\right) + 3(1-x-y-z=)(1-3\omega)\right) - 2\right) + \\
&+yz, \\
z' &= 6z\left(\frac{1}{2} - \frac{z}{18(4x+z^2)}\left(-6x + 12y + \frac{z^2 b}{2\xi A} + \frac{yc}{\xi A} + 3(1-x-y-z)(1-3\omega)\right)\right) + \frac{z^2}{6\xi}\frac{b}{A} - \\
&-z\left(\frac{2x}{3(4x+z^2)}\left(-6x + 12y + \frac{z^2}{4\xi A}\left(2b - \frac{yc}{x}\right) + 3(1-x-y-z)(1-3\omega)\right) - 2\right) + z^2, \\
A' &= \frac{z}{6\xi}(b+1) + Az.
\end{aligned} \tag{8}$$

In what follows, we shall investigate the autonomous system (8) for fixed points. We would specially be interested in stable solutions of interest to late time cosmic acceleration.

3 Stationary points and their stability : $B(\phi) = \phi^N$, $V(\phi) = V_0\phi^n$

In this special case, $b = B''(\phi)\phi/B'(\phi) = N-1$ and $c = V'(\phi)\phi/V(\phi) = n$. Our autonomous system was written keeping this simple case in mind.

We shall find stationary points equating to zero the left-hand sides of the system (8). Their stability will be established using the sign of the corresponding eigenvalues which we shall obtain numerically. We begin our discussion from the case, $N \neq 2$ ($b \neq 1$). In the case of $N = 2$, a simple additional relation exists between x and z and we also investigated corresponding system with the same method (obtained solutions in this case coincide with ones found in [15], these results are written down in the Table 1.). In the general case the form of this relation is more involved algebraically and its substitution into our system leads to cumbersome equations. We, therefore, prefer not to make use of it, we would rather check resulting solutions for consistency.

3.1 The case of $b \neq 1 (N \neq 2)$.

After solving the system of algebraic equations obtained after equating the left hand sides of (8) to zero, we find the following stationary points:

1. Vacuum stationary line: $x = 1, y = 0, z = 0, A = -\frac{b+1}{6\xi}, \Omega = 0$
This is a vacuum solution for which the corresponding eigenvalues are given by,

$$\lambda_1 = 3 - 3\omega, \quad \lambda_2 = 6, \quad \lambda_3 = 0, \quad \lambda_4 = 0. \tag{9}$$

As one of these eigenvalues is positive, the stationary line is unstable for any value of ξ and ω.

The time dependence $a(t)$, $\phi(t)$, $\rho(t)$ can be found using the combinations of coordinates of a stationary point. For point 1 we get

$$a(t) = a_0|t-t_0|^{\frac{1}{3}}, \tag{10}$$

$$|\phi(t)|^{\frac{2-N}{2}} = \pm\frac{(2-N)\sqrt{6|\xi|}}{6}\ln|\tfrac{t-t_0}{t'-t_0'}|, \tag{11}$$

where $t_0 = const$, $t' = const$, $t_0' = const$. This solution exists for $0 < N < 2$. We can now write the expression for the quantity G_N using the coordinates of the stationary point, $A = B'(\phi)\phi/(1-6\xi B(\phi)) = N\phi^N/(1-6\xi\phi^N) = N/(\phi^{-N}-6\xi)$, we find that $\phi^N = A/(N+6\xi A)$ leading to $G_N = 6/8\pi(1-6\xi\phi^N) = 6(N+6\xi A)/8\pi N$ which is positive when $\xi > 0, A < -N/6\xi$ or $\xi < 0, A > -N/6\xi$. We note that at all stationary points either $t \to t_0$ or $t \to \infty$. And all functions of time ($a(t)$, $\phi(t)$, $H(t)$, $\rho(t)$) either go to zero or infinity or become constants in this limit.

Since the stationary line reduces to a stationary point for which $A = -N/6\xi$ ($\phi(t) \to \infty$), it follows that $G_N \to 0$. We also note that the vacuum solution does not contain parameters, ω, n and N.

2. $x = 0, y = 0, z = 1, A = -\frac{b+1}{6\xi} = -\frac{N}{6\xi}, \Omega = 0$

The eigenvalues are,

$$\lambda_1 = \frac{b-1}{b+1} = \frac{N-2}{N},\quad \lambda_2 = \frac{5+5b-c}{b+1} = 5-\frac{n}{N},\quad \lambda_3 = 2-3\omega,\quad \lambda_4 = 1. \tag{12}$$

This point is unstable because λ_4 is positive for any ξ, ω. We can compute $a(t)$

$$a(t) = a_0|t-t_0|^{\frac{1}{2}}, \tag{13}$$

$$\phi(t) = \phi_0|t-t_0|^{-\frac{1}{2N}}. \tag{14}$$

Consistency analysis shows that this solution exists for $N > 2$ and $n < 5N$, so it never coexists with the solution discussed above under point **1**. They both correspond to the situation in which the scalar field potential is negligible. Power indexes in this solutions do not depend on ξ, ω, n, however the function $\phi(t)$ contains dependence on N. We also note that since $A = -N/6\xi$ ($\phi(t) \to \infty$ for $t \to t_0$) in the present case, G_N vanishes asymptotically at the stationary point.

3. $x = 0, y = 0, z = -1+3\omega,\quad A = -\frac{b+1}{6\xi} = -\frac{N}{6\xi},\quad \Omega = 2-3\omega$

The corresponding eigenvalues are,

$$\begin{aligned}
\lambda_1 &= \tfrac{(1-b)(1-3\omega)}{b+1},\\
\lambda_2 &= \tfrac{c(1-3\omega)+3(b+1)(1+\omega)}{b+1} < 0, \text{for } c > 3(b+1), \omega_0 < \omega < 1,\\
\lambda_3 &= -2+3\omega < 0,\ \text{for } \omega < \tfrac{2}{3},\\
\lambda_4 &= -1+3\omega < 0,\ \text{for } \omega < \tfrac{1}{3}.
\end{aligned} \tag{15}$$

where $\omega_0 = -\frac{3(b+1)+c}{3(b+1-c)} > \frac{1}{3}$ for $b > 0$, $c > 3(b+1)$.

As regions where λ_2 and λ_4 are negative do not intersect, this point is unstable (either a saddle or a repulsive node).

For obtaining $a(t)$, $\phi(t)$, we note that $Y_{stat} = 0$, $\beta = \frac{1-3\omega}{b+1} = \frac{1-3\omega}{N}$. This tells us that,

$$a(t) = a_0|t-t_0|^{\frac{1}{2}}, \tag{16}$$

$$\phi(t) = \phi_0|t-t_0|^{\frac{1-3\omega}{2N}}. \tag{17}$$

The time dependence of the matter density is obtained

$$\rho(t) = \rho_0 |t - t_0|^{-\frac{3(1+\omega)}{2}} \tag{18}$$

Power indexes of this solution do not contain parameters ξ, n and they depend on ω, N.
It follows from the definition of Ω that it must be positive to ensure that $\rho > 0$ and$G_N > 0$. Since $\Omega = 2 - 3\omega$ in the present case, we should have $\omega < 2/3$ to avoid a pathological situation. We also note that since $A = -N/6\xi$ ($\phi(t) \to \infty$), the effective Newtonian constant G_N vanishes at the stationary point.

4.

$$\begin{aligned} x &= 0, y = \frac{(b+1)(c(1-3\omega)+3(b+1)(\omega+1))}{2c^2}, \\ z &= \frac{3(b+1)(\omega+1)}{c}, A = -\frac{b+1}{6\xi} = -\frac{N}{6\xi}, \\ \Omega &= \frac{2c^2-(b+1)(3(\omega+1)(b+c+1)+4c)}{2c^2} \end{aligned} \tag{19}$$

The corresponding eigenvalues are,

$$\begin{aligned} &\lambda_1 = \tfrac{3(\omega+1)(b-1)}{c}, \\ &\lambda_{2,3} = \tfrac{3(\omega+1)(b+1)^2+3c(\omega-1)(b+1))}{4c(b+1)} \pm \tfrac{\sqrt{(b+1)(f_1(b,c)\omega+f_2(b,c)+9\omega^2 f_3(b,c))}}{4c(b+1)}, \\ &\lambda_4 = \tfrac{3(\omega+1)(1+b)}{c}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} f_1(b,c) &= -210c^2(b+1)+162(b^3+1)+192b^2c+192c+486b(b+1)+384bc+48c^3, \\ f_2(b,c) &= 81(1+b^3)+(243b+17c^2)(1+b)+174c(1+b^2)-16c^3+348bc, \\ f_3(b,c) &= (1+b)(b+1+c)(9(b+1)-7c). \end{aligned}$$

and λ_1, λ_4 are positive for $-1 < \omega \leqslant 1$ therefore this stationary point is unstable for any ξ.

Analogous to previous points we get

$$\begin{aligned} a(t) &= a_0 |t-t_0|^{-\frac{2n}{3(N-n)(\omega+1)}}, \\ \phi(t) &= \phi_0 |t-t_0|^{\frac{2}{(N-n)}}, \\ \rho(t) &= \rho_0 |t-t_0|^{\frac{2n}{N-n}}. \end{aligned} \tag{21}$$

We note that power indices of this solution depend on ω, N, n and do not depend on the coupling constant ξ.

For $b+1 = c$ ($N = n$), the power index of functions $a(t)$ and $\phi(t)$ is infinite and power-law solutions cease to exist. In this case the coordinates of the fixed point 4 are, $x = 0, y = 2, z = 3(\omega+1), A = -\frac{b+1}{6\xi}, \Omega = -(3\omega+4)$. We note that $\Omega < 0$ for $\omega \in [-1, 1]$ and that from the definition of Ω in (6), it follows that either $\rho > 0$, $G_N < 0$ or $\rho < 0$, $G_N > 0$ which doesn't correspond to the real Universe. We find in this case that

$$\begin{aligned} a(t) &= a_0 e^{H_0(t-t_0)}, \\ \phi(t) &= \phi_0 e^{-\frac{3H_0(1+\omega)(t-t_0)}{N}}, \\ \rho(t) &= \rho_0 e^{-3H_0(1+\omega)(t-t_0)}. \end{aligned} \tag{22}$$

As for the constants H_0 and ρ_0, we substitute (22) in definitions (6) of y, Ω taking into account that $N = n$, $y = 2$ and $\Omega = -(3\omega+4)$. We then have for this exponential solution $y \to -\frac{V_0}{3{H_0}^2\xi}$

for $t \to t_0$ when $H_0 > 0$ (or $t \to t_0$ when $H_0 < 0$). Therefore, $H_0{}^2 = -\frac{V_0}{3\xi y} = -\frac{V_0}{6\xi}$, where $\xi < 0$, $V_0 > 0$ or $\xi > 0$, $V_0 < 0$, and $\rho_0 = -3\Omega H_0{}^2 \phi_0{}^N \xi = -\frac{V_0 \phi_0{}^N(3\omega+4)}{2}$.
We next consider the behavior of quantities $T = T_\phi + T_m$ and $R = 6T_{eff} = 8\pi G_N T$ for the case $N = n$. Substituting the so obtained exponential solution, $a(t) = a_0 e^{H_0(t-t_0)}$, $\phi(t) = \phi_0 e^{-\frac{3H_0(1+\omega)(t-t_0)}{N}}$ in (4), we find

$$
\begin{aligned}
&T_{eff} = \tfrac{R}{6} = 2H_0{}^2 = const,\\
&G_N = \frac{6}{8\pi(1-6\xi\phi_0{}^N e^{-3H_0(1+\omega)(t-t_0)})} \propto e^{3H_0(1+\omega)(t-t_0)}\\
&\\
&T = T_\phi + T_m\\
&= e^{-3H_0(1+\omega)(t-t_0)}\left(\phi_0{}^N(4V_0 - 27\xi(1+\omega)H_0{}^2 + 27\xi(1+\omega)^2 H_0{}^2) + \rho_0(1-3\omega)\right) \propto e^{-3H_0(1+\omega)(t-t_0)}
\end{aligned}
\tag{23}
$$

for $t \to t_0$, $H_0 > 0$ (or $t \to \infty$, $H_0 < 0$).

It is therefore clear from the aforesaid that $G_N(t)$ grows as an exponent whereas $T(t)$ decreases with the same rate thereby leading a constant product $G_N T$. A remark about the exponentially expanding solution is in order. The solution though has features similar to de Sitter solution but does not really qualify for a true de Sitter as G_N is not constant in this case. We shall say more about this point in the discussion of the vacuum solution to follow.

5. Vacuum solution

This solution corresponds to the following fixed point,

$$
x = 0,\ \ y = \frac{5+5b-c}{b+1+c},\ z = -\frac{2(2+2b-c)}{b+1+c},\ \ A = -\frac{b+1}{6\xi} = -\frac{N}{6\xi},\ \ \Omega = 0. \tag{24}
$$

The corresponding eigenvalues in this case are given by,

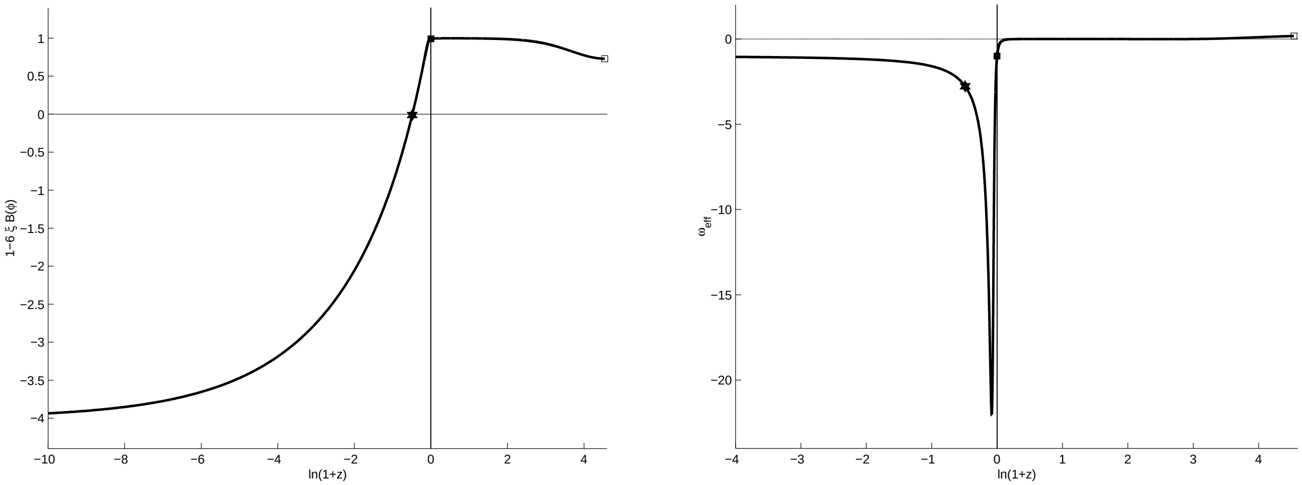
$$
\begin{aligned}
&\lambda_1 = -\tfrac{2((b-1)(2(b+1)-c))}{(b+1)(b+1+c)} < 0,\ \text{for } b > 1 \text{ and } c < 2(b+1),\\
&\lambda_2 = -\tfrac{2(2+2b-c)}{b+1+c} < 0,\ \text{for } c < 2(b+1),\\
&\lambda_3 = -\tfrac{5+5b-c}{b+1} < 0,\ \text{for } c < 5(b+1).\\
&\lambda_4 = -\tfrac{(b+1)(3(b+1)(\omega+1)+c(7+3\omega))-2c^2}{(b+1)(b+1+c)} < 0,\ \text{for } c < 2(b+1) \text{ when } \omega \in [-1;1],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \text{for } c = 2(b+1) \text{ when } \omega \in (-1;1],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \text{for } 2(b+1) < c \leqslant \tfrac{6(b+1)}{\sqrt{35}-5},\ \text{when } \omega \in (\omega_0;1],
\end{aligned}
\tag{25}
$$

where $\omega_0 = \frac{2c^2-7c(b+1)-3(b+1)^2}{3(b+1)(b+1+c)}$. The negativity of eigenvalues for this vacuum point show that it is stable for $c < 2(b+1)$.
We find the following expressions for $a(t)$ $\phi(t)$,

$$
\begin{aligned}
a(t) &= a_0|t-t_0|^{\frac{(N+n)N}{(2N-n)(N-n)}},\\
\phi(t) &= \phi_0|t-t_0|^{\frac{2}{N-n}}
\end{aligned}
\tag{26}
$$

This solution contains parameters N, n but is independent of ξ and ω. We note that power indexes in (26) are negative for $N < n < 2N$ and, therefore, $a(t)$, $\phi(t)$ diverge leading to "Big Rip" singularity at $t = t_0$. This result is generalization for the analogous vacuum solution $a(t) = a_0|t-t_0|^{\frac{2(\xi(2+n)-1)}{\xi(n-2)(n-4)}}$, $\phi(t) = \phi_0|t-t_0|^{\frac{2}{2-n}}$ obtained in the case of $N = 2$ [12], [15].

Let us further investigate the nature of the fixed point. For $b+1 = c$ ($N = n$) and also for $2(b+1) = c$ ($2N = n$) power indexes of functions $a(t)$ and $\phi(t)$ diverge and power-law solutions should transform into exponential ones. Indeed, the coordinates of the fixed point in these cases



(a) **(b)**

Figure 1: The left panel (a) shows the evolution of $(1-6\xi B(\phi))$ versus redshift z, where $G_N = 6/8\pi(1-6\xi B(\phi))$. The right panel (b) shows the evolution of w_{eff} versus redshift z. Both the figures correspond to the case of non-minimal coupling with $N=2$, $n=5$, $\xi=1/2$. We had chosen appropriate initial conditions, $H=2.034, \phi=-0.300, \dot{\phi}=-0.001$ and $\xi=0.5000$ (corresponding to $w_m=0$ initially) to obtain $w_{eff}\simeq -1$ at the present epoch. The black dot in both the figures designates the present epoch which occurs in the regime of $G_N>0$, the effective Newtonian constant changes sign thereafter in future. The square marks the epoch where G_N turns negative.

are given by, $x=0, y=2, z=-1, A=-\frac{N}{6\xi}, \Omega=0$ and $x=0, y=1, z=0, A=-\frac{N}{6\xi}, \Omega=0$ respectively for $N=n$ and $2N=n$. In case of $N=n$, we find that

$$\begin{aligned} a(t) &= a_0 e^{H_0(t-t_0)}, \\ \phi(t) &= \phi_0 e^{\frac{H_0(t-t_0)}{N}}. \end{aligned} \tag{27}$$

We can find out H_0, using the definition of the coordinate y from (6), $y\to -\frac{V_0}{3{H_0}^2\xi}$ and ${H_0}^2 = -\frac{V_0}{3\xi y} = -\frac{V_0}{6\xi}$, where $\xi<0$, $V_0>0$ or $\xi>0$, $V_0<0$.

In case of $n=2N$, the fixed point under consideration does not lead to any physically admissible regime (our numerical work shows that all trajectories in this case lead to oscillations near the minimally coupled field solution). It is interesting to note that in the absence of the standard curvature term in the action $(F(\phi)R=\phi^N R)$, a family of de Sitter solutions exists [20] for arbitrary values of field ϕ.

As for $n=N$, the solution corresponds to $\dot{H}=0$ and $w_{eff}=-1$ taken usually as the definition for de-Sitter solution tacitly assuming that Newtonian gravitational constant is a true constant of nature. In case, ϕ is constant, the constancy of G_N is trivially satisfied. However, in the model under consideration, we have an interesting vacuum solution with $\dot{H}=0$ and an exponentially expanding solution for ϕ which corresponds to an exponentially decreasing/increasing (depending upon the sign of H_0) the effective Newtonian gravitational constant, G_N. The true de-Sitter corresponds to $\dot{H}=0$ and $\phi=const$ implying $G_N=const$. Let us note that in this case, choosing $\phi_0=0$, one might think to obtain de Sitter solution but the latter implies $H_0=0$. Clearly, this solution does not qualify for a genuine de Sitter.

In fact, the true de-Sitter solution is not captured by the autonomous variables, we have used. In this case, our autonomous system is not suitable for the study of de-Sitter solution corresponding to, $x=0$, $y=1$, $z=0$, $A=-\frac{n}{12\xi}$, $\Omega=0$ as the combination, $4x+z^2$ appearing in the denominator of dynamical system vanishes identically. In this special, we go back to the original variables and make use of the system of equations (3), (4), (5). Investigating this system numerically for $N = 2, 3, 4, 5, 6, 7$; $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$ we find stability conditions of de-Sitter solution which are written in the Table 1 .

Table 1. Cosmological solutions and their character of stability for cases $N \neq 2$ and $N = 2$.

<table>
<tr><th>Kind of solution</th><th>$N = 2$</th><th>$N \neq 2$</th></tr>
<tr><td>De Sitter solution
(doesn't exist for $n > 2N$, where n − even.)
$a(t) = a_0 e^{H_0(t-t_0)}$,
$\phi(t) = \phi_0$.</td><td colspan="2">$H_0{}^2 = -\frac{V_0 n \phi_0{}^{n-N}}{6\xi N}$, $\phi_0{}^N = \frac{n}{6\xi(n-2N)}$.
Stable only for
1). $n \geqslant 2N+1$, **where** $N = 2, 4, 6, ...$,
where $N = 2, 4, 6, ...$, $n = 5, 7, 9, ...$
2). $\xi > \xi_0 > 0$,
3). $H_0 > 0$, $\phi_0 < 0$.</td></tr>
<tr><td rowspan="5">Power-low solutions
$a(t) = a_0\vert t-t_0\vert^{\alpha}$,
$\phi(t) = \phi_0\vert t-t_0\vert^{\beta\alpha}$.</td><td rowspan="2">$\alpha_{1,2} = \frac{1}{3-12\xi\pm2\sqrt{6\xi(6\xi-1)}}$
$\beta\alpha_{1,2} = -\frac{6\xi\pm\sqrt{6\xi(6\xi-1)}}{3-12\xi\pm2\sqrt{6\xi(6\xi-1)}}$.
Unstable.</td><td>$\alpha = \frac{1}{2}$, $\beta\alpha = -\frac{1}{2N}$.
Unstable.</td></tr>
<tr><td>$\alpha = \frac{1}{3}$, $\vert\phi(t)\vert^{\frac{2-N}{2}} = \pm\frac{(2-N)\sqrt{6\vert\xi\vert}}{6}\ln\vert\frac{t-t_0}{t'-t'_0}\vert$.
Unstable.</td></tr>
<tr><td>$\alpha = \frac{2(4\xi+\omega-1)}{3\omega^2+16\xi-3}$,
$\beta\alpha = \frac{4\xi(1-3\omega)}{3\omega^2+16\xi-3}$
Unstable.</td><td>$\alpha = \frac{1}{2}$, $\beta\alpha = \frac{1-3\omega}{2N}$.
Unstable.</td></tr>
<tr><td colspan="2">$\alpha = -\frac{2n}{3(N-n)(\omega+1)}$, $\beta\alpha = \frac{2}{N-n}$
Unstable.</td></tr>
<tr><td>$\alpha = \frac{2(\xi(2+n)-1)}{\xi(n-2)(n-4)}$,
$\beta\alpha = \frac{2}{2-n}$
Stability depends on ξ, n.</td><td>$\alpha = \frac{(N+n)N}{(2N-n)(N-n)}$,
$\beta\alpha = \frac{2}{N-n}$.
Stable for $n < 2N$.</td></tr>
<tr><td rowspan="2">Exponential solutions
(**exist only for $N = n$.**)
$a(t) = a_0 e^{H_0(t-t_0)}$,
$\phi(t) = \phi_0 e^{\beta H_0(t-t_0)}$.</td><td>$H_0{}^2 = \frac{V_0(4\xi-1)^2}{3(96\xi^2-34\xi+3)}$
$\beta = \frac{2\xi}{4\xi-1}$
Stability depends on ξ, n.</td><td>$H_0{}^2 = -\frac{V_0}{6\xi}$.
$\beta = \frac{1}{N}$.
Stable.</td></tr>
<tr><td>$H_0{}^2 = -\frac{8V_0}{3(16\xi-3+3\omega^2)}$
$\beta = -\frac{3(1+\omega)}{2}$.
Unstable.</td><td>$H_0{}^2 = -\frac{V_0}{6\xi}$.
$\beta = -\frac{3(1+\omega)}{N}$.
Unstable.</td></tr>
</table>

We also note that for de-Sitter solution the effective Newtonian gravitational constant $G_N = 6/(8\pi(1 - 6\xi B(\phi)) = 3(2N - n)/(8\pi N)$ is positive only if $n < 2N$, which means that graviton is ghost thereby leading to instability. Before approaching the attractor, the system passes through a phantom phase and parameters in the theory can easily be adjusted such that we obtain the observed value of equation of state parameter, $w_{eff} \simeq -1$ at present with $G_N > 0$ followed by a brief phantom phase before approaching the stable de Sitter fixed point ultimately pushing the ghost dominated regime to future (see Fig.1). It is also possible to set the phantom phase at the present epoch. We have carefully managed to shift the ghost regime to future by adjusting the parameters in the model. We should, however, admit that such a model of transient dark energy suffers from ugly fine tuning problem.

4 Conclusion

In this paper, we have revisited cosmological dynamics of non-minimally coupled scalar field system. We presented detailed investigation of dynamics of the underlying system in case of $F(\phi)R = (1-\xi B(\phi))R$ coupling with $B(\phi) = \phi^N$ and $V(\phi) = V_0\phi^n$ ($N \geq 2$) using a convenient set of autonomous variables. We studied asymptotic regimes of solutions in the model. In case of the vacuum solution, we found a very interesting solution for which $\dot{H} = 0$ ($w_{eff} = -1$) and the scalar field ϕ increasing exponentially giving rise to exponentially decreasing G_N − effective Newtonian gravitational constant. Such a solution does not qualify for de Sitter for which G_N should be held constant.
The autonomous variables used in this paper though convenient in general but miss certain important features of the dynamics. The description fails to capture the de Sitter solution as the combination of variables $4x + z^2$ appearing in the denominator of the autonomous system vanishes identically in this case. The investigation of this solution took us back to the original variable in the evolution equations. We found that in case of de Sitter solution, $G_N > 0$ provided that $n < 2N$. On the other hand, our numerical investigations showed that the solution under consideration is stable only for $n \geq 2N + 1$ (we checked for lower values of $N \geq 2$). For initial conditions of matter dominated universe, the system enters the phase of acceleration consistent with observation at present followed by a brief phantom phase thereafter which continues till de Sitter is reached, see Fig.1(b). During the phantom phase G_N changes sign from positive to negative thereby making the universe ghost dominated in future, see Fig.1. It is possible to set parameters in the model such that phantom phase occurs at present epoch corresponding to $G_N > 0$ compilable with observed values of the equation of state and fractional density parameter pushing the ghost dominated phase to future which no body has yet seen. Incidentally, similar features of equation of state are shared by the braneworld model discussed in Ref. [20].

The de Sitter solution for the case with $N = 2$ also shares the aforementioned features. In other cases, we have shown that the solutions obtained earlier are continued for values of $N > 2$. We have investigated in detail the asymptotic regimes of these solutions in all cases including the one corresponding to $N = 2$

We have shown that the non-minimally coupled scalar field system can account for late time cosmic acceleration. We should, however, emphasize that dark energy in this scenario appears as a transient phenomenon which involves extra fine tuning.

A final remark about the ghost dominated evolution in the model is in order. In the framework of the simple set up of non-minimal scheme discussed here, there exists no consistent de Sitter solution such that G_N remains positive throughout the evolution. It would really be interesting to explore generic functional forms of the coupling function and the field potential to check for a well behaved de Sitter solution.

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