PPN Formalism in Higher Order Curvature Gravity. Spherically Symmetric Case

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Abstract

The influence of the dilatonic scalar field on the Parameterized Post-Newtonian expansion of a static, asymptotically flat, spherically-symmetric Gauss-Bonnet solution is considered. We present analytical and numerical expressions for the dilatonic magnitude and demonstrate that at the present time the theoretical limit for "dilatonic charge" obtained 20 years ago remains much more strong than one from existing experimental data.

1 Parameterized post-Newtonian formalism

Parameterized post-Newtonian formalism was constructed for comparing different extended theories of gravity and selecting the most probable ones. First attempt to create such a framework was taken by Eddington in 1922 [1]. Robertson in 1962 [2], Schiff in 1967 [3] and Nordtvedt in 1968 [4, 5] continued to develop this idea. The formalism was completed by Will and Thorn in 1971 [6, 7] and this version is considered to be a standard one nowadays.

For using the Parameterized post-Newtonian the one should stay in within the post-Newtonian limit i.e. the weak field approximation, an asymptotically flat space-time and small velocities of matter so that it would obey the state equation for perfect fluid. As the space-time is considered to be asymptotically flat, the metric tensor can be represented as the perturbative expansion around Minkowski spacetime [8]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

$$h_{00} \sim O(2) + O(4), h_{0j} \sim O(3), h_{ij} \sim O(2), \tag{2}$$

$$U \sim v^2 \sim p/\rho \sim \Pi \sim O(2). \tag{3}$$

The metric can be written down in the following view:

$$\begin{aligned}
g_{00} =& -1 + 2U - 2\beta U^2 - 2\xi \Phi_W + (2\gamma + 2 + \alpha_3 + \varsigma_1 - 2\xi)\ \Phi_1 + \\
&+ 2\ (3\gamma - 2\beta + 1 + \varsigma_2 + \xi)\ \Phi_2 + 2\ (1 + \varsigma_3)\ \Phi_3 + 2\ (3\gamma + 3\varsigma_4 - 2\xi)\ \Phi_4 - \\
&- (\varsigma_1 - 2\xi)\ \mathcal{A} - (\alpha_1 - \alpha_2 - \alpha_3)\ w^2 U + \alpha_2 w^i w^j U_{ij} + (2\alpha_3 - \alpha_1)\ w^i V_i, \\
g_{0i} =& -\frac{1}{2}\ (4\gamma + 3 + \alpha_1 - \alpha_2 + \varsigma_1 - 2\xi)\ V_i - \frac{1}{2}\ (1 + \alpha_2 - \varsigma_1 + 2\xi)\ W_i - \frac{1}{2}\ (\alpha_1 - 2\alpha_2)\ w^i U - \\
&- \alpha_2 w^i U_{ij}, \\
g_{ij} =& \ (1 + 2\gamma U)\ \delta_{ij},
\end{aligned} \tag{4}$$

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where U is the gravitational potential taken with the opposite sign, U_{ij} is its tensor form, w is the coordinate velocity of the PPN frame, Φ_1, Φ_2, Φ_3, Φ_4, Φ_W, $\mathcal{A}$, V_i and W_i are so called post-Newtonian potentials and β, γ, ξ, α_1, α_2, α_3, α_4, ς_1, ς_2 and ς_3 are post-Newtonian parameters that characterize the gravity model. As the matter is considered to be a perfect fluid its stress-energy tensor has the following view [8]:

$$T^{00} = \rho \left(1 + \Pi + v^2 + 2U\right) \tag{5}$$

$$T^{0i} = \rho \left(1 + \Pi + v^2 + 2U + \frac{p}{\rho}\right) v^i \tag{6}$$

$$T^{ij} = \rho \left(1 + \Pi + v^2 + 2U + \frac{p}{\rho}\right) v^i v^j + p\delta^{ij}(1 - 2\gamma U) \tag{7}$$

The post-Newtonian parameters are measured experimentally very well (Table 1). These values generally depend on the model. So it is possible to select between different theories of gravity by comparing their PPN-parameters with the experimental data.

PPN-parameter	**Physical meaning**	**Current value**	**Effects**
$\gamma - 1$	measure of space curvature produced by unit mass	2.3×10^{-5}	time delay, light deflection
$\beta - 1$	measure of non-linearity in gravitational superposition	1.1×10^{-4}	Nordtvedt effect, perihelion shift
ξ	measure of existence of preferred location effects	1×10^{-3}	Earth tides
α_1	measure the existence of preferred frame effects	1×10^{-4}	orbit polarization
α_2		4×10^{-7}	spin precession
α_3		4×10^{-20}	self-acceleration
ς_1	measure of the failure of conservation laws of energy momentum and angular momentum	2×10^{-2}	—
ς_2		4×10^{-5}	binary pulsar acceleration
ς_3		1×10^{-8}	Newton's 3rd law
ς_4		6×10^{-3}	—

Table 1: Experimental values of PPN-parameters.

2 Dilatonic Gauss-Bonnet

The purpose of this work was to research into the PPN formalization of the Gauss-Bonnet model with the scalar field. The corrsponding 4-dimensional low-energy effective string action with second order curvature correction can be written down as following [9, 10]:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[-R + 2\partial_\mu \phi \partial^\mu \phi + e^{-2\phi} S_{GB}\right], \tag{8}$$

where $S_{GB} = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$ is the Gauss-Bonnet term and the dilaton scalar field can be approximated like $\phi = D/r + \dots$.

Spherically symmetrical solutions are not very popular in PPN research. In 1978 Karlhede studied the post-Newtonian formalism Schwarzschild metric [11]. He found obtained a coordinate system in which the PPNmetric is linear that allowed to deal with the solutions in a non-isotropic form.

The solution for the Gauss-Bonnet action with the scalar field was found by Mignemi and Stewart in 1993 [12] and studied in 1997 by Alexeyev and Pomazanov [9]. As a result a solution for spherically symmetrical BH was obtained by the method of successive approximations and it was shown that the dilatonic charge $D \sim 1/M$ for small higher order corrections. In 2007 Sotiriou and Barausse considered the cosmological solution for action with dilaton and Gauss-Bonnet term that seemed to be indistinguishable from general relativity at the post-Newtonian order [13].

3 Results

As the concerned solution should be asymptotically Schwarzschild at the infinity the metric

$$ds^2 = -\left(1 - \frac{a}{r}\right) dt^2 + \frac{dr^2}{(1 - a/r)} + (r^2 - br)\, d\Omega \tag{9}$$

can be written down in the following isotropic form

$$ds^2 = -\left(1 - \frac{a}{\tilde{r}} + \frac{a\,(a+b)}{2\tilde{r}^2}\right) dt^2 + \left(1 + \frac{a}{\tilde{r}}\right) (d\tilde{r}^2 + \tilde{r}^2 d\Omega). \tag{10}$$

If $a = 2M$ and $b = 0$ we obtain the Schwarzschild solution itself. Therefore

$$\begin{aligned} g_{00} &= -\,1 + 2U + \ldots, & (11)\\ g_{ij} &= (1 + 2\gamma U)\delta_{ij}, & (12)\\ U &= M/r \;\rightarrow\; \gamma = 1, & (13) \end{aligned}$$

for all Schwarzschild-like metrics.

For the second and third order by the matter velocity v the solution does not differ from the GR case [8]:

$$\begin{aligned} h_{00} &= 2U, & (14)\\ h_{ij} &= 2U\delta_{ij}, & (15)\\ h_{0i} &= -\frac{7}{2}\, V_i - \frac{1}{2}\, W_i = 0. & (16) \end{aligned}$$

In the forth order the equations of motion have the following view [13]:

$$\begin{aligned} G_{\mu\nu} &= 8\pi \left(T^m_{\mu\nu} + T^\phi_{\mu\nu} + T^{GB}_{\mu\nu}\right), \\ T^\phi_{\mu\nu} &= \frac{1}{8\pi}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\, g_{\mu\nu}\,\partial^\rho\phi\,\partial_\rho\phi\right), \\ T^{GB}_{\mu\nu} &= \frac{1}{16\pi}\Big[2\,(\nabla_\mu\nabla_\nu e^{-2\phi})\, R - 2\, g_{\mu\nu}(\Box e^{-2\phi})\, R - 4\,(\nabla^\rho\nabla_\mu e^{-2\phi})\, R_{\nu\rho} - \\ &\quad - 4\,(\nabla^\rho\nabla_\nu e^{-2\phi})\, R_{\mu\rho} + 4\,(\Box e^{-2\phi})\, R_{\mu\nu} + 4\, g_{\mu\nu}(\nabla^\rho\nabla^\sigma e^{-2\phi})\, R_{\rho\sigma} - \\ &\quad - 4\,(\nabla^\rho\nabla^\sigma e^{-2\phi})\, R_{\mu\rho\nu\sigma}\Big] \end{aligned} \tag{17}$$

and $T^m_{\mu\nu}$ is the matter stress-energy tensor. Therefore

$$R_{00} = -\,\frac{1}{2}\,\nabla^2(h_{00} + 2U^2 - 8\Phi_2) = 4\pi\,(\rho + 2v^2\rho - 2U\rho + \Pi\rho + 3p) - \frac{5}{8}\,\frac{g_{00}}{g_{11}}\left(\frac{\partial\phi}{\partial r}\right)^2 \tag{18}$$

and the 4th order correction for the metric tensor is

$$
\begin{aligned}
h_{00} &= 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 - \frac{5}{2}\frac{\pi D^2}{r^2} = \\
&= 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\left(1 - \frac{5}{4}\frac{\pi D^2}{r^2\ \Phi_3}\right)\Phi_3 + 6\Phi_4
\end{aligned}
\tag{19}
$$

The PPN potential Φ_3 can be evaluated as

$$
\Phi_3 = \int \frac{\rho' \Pi'}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \sim -\frac{M^2}{2r^2}. \tag{20}
$$

So it is possible to find the influence of the scalar field on the parameter ς_3:

$$
\varsigma_3 = \frac{5\pi D^2}{2M^2}. \tag{21}
$$

Consequently we can find the experimental limit on the value of dilatonic charge from the PPn data:

$$
\varsigma_3 = 10^{-8} \;\Rightarrow\; D \leq 3.6 \times 10^{-5} M \tag{22}
$$

and compare it with the theoretical limit $D \sim 1/M$ found by Mignemi [12].

4 Conclusions

In this work a spherically symmetric case for dilaton-Gauss-Bonnet gravity is considered. Influence of the dilaton term was found in explicit form and the upper limit on magnitude of dilatonic charge was obtained from the experimental PPN data. It is obvious that the theoretical limits on contributions of higher order curvature corrections and scalar field are still much more strict than the experimental ones.

References

[1] A. S. Eddington *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1922).

[2] H. P. Robertson *Space Age Astronomy* 228 (Academic Press, New York, 1962).

[3] L. I. Schiff *Lectures in Applied Mathematics* 105 (American Mathematical Society, Providence, R.I., (1967).

[4] K. Jr. Nordtvedt, *Phys. Rev.* **169** 1014 (1968).

[5] K. Jr. Nordtvedt, *Phys. Rev.* **169** 1017 (1968).

[6] K. S. Thorn and C. M. Will, *Astrophys. J.* **163** 595 (1971).

[7] C. M. Will, *Astrophys. J.* **163** 611 (1971).

[8] C. M. Will *Theory and experiment in gravitational physics* (Cambridge University Press, Cambridge, 1981).

[9] S. O. Alexeyev and M. V. Pomazanov *Phys. Rev.* **D 55** 2110 (1997).

[10] S. O. Alexeyev, A. Barrau and K. A. Rannu *Phys. Rev.* **D 79** 067503 (2009).

[11] A. Karlhede, USIP Report **78**, 13 (1978).

[12] S. Mignemi and N. R. Stewart, *Phys. Rev.* **D 47**, 5259 (1993).

[13] T. Sotiriou and E. Barausse, *Phys. Rev.* **D 75**, 084007 (2007).