Anisotropic Deformations and Phase Transition of Neutron Star Matter.

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Abstract

The Skyrme model has been shown to be a low energy, effective field theory for QCD. When coupled to a gravitational field it can be used to describe neutron stars. The solution of the Skyrme model with the lowest energy per baryon is the lattice of α-like particles. We use such a lattice as a building block to construct low energy neutron star configurations, allowing the crystal to be strained anisotropically. Below 1.49 solar masses the stars' crystal deforms isotropically and above this critical mass, it undergoes anisotropic strain. We show that the maximum mass allowed for a neutron star is 1.90 solar masses, in close agreement with the most massive neutron star found so far. The computed solutions have radii that match the experimentally estimated values of approximately 10km.

1 Introduction

Neutron stars are stars that due to the intense self gravitational pressure have collapsed to the point where all electrons are squeezed into nuclei. The results is a large cluster of neutrons with a typical radius of about 10km. Their mass has been measured to be of up to twice the mass of the sun and they are also know as pulsar: neutron stars spin at a frequency varying between 0.1 and 1000Hz. By a process not yet fully understood, this leads to the emission intense radiation in a narrow cone at an angle with the axis or rotation of the star. When the line of sight between the star and the earth crosses that cone, the emitted radiation reaches the earth. The star is then see as emitting burst of light with a period equal to the period of rotation of the star.

A neutron star can be viewed as a gigantic nuclei that is electrically neutral but is strongly affected by the gravitational field that it generates. This system should be described by a unified theory of QCD and General Relativity, but for a lack of such theory, one can try to use the Skyrme model coupled to a gravitational field theory.

Originally proposed by Skyrme in 1961 [1], [2] the Skyrme model is a nonlinear theory of pions describing strong interactions. Later Witten[3] showed it to be an approximate, low energy, effective field theory for QCD in the limit of large number of quark colours.

Each solution of the Skyrme model is characterised by an integer valued topological charge which can be identified with the baryon number B. The simplest solution has $B = 1$ and it is made out of a so-called Skyrmion which is interpreted as a proton or a neutron. At the semi-classical level, the Skyrme model does not distinguish between a neutron and a proton and as the model does not include the electroweak interaction, all Skyrmions are electrically neutral.

The $B = 1$ solution of the Skyrme model is the only exact stable solution that can be computed easily [1]. Solutions with larger baryon number have been computed numerically

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[4],[5]. These solutions have been shown to describe various nuclei successfully [6]. One of these solution, $B = 4$, correspond to an α particle and, as shown in figure 1.a, it has a cubic symmetry.

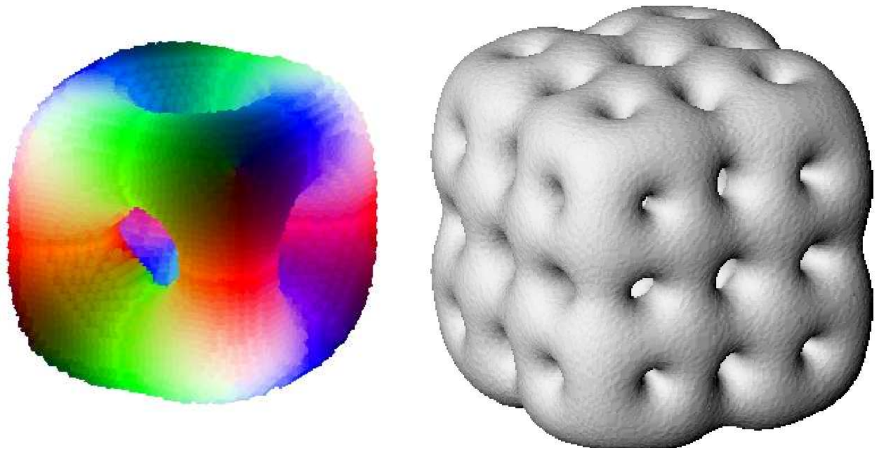

Figure 1: *Surface of constant energy for Skyrmion solutions. a)* $B = 4$ *b)* $B = 32$.

This cubic solution can then be arranged to form solutions with a larger topological charge (figure 1.b). One can then takes an infinite number of α-like solutions to form a cubic lattice and this solutions has been shown[7] to be the solution with the lowest energy per baryon. Being the lowest energy per baryon solution, this lattice looks as the best building block to describe a neutron star.

Before we proceed, we must first estimate if a neutron star should be considered as fluid of Skyrmion rather than a solid. The temperature of a neutron star, a few years after its creation, cools down to a temperature of around 100eV $\approx 10^6 K$ [8]. While this energy is very high, compared to the binding energy of an electron around a nucleus, it is small from a nuclear point of view. The lowest excited state of an α particle, for example, is 23.3MeV [9]. From a more theoretical point of view, the lowest vibration mode of a $B = 4$ Skyrmion is of the order of 100MeV [10],[11]. Moreover, Walhout [12] showed that even under intense gravitational energy, the excitation energy of a lattice of $B = 1$ Skyrmions is also of the order of 100MeV.

This all supports the view that neutron star must be considered as a solid phase rather than a liquid or a gas and that the thermal energy will only excite acoustic phonon modes. As a result, it is sensible to model a neutron star as a lattice of $B = 4$ Skyrmions.

Before we proceed we must also estimate the height of it atmosphere At the surface of a neutron star twice the mass of the sun, the gravitational acceleration is $g \approx 2.6 \times 10^{12}\mathrm{ms}^{-2}$. As a result, the average height that an α particle with a thermal energy of 100eV will be able to jump is of the order of $1mm$, in other words much smaller than the radius of the star. We thus see that the neutron star atmosphere is extremely thin and consider the star as a solid ball in our model.

To model the neutron star as a Skyrmion star, we will proceed as follows. We will start from the the equations of state computed by Castillejo *et al.* [7] for the $B = 4$ crystal when the lattice is deformed asymmetrically. Then, following Walhout [12], we will use a Tolman-Oppenheimer-Volkoff (TOV) equation [13], [14], generalising it to anisotropic matter [15]. The TOV equation describes the static equilibrium between the gravitational forces self-generated by the star matter and the matter forces within the spherically symmetric star.

We will then combine the equations of state of the Skyrme crystal with the TOV equation to find configurations that are spherically symmetric distributions of anisotropically deformed matter in static equilibrium. These will correspond to low energy configurations for neutron stars. We will solve these equations numerically for large stars and show that below a critical mass of 1.49 solar masses ($M_\odot = 1.98892 \times 10^{30}$kg) all neutron/Skyrmion stars are made out of an isotropically strained crystal. Then we will show that at this critical mass, there is a phase transition and that heavier stars are made out of an anisotropically deformed crystal that is less strained radially than tangentially. Moreover, we will show that these stars can have a mass of up to $1.90 M_\odot$. We will also investigate the impact of adding a mass term to the Skyrme model and describe what happens to a star when its mass is increased above its maximum value.

Using Skyrmions to model neutron stars is not new and has been performed previously in several ways[12][16][17] where the star was considered as a fluid. Our model differ in that we consider the star as a solid crystal allowed to deform anisotropically, *i.e.* be compressed differently in the radial and tangential directions. In our previous papers [18],[19], we computed minimal energy Skyrmion stars made out of layers of 2 dimensional Skyrme lattices using the rational map ansatz [20]. This resulted in relatively small stars with a maximum mass of $0.574 M_\odot$ mainly because the multi-layer ansatz that we used was energetically costly.

2 Skyrme Crystals

The Skyrme model [1], [2] is described by the Lagrangian

$$\mathcal{L}_{Sk} = \frac{F_\pi^2}{16}\mathrm{Tr}(\nabla_\mu U \nabla^\mu U^{-1}) + \frac{1}{32e^2}\mathrm{Tr}[(\nabla_\mu U)U^{-1}, (\nabla_\nu U)U^{-1}]^2 + \frac{m_\pi^2 F_\pi^2}{8}\mathrm{Tr}(U-1), \qquad (1)$$

where here U, the Skyrme field, is an $SU(2)$ matrix and F_π, e and m_π are the pion decay constant, the Skyrme coupling and the pion mass term respectively. In the Lagrangian (1) the ∇ are ordinary partial derivatives in the absence of a gravitational field and become covarient derivatives when the Skyrme field is coupled to gravity.

Being finite energy maps from $\mathbb{R}^3$ to S^3, the Skyrme solutions can be defined as maps from S^3 to S^3 and be characterised by the topological charge B identified with the baryon number. In what follows, we will consider massless pion, $m_\pi = 0$, but we will describe the effects of massive pions in section 4.3.

The two other Skyrme parameters, F_π and e can be obtained in different ways in the absence of gravitational fields. Skyrme first evaluated them by taking the experimental value of the pion decay constant $F_\pi = 186$MeV and then fitting the mass of a Skyrmion to that of a proton and obtained $e = 4.84$. Later Adkins, Nappi and Witten [21] quantised the $B = 1$ Skyrmion to fit the parameter values to the mass of the nucleon and the delta excitation and obtained $F_\pi = 129$MeV and $e = 5.45$. These later values were the ones used by Castillejo *et al.* [7] to compute the energy of the deformed $B = 4$ crystal and we will thus use them too.

Castillejo *et al.* [7] also computed the energy of dense Skyrmion crystals when the face-centred cubic lattice was strained. Two types of deformation where considered: pure compression or dilations $x \to \sigma x$, $y \to \sigma y$, $z \to \sigma z$ and volume preserving strain along one direction: $x \to rx$, $y \to ry$, $z \to z/r^2$. The parameter $p = r - 1/r$ describes the deviation away from the face-centred cubic lattice symmetries which have $p = 0$.

The numerical solutions found in [7] provide an equation for the dependence of the energy of a single Skyrmion, $E(L, p)$, on its size, $L = n^{-1/3} = \sigma^3 L_0$, where n is the Skyrmion number density and L_0 the lattice size of the unstrained lattice:

$$E(L,p) = E_{p=0}(L) + E_0[\alpha(L)p^2 + \beta(L)p^3 + \gamma(L)p^4 + \delta(L)p^5 + ...], \qquad (2)$$

where the coefficients are given by

$$E_{p=0}(L) = E_0\left[0.474\left(\frac{L}{L_0}+\frac{L_0}{L}\right)+0.0515\right], \quad (3)$$

$$\alpha(L) = 0.649 - 0.487\frac{L}{L_0} + 0.089\frac{L_0}{L}, \quad (4)$$

$$\beta(L) = 0.300 + 0.006\frac{L}{L_0} - 0.119\frac{L_0}{L}, \quad (5)$$

$$\gamma(L) = -1.64 + 0.78\frac{L}{L_0} + 0.71\frac{L_0}{L}, \quad (6)$$

$$\delta(L) = 0.53 - 0.55\frac{L}{L_0}. \quad (7)$$

Here $E_0 = 727.4$MeV and $L_0 = 1.666\times10^{-15}$m. Notice that for any value of L the minimum energy occurs at the face-centred cubic lattice configuration, $p = 0$, and the global minimum is reached for $L = L_0$.

3 TOV Equation for Skyrmion Stars

Using equation (2) relating the energy of a Skyrmion to its size and aspect ratio we can investigate how a neutron star can be described using a Skyrme crystal and how this crystal is deformed under the high gravitational field it experiences.

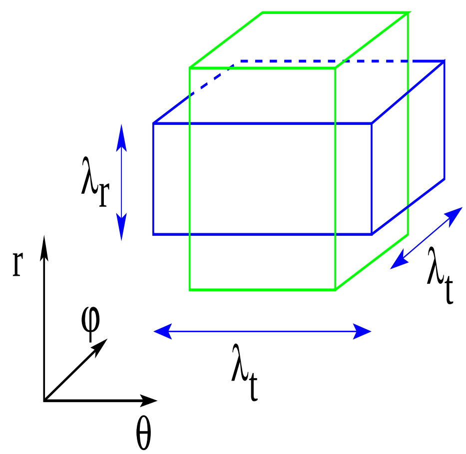

Figure 2: *a) Lattice cell parametrisation.*

In what follows, we denote λ_r as the Skyrmion length in the radial direction of the star and λ_t as the Skyrmion length in the tangential direction (figure 2). These parameters and the parameters L and p used in (2) are related as follows

$$L = (\lambda_r\lambda_t\lambda_t)^{\frac{1}{3}}, \text{ and } p = \left(\frac{\lambda_t}{\lambda_r}\right)^{\frac{1}{3}} - \left(\frac{\lambda_r}{\lambda_t}\right)^{\frac{1}{3}}. \quad (8)$$

To construct a neutron star we consider a spherically symmetric distribution of matter in static equilibrium with a stress tensor that is allowed to be locally anisotropic. As a result of the spherical symmetry the stress tensor, T^μ_ν, is diagonal and all the components are functions of the radial coordinate r. We can thus write this stress tensor as

$$T^\mu_\nu = \mathrm{diag}(\rho(r), p_r(r), p_\theta(r), p_\phi(r)), \quad (9)$$

and consider that, again due to spherical symmetry, $p_\theta(r) = p_\phi(r)$. We can thus write $p_t(r) = p_\theta(r) = p_\phi(r)$ where $p_r(r)$ and $p_t(r)$ describe the stresses in the radial and tangential directions respectively. Moreover we define $\rho(r)$ as the mass density.

We now use the generalised TOV equation [13], [14] derived by Bowers and Liang [15] to describe a spherically symmetric star composed of anisotropically deformed matter in static equilibrium.

One starts from the metric for the static spherically symmetric distribution of matter written in Schwarzschild coordinates

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{10}$$

where $e^{\nu(r)}$ and $e^{\lambda(r)}$ are functions of the radial coordinate that need to be determined. One then must impose that the combination of this metric and the matter distribution, described by the stress tensor (9), satisfies Einstein's equations

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}, \tag{11}$$

where we have set $G = c = 1$. This leads to the following equations

$$\begin{aligned} e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} &= 8\pi\rho & (12)\\ e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} &= 8\pi p_r & (13)\\ e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\left(\nu'\right)^2 + \frac{(\nu' - \lambda')}{2r}\right) &= 8\pi p_t\ . & (14) \end{aligned}$$

We can rewrite equation (12)

$$(re^{-\lambda})' = 1 - 8\pi\rho r^2 \tag{15}$$

and if we integrate it we get

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{16}$$

where $m = m(r)$ is defined as the gravitational mass contained within the radius r

$$m = \int_0^r 4\pi r^2 \rho dr. \tag{17}$$

Substituting equation (16) for $e^{-\lambda}$ into equation (13) we find

$$\frac{1}{2}\nu' = \frac{m + 4\pi r^3 p_r}{r(r - 2m)}. \tag{18}$$

Differentiating equation (13) with respect to r and adding it to equation (14) we get the generalised TOV equation that we will use to find suitable neutron star configurations

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{\nu'}{2} + \frac{2}{r}(p_t - p_r)\ . \tag{19}$$

Now, substituting (18) into (19), we get

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{m + 4\pi r^3 p_r}{r(r - 2m)} + \frac{2}{r}(p_t - p_r)\ . \tag{20}$$

To solve this generalised TOV equation two equations of state need to be specified, $p_r = p_r(\rho)$ and $p_t = p_t(\rho)$, where, as argued above, we are allowed to assume that the temperature of the start is 0.

We must then specify the following boundary conditions: First, the solution must be regular at the origin and so $m(r) \to 0$ as $r \to 0$. Then p_r must be finite at the centre of the star implying that $\nu' \to 0$ as $r \to 0$. Moreover, the gradient dp_r/dr must be finite at the origin too and so $(p_t - p_r)$ must vanish at least as rapidly as r when $r \to 0$. This implies that we need to impose the boundary condition $p_t = p_r$ at the centre of the star.

As the radial stress for the Skyrmions on the surface of the star will be negligibly small the radius of the star, R, can be determined by the condition $p_r(R) = 0$. The equations, however, do not impose that $p_t(R)$ vanishes at the surface. We must also impose the condition that to be physically relevant solutions must all have $p_r, p_t \geq 0$ for $r \leq R$. Moreover, as an exterior vacuum Schwarzschild metric can always be matched with our metric, the star is allowed to have a sharp edge at its surface, as one would expect from a solid star.

As the star as a zero temperature the equations of state for the neutron star configurations can be calculated from equation (2) which depends on the lattice scale L, and aspect ratio, p, which are both functions of the radial distance form the centre of the star, r. Using the theory of elasticity, we find that the radial and the tangential stresses are related to the energy per Skryrmion, Eq (2), as follows

$$p_r = -\frac{1}{\lambda_t^2}\frac{\partial E}{\partial \lambda_r}, \text{ and } p_t = -\frac{1}{\lambda_r}\frac{\partial E}{\partial \lambda_t^2}. \tag{21}$$

The mass of the star is given by

$$M_G = m(R) = m(\infty) = \int_0^R 4\pi r^2 \rho dr, \tag{22}$$

where R is the total radius of the star and

$$\rho = \frac{E}{\lambda_r \lambda_t^2 c^2}. \tag{23}$$

One can now minimise M_G for stars of various baryon number using the generalised TOV equation (20) and the two equations of state (21).

As the mass of the star is a function of λ_r and λ_t which both depend on r, we can minimise M_G by assuming a profile for $\lambda_t(r)$ and compute M_G for this profile as described below. We can then determine the configuration of the neutron star, with a specific baryon charge, by minimising M_G over the field λ_t. This can be easily done using a simulated annealing algorithm.

Notice that at the origin, one can use (21) to determine $p_r(0)$ and $p_t(0)$ from the initial values of $\lambda_r(0)$ and $\lambda_t(0)$. Using this, one can proceed as follows. Knowing $\lambda_r(r)$ and $\lambda_t(r)$ one computes $\rho(r)$ using (23) and $m(r)$ using (17). Then, knowing $p_r(r)$, $p_t(r)$, $\rho(r)$ and $m(r)$ one integrates (20) by one step to determine $p_r(r + dr)$. One then uses (21) to determine $\lambda_r(r + dr)$ and as the profile for $\lambda_t(r)$ is fixed, one proceeds with the next integration step.

One then integrates (20) up to the radius R for which $p_r(R) = 0$; this sets the radius of the star. In our integration, we used a radial step of 50m.

Finally, one must evaluate the total baryon charge of the star using

$$B = \int_0^R \frac{4\pi r^2 n(r)}{(1 - \frac{2Gm}{c^2 r})^{1/2}} dr, \tag{24}$$

where

$$n(r) = \frac{1}{\lambda_r(r)\lambda_t(r)^2} \tag{25}$$

and rescale λ_t to restore the baryon number to the desired value. One then repeats the integration procedure until the baryon charge reaches the correct value without needing any rescaling.

4 Results

4.1 Stars Made of Isotropically Deformed Skyrme Crystal

After computing the minimum energy configuration of a Skyrmion star for a large range of baryon number we found that below the critical value $B_c = 2.61 \times 10^{57}$, equivalent to $1.49 M_\odot$, all the stars are made out of crystals that are isotropically deformed, *i.e* $\lambda_t(r) = \lambda_r(r)$, across the whole radius of the star. To confirm this result, we have solved the TOV equations for isotropic starts and found that no such solution exits above B_c.

It can be shown that this indeed has to be the case as one can prove that if it is possible to find an isotropic Skyrme crystal solution then that solution will be the minimum energy configuration[22].

The proof however does not rule out the existence of anisotropic Skyrme crystal solutions with baryon numbers for which there does not exist an isotropic Skyrme crystal solution. We will discuss such configurations in the next section.

We have also computed the quantity

$$S(r) = e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}, \tag{26}$$

which appears in the static, spherically symmetric metric (10). Its root correspond to singularities in the metric, or in other words, to horizons. We have thus computed its minimum value S_{min} and observed that even for the largest isotropically deformed star $S_{min} = 0.5578$ and so all the isotropically deformed stars are far from an horizon.

The neutron star solutions which have masses larger than the mass of the Sun have radii of about 10km, which very much matches the experimental estimates of the radii of observed neutrons stars. Notice also that the largest neutron star, in our model, has a mass of approximately $1.28 M_\odot$, and above that value, the radius of the stars decreases with their mass.

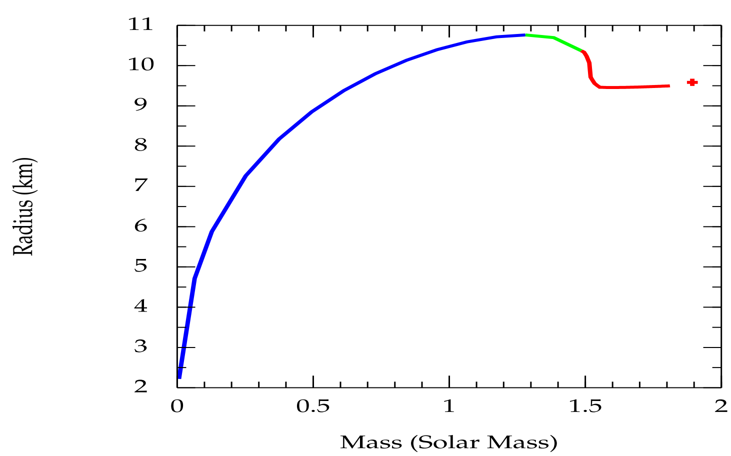

Figure 3: *Radius of the neutron star solutions as a function of their mass (solid line), and that of the maximum mass solution (cross).*

4.2 Stars Made of Anisotropically Deformed Skyrme Crystal

The critical Baryon mass B_c correspond to a phase transition between isotropically deformed star crystals and anisotropically deformed ones. We were able to compute such minimising configurations for starts up to $B = 3.25 \times 10^{57}$, corresponding to $1.81 M_\odot$. Above that value our numerical energy minimisation procedure became difficult to implement but we were able to compute anisotropic Skyrme crystal solutions up to a baryon number of 3.41×10^{57}, equivalent to $1.90 M_\odot$ by maximising the baryon number. This maximum baryon number solution is unique as any modification to it results in a decrease in the baryon number, hence it is the minimum energy solution. Above this baryon number, solutions do not exist.

We found that the anisotropically deformed crystal configurations are energetically favourable as the energy per baryon decreases as the total baryon number increases, indicating stable solutions.

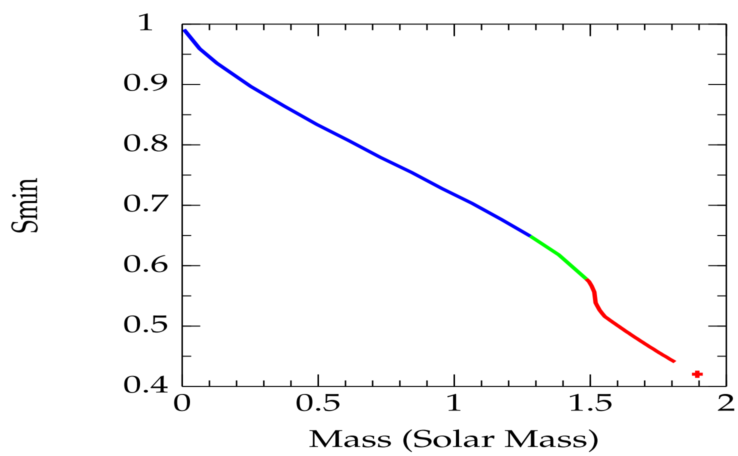

Figure 4: *S_{min} of the neutron star solutions as a function of their mass. The maximum mass solution is shown as a cross.*

Figure 3 shows a plot of the star radius as a function of its mass in units of $M_\odot$ for all the stars that we found, isotropic and anisotropic. For small stars, the radius increases with their mass until the maximum of $10.8km$ at $1.28M_\odot$. The radius then decreases up to the critical mass $1.49M_\odot$ above which star crystals becomes anisotropically deformed. The radius then decreases further to reach a plateau value of $9.5km$ just over $1.5M_\odot$.

We can see that the configurations we have constructed do not collapse into a black hole by noticing that the values of S_{min} are always positive, and monotonically decreasing, as shown in figure 4. Despite a sharp decrease just over $1.5M_\odot$, *i.e.* just above the critical mass, S_{min} always remains positive, indicating that no black hole is formed.

In all previous work[12],[16], [17] on Skyrmion star, it has always been assumed that neutron star matter was a fluid and thus deformed isotropically and a maximum mass was then derived using that assumption. We have argued that anisotropic deformations must be considered as well and doing so one finds that the mass range over which solutions can be found increases by about 28% above the maximum mass found for the isotropic case.

The maximum star mass that we obtained for our Skyrme crystal model is approximately $1.90M_\odot$ and the recent discovery of a $1.97 \pm 0.04\, M_\odot$ neutron star [23], the highest neutron star mass ever determined, makes this an encouraging finding, especially when we consider that

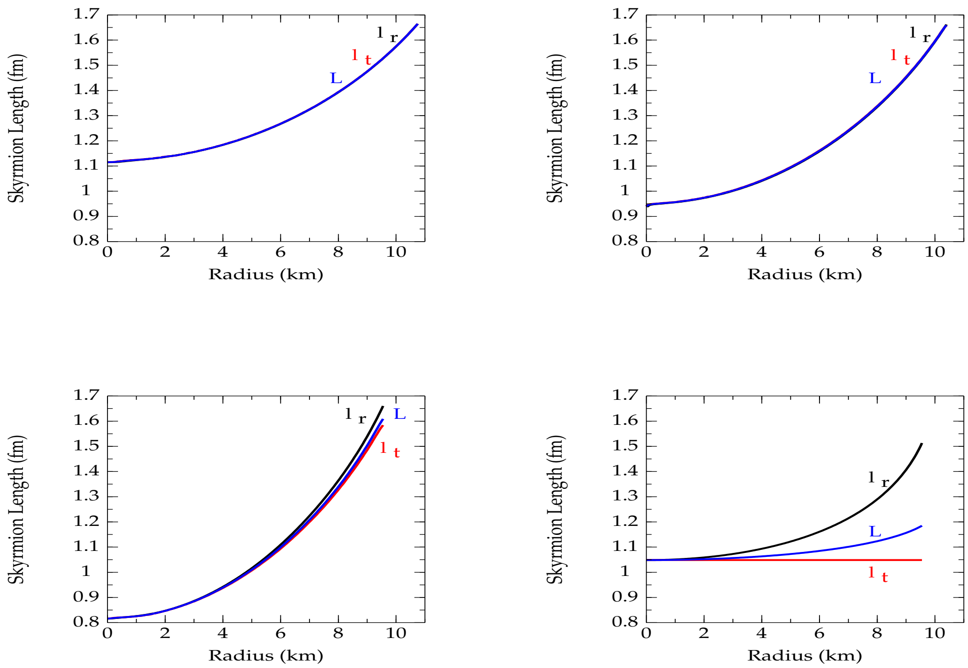

Figure 5: Skyrmion lengths $\lambda_r(r)$ (solid line), $\lambda_t(r)$ (dashed line) and $L(r)$ (dotted line) for a) Largest neutron star ($R = 10.8$km): $M = 1.28 M_\odot$ b) Heaviest isotropic neutron star: $M = 1.49 M_\odot$ (all lengths coincide as they are made of isotropically deformed crystal); c) Densest neutron star: $M = 1.54 M_\odot$; d) Heaviest neutron star: $M = 1.90 M_\odot$.

including the effects of rotation into our model will increase the maximum mass found, by up to 2% for a star with a typical 3.15ms spin period [24].

Figure 5 shows a selection of plots of the Skyrmion lengths λ_r and λ_t and the Skyrmion size L, equation (8), over the radius of the star for four special stars: the largest star, with radius $R = 10.8$km and mass $M = 1.28M_\odot$ (figure 4.2); the heaviest isotropically deformed star $M = 1.49M_\odot$ (figure 4.2); the densest neutron star, $M = 1.54M_\odot$ (figure 4.2) and the heaviest neutron star, $M = 1.90M_\odot$ (figure 4.2). The first two figures correspond to isotropically deformed crystals, while the last two correspond to anisotropically deformed ones. One notices that the amount of anisotropy increases as the mass increases (the divergence between λ_r and λ_t increases). In what follows we will use these four special stars as examples to illustrate various properties of the neutron stars.

As the maximum mass is approached the gradient of the profile of tangential Skyrmion lengths over the radius of the star becomes smaller as the mass increases. Moreover physically meaningful stars composed of anisotropically deformed crystal must have $d\lambda_t/dr \geq 0$ [25] confirming that the minimum energy solution for the maximum mass found, $1.90M_\odot$, for anisotropic Skyrme crystal solutions is the configuration with a constant tangential Skyrmion length as illustrated in figure 4.2.

As shown above, the generalised TOV equation imposes that the sizes of the Skyrmions are equal in all directions at the centre of the star, but away from the centre we find that the amount of Skyrmion anisotropy increases as we move towards the edge of the star, reaching the maximum at the edge. We thus observe that the Skyrmions are more deformed in the tangential direction in agreement with the value of the aspect ratio, p, being negative over the values where $\lambda_r \neq \lambda_t$.

As expected, the profiles for λ_r and λ_t show that the mass density at the centre of the star is higher than at the edge, decreasing monotonically as the radial distance increases. This is shown by figure 6 for the largest, heaviest isotropic, densest and maximum mass solutions.

In figure 7 one can see how the lengths of the Skyrme crystal λ_r and λ_t vary with the mass of the star both at the centre ($r = 0$) and the edge of the star ($r = R$). For isotropically deformed stars, $\lambda_r(R) = \lambda_t(R)$ is constant and corresponds to the minimum energy Skyrme crystal in the absence of gravity. Not surprisingly, $\lambda_r(0) = \lambda_t(0)$ decreases steadily as the mass of the star increases, showing that the density at the centre of the star increases. Once the phase transition has taken place and the star is too heavy to remain isotropically deformed, we observe that $\lambda_r(0) = \lambda_t(0)$ drops sharply to a local minimum, reached for $M \approx 1.54M_\odot$. Meanwhile, $\lambda_r(R)$ and $\lambda_t(R)$ remain nearly identical. Beyond the minimum of $\lambda_{r,t}(0)$, $\lambda_r(R)$ and $\lambda_t(R)$ start to diverge sharply; $\lambda_r(R)$ decreases slightly in value while $\lambda_t(R)$ decreases rapidly. These stars are thus much more compressed in the tangential direction than in the radial one, as seen on figure 4.2, $\lambda_t(R) = \lambda_t(0)$ for the maximum mass neutron star.

The expression for the energy of the Skyrme crystal eq. (2) allows us to compute the speed of sound in the star along the radial direction (z in (2))

$$v_r = \left(\frac{dp_r}{d\lambda_r} \left(\frac{d\rho}{d\lambda_r} \right)^{-1} \right)^{1/2}. \tag{27}$$

First of all it is interesting to observe that v_r is amazingly large in the pure Skyrme crystal in the absence of any gravitational field: $v = 0.57\,c$. This is the speed of sound at the surface of a neutron star when it is deformed isotropically. One sees From figure 8 that v_r increases as one moves towards the centre of the star. Moreover, as v_r is directly related to the density of the star, it is not surprising to find that the maximum radial speed obtained, $v_r = 0.78c$, is reached at the centre of the densest neutron star, *i.e.* the one with $M = 1.54M_\odot$. Notice also that, as one expects it, $v_r < c$ everywhere.

We would like to point out that the minimum value of the aspect ratio, p, for the minimum

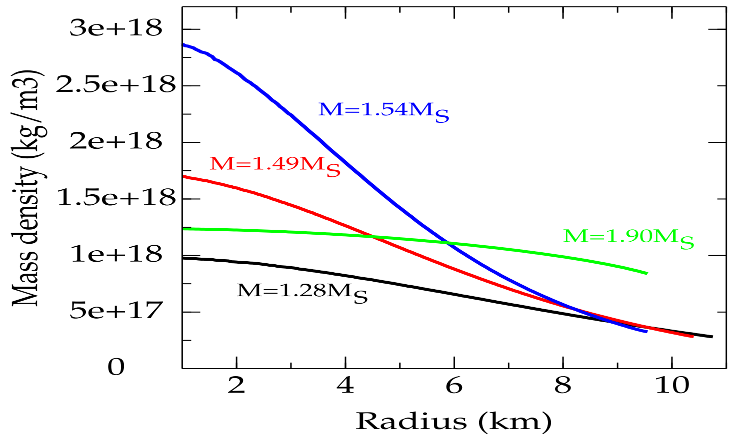

Figure 6: Mass density $\rho(r)$ for: a) Largest neutron star ($R = 10.8$km): $M = 1.28 M_\odot$ (solid line) b) Heaviest isotropic neutron star: $M = 1.49 M_\odot$ (dashed line); c) Densest neutron star: $M = 1.54 M_\odot$ (dotted line); d) Heaviest neutron star: $M = 1.90 M_\odot$ (dash dotted line).

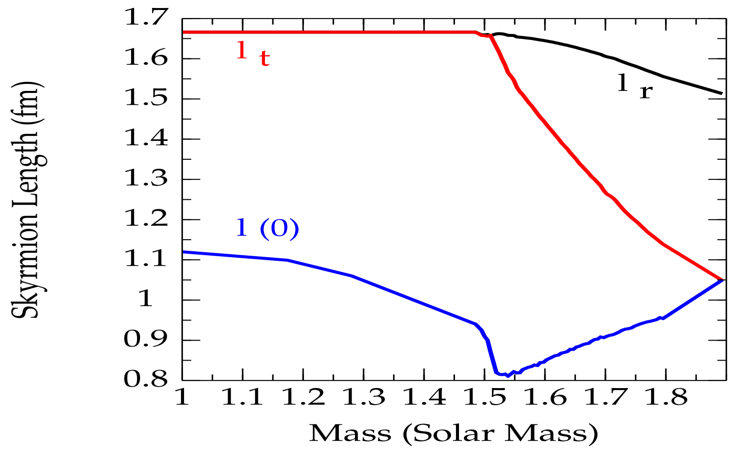

Figure 7: *Skyrmion lengths at the edge of the star,* $\lambda_r(R)$ *(solid line) and* $\lambda_t(R)$ *(dashed line), and at the centre of the star,* $\lambda_r(0) = \lambda_t(0)$ *(dotted line), as a function of the star mass.*

energy configurations found is -0.283 and the minimum value of L is 8.11×10^{-16}, both of which are within the valid range of values for equation (2) [7].

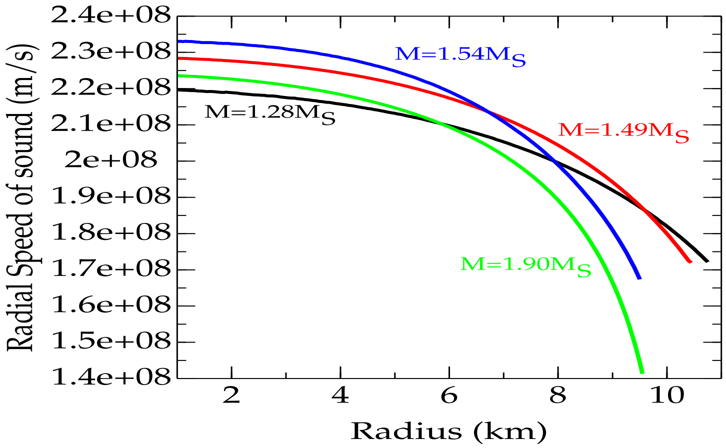

Figure 8: Radial speed of sound, $v_r(r)$ for a) Largest neutron star ($R = 10.8$km): $M = 1.28M_\odot$ (solid line) b) Heaviest isotropic neutron star: $M = 1.49M_\odot$ (dashed line); c) Densest neutron star: $M = 1.54M_\odot$ (dotted line); d) Heaviest neutron star: $M = 1.90M_\odot$ (dash dotted line).

4.3 Inclusion of the Pion Mass

So far we have assumed that the pion mass in the model was zero. The inclusion of a non-zero pion mass can be considered by including the pion mass term,

$$\int \frac{m_\pi^2 F_\pi^2}{8} \mathrm{Tr}(U-1) d^3x, \tag{28}$$

in the static Skyrme Lagrangian (1), where U is the Skyrme field, F_π is the pion decay constant and m_π is the pion mass. Using the cubic lattice of α-like Skyrmions considered above one finds that $\mathrm{Tr}(U-1) = -2$, meaning that the energy E_π arising from the pion mass term reduces to

$$E_\pi = \frac{1}{4} m_\pi^2 F_\pi^2 L^3, \tag{29}$$

an energy term proportional to the volume of the Skyrmions.

Figure 9 shows that the inclusion of the pion mass $m = 138$MeV decreases the maximum mass of the star by a very small amount from 1.49 to $1.47M_\odot$ while also slightly decreasing the central density at which this occurs.

Including a pion mass of $m = 138$MeV in the simulated annealing process used to find the maximum baryon number for the anisotropic Skyrme crystal solutions results in a maximum baryon number of 3.34×10^{57}, equivalent to $1.88M_\odot$, a decrease of $0.02M_\odot$ from the maximum mass found in the case without a pion mass.

5 Conclusions

Neutron stars are stars which have collapsed under their own gravitational pull. The electrons, instead of orbiting the atoms, are forced to merge with the nuclei, resulting in extremely dense stars made out of neutrons. Their temperature, from a nuclear point of view, is very low and this implies that they must be considered as a solids rather than a fluids. Moreover, the gravitational pull of the star is so strong that the “atmospheric” fluid one might expect at the surface is of negligible height.

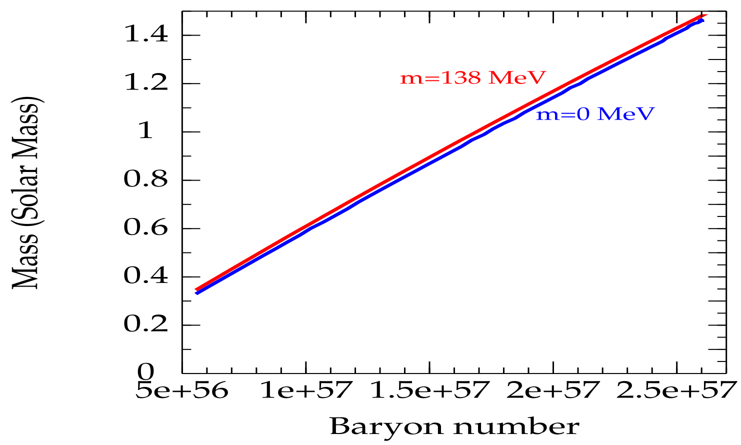

Figure 9: *Mass of the star as a function of the size of the Skyrmions at the centre, L_0, for zero pion mass (solid line) and $m = 138MeV$ (dashed line).*

The Skyrme model, known to be a low energy effective field theory for QCD [3], is then an ideal candidate to describe neutron stars if one modifies it to add gravity. The minimum energy configuration of large numbers of Skyrmions is a cubic crystal made of $B = 4$ Skyrmions corresponding to a crystal of α-like particles. Using these solutions as building blocks we have described the neutron star by combining the deformation energy computed in [7] and a generalised version of the TOV equation [13], [14], [15] which describes the static equilibrium between matter forces, within a solid or fluid, and the gravitational forces self-generated by the matter for a spherically symmetric body.

We showed that the star must be considered as a solid that could potentially deform itself anisotropically. We then found that below $1.49M_\odot$, all stars were made of a crystal deformed isotropically, *i.e.* the radial strain was identical to the tangential one. Above that critical value, the neutron star undergoes a critical phase transition and the lattice of Skyrmions compresses anisotropically: the Skyrmions are more compressed tangentially than radially. Stars were shown to exist up to a critical mass of $1.90M_\odot$, a result that closely matches the recent discovery of Demorest et al. [23] who measured the mass of the heaviest neutron star found to date, PSR J1614-2230, to be $1.97M_\odot$. We also observed that the maximum radius for a Skyrmion star was approximately 11km, a figure that matches well the experimental estimations.

In our model we did not consider the rotational energy of the star which is approximated at about 2% of its total energy. If we included that extra energy, our upper bound would thus just fit above the mass of PSR J1614-2230.

6 Acknowledgements

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