Monodromies and functional determinants in the CFT driven quantum cosmology

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Abstract

We discuss the calculation of the reduced functional determinant of a special second order differential operator $\boldsymbol{F} = -d^2/d\tau^2 + \ddot{g}/g$, $\ddot{g} \equiv d^2g/d\tau^2$, with a generic function $g(\tau)$, subject to periodic boundary conditions. This implies the gauge-fixed path integral representation of this determinant and the monodromy method of its calculation. Motivations for this particular problem, coming from applications in quantum cosmology, are also briefly discussed. They include the problem of microcanonical initial conditions in cosmology driven by a conformal field theory, cosmological constant and cosmic microwave background problems.

1. Introduction

Here we consider the class of problems involving the differential operator of the form

$$\boldsymbol{F} = -\frac{d^2}{d\tau^2} + \frac{\ddot{g}}{g},. \tag{1.1}$$

where $g = g(\tau)$ is a rather generic function of its variable τ. From calculational viewpoint, the virtue of this operator is that $g(\tau)$ represents its explicit basis function – the solution of the homogeneous equation,

$$\boldsymbol{F}g(\tau) = 0, \tag{1.2}$$

which immediately allows one to construct its second linearly independent solution

$$\Psi(\tau) = g(\tau)\int\limits_{\tau^*}^{\tau}\frac{dy}{g^2(y)} \tag{1.3}$$

and explicitly build the Green's function of $\boldsymbol{F}$ with appropriate boundary conditions. On the other hand, from physical viewpoint this operator is interesting because it describes long-wavelength perturbations in early Universe, including the formation of observable CMB spectra [1, 2], statistical ensembles in quantum cosmology [3], etc. In particular, for superhorizon cosmological perturbations of small momenta $k^2 \ll \ddot{g}/g$ their evolution operator only slightly differs from (1.1) by adding k^2 to its potential term, whereas in the minisuperspace sector of cosmology, corresponding to spatially constant variables, the operator has exactly the above form.

Up to an overall sign, this operator is the same both in the Lorentzian and Euclidean signature spacetimes with the time variables related by the Wick rotation $\tau = it$. In the Euclidean case it plays a very important role in the calculation of the statistical sum for the microcanonical ensemble in cosmology. This ensemble realizes the concept of cosmological initial conditions by generalizing the notion of the no-boundary wavefunction of the Universe [4] to the level of a special quasi-thermal state and puts it on the basis of a consistent canonical quantization [5, 6]. This concept is very promising both from the viewpoint of foundations of quantum cosmology and their applications within the cosmological constant, inflation and dark energy problems [3, 5, 6, 7, 8].

The recently suggested alteration in foundations of quantum cosmology – the theory of initial conditions for the early Universe – consists in a qualitative extension of the class of its initial quantum states. Instead of a usually accepted pure state, like the no-boundary one, it is assumed that the cosmological state can be mixed and characterized by the density matrix [5]. Under a natural and most democratic assumption of the microcanonical distribution, this density matrix and its statistical sum can be rendered the form of the Euclidean quantum gravity path integral [6]. Its calculation then shows if it is dominated either by the contribution of a pure state or a mixed statistical ensemble. Thus the dilemma of pure vs mixed state, rather than being postulated, gets solved at the dynamical level according to the matter content of the model.

For models dominated by heavy massive fields this microcanonical ensemble reduces to the pure vacuum no-boundary or tunneling state [8], whereas for massless conformally invariant fields the situation becomes even more interesting. In this case of the CFT driven cosmology the microcanonical ensemble incorporates a possible solution of the cosmological constant problem – the restriction of the range of the primordial Λ by a new quantum gravity scale which is encoded in the conformal anomaly of the underlying CFT [5, 6]. Moreover, it contains a mechanism of formation of the red tilted CMB spectrum complementary (or maybe even alternative) to the conventional mechanism based on primordial vacuum fluctuations in the early inflationary Universe [9]. As it was first observed in [7] this follows from a simple fact that thermal corrections to the CMB spectrum enhance its infrared part. This density matrix construction and its application in the CFT driven cosmology are briefly considered in Sects. 2 and 3.

In this context the operator (1.1) arises in the one-loop approximation for the cosmological statistical sum with τ playing the role of the Euclidean time, and the properties of this operator essentially differ from those of the Lorentzian dynamics. In the latter case the function g is a monotonic function of time because of the monotonically growing cosmological scale factor, whereas in the Euclidean case $g(\tau)$ is periodic just as the scale factor $a(\tau)$ itself and, moreover, has zeroes (roots) at turning points of the Euclidean evolution with $\dot{a} = 0$, because $g(\tau) \propto \dot{a}(\tau)$. This does not lead to a singular behavior of $\boldsymbol{F}$ because $\ddot{g}$ also vanishes at the zeroes of g [3, 5], and the potential term of (1.1) remains analytic (both g and $\ddot{g}$ simultaneously have a *first-order* zero). Nevertheless, the calculation of various quantities associated with this operator becomes cumbersome due to the roots of $g(\tau)$ – in particular, the basis function (1.3) becomes singular at each of these roots and cannot be extended beyond any of them. Among such quantities is the functional determinant of $\boldsymbol{F}$ which determines the one-loop contribution to the statistical sum of the CFT driven cosmology of [3]. Since this operator has an obvious zero mode which is the function $g(\tau)$ itself, the functional determinant of $\boldsymbol{F}$ should, of course, be understood as calculated on the subspace of its nonzero modes. The construction of this *restricted* functional determinant denoted below by $\mathrm{Det}^*\boldsymbol{F}$ is presented in Sect. 4.

The calculation of $\mathrm{Det}^*\boldsymbol{F}$ begins with the remark that there exist several different methods for restricted functional determinants. When the whole spectrum of the operator is known this is just the product of all non-zero eigenvalues. With the knowledge of only the zero mode, one can use the regularization technique [10] or contour integration method [11, 12, 13] to extract the regulated zero-mode eigenvalue from the determinant and subsequently take the regularization off. Here we use another approach to the definition of $\mathrm{Det}^*\boldsymbol{F}$ based on the Faddeev-Popov gauge-fixing procedure for the path integral in quantization of gauge theories [14]. In Sect. 4 we interpret the zero mode of $\boldsymbol{F}$ as a generator of the gauge invariance transformation of the relevant action, so that the reduced functional determinant arises as a result gauge-fixed Gaussian path integration. This allows one to express it in terms of the Green's function of the operator $\boldsymbol{F}$ on the subspace of its non-degeneracy. Remarkably, this Green's function follows from very simple and clear identical transformations under the path integration sign, rather than from verbose explanations one usually encounters in numerous works on the treatment of soliton or instanton zero modes.

In Sect. 5 we go over to the calculation of $\mathrm{Det}^*\boldsymbol{F}$ by the monodromy method. First we introduce the *monodromy* of the second basis function $\psi(\tau)$ of $\boldsymbol{F}$, which is linearly independent of $g(\tau)$. In particular, we derive the answer for this monodromy for the so-called multiple nodes case, when the function $g(\tau)$ within its period range has an arbitrarily high even number $2k$ of roots, $g(\tau_i) = 0$,

$i = 1, 2, ..., 2k$.[1] The monodromy is presented as an additive sum of contributions of segments of the time variable, $\tau_{i-1} \leq \tau \leq \tau_i$, separating various pairs of neighboring roots. Each contribution is given by a closed integral expression in terms of $g(\tau)$ on an underlying segment. Then, by the variational method for the functional determinant, we express $\mathrm{Det}^* \boldsymbol{F}$ in terms of this monodromy. Sect.6 presents the summary of obtained results and their discussion.

2. Euclidean quantum gravity from the physical theory in Lorentzian spacetime

The physical setting in Lorentzian signature spacetime starts with the definition of the microcanonical ensemble in canonically quantized gravity theory. In cosmology the corresponding density matrix $\hat\rho = \rho(\varphi, \varphi')$ was suggested in [6] as a formal projector

$$\hat\rho \sim \prod_\mu \delta(\hat H_\mu) \tag{2.1}$$

on the subspace of physical states satisfying the system of the Wheeler-DeWitt equations

$$\hat H_\mu(\varphi, \partial/i\partial\varphi)\, \rho(\varphi, \varphi') = 0, \tag{2.2}$$

where $\hat H_\mu$ denotes the operator realization of the full set of the gravitational Hamiltonian and momentum constraints $H_\mu(q, p)$. The formal product in (2.1) runs over the condensed index μ signifying a collection of discrete labels along with continuous spatial coordinates, $\mu = (\perp, a, \mathbf{x})$, $a = 1, 2, 3$. The phase space variables (q, p) include the collection of spatial metric coefficients and matter fields $q = \big(g_{ab}(\mathbf{x}), \phi(\mathbf{x})\big)$ and their conjugated momenta p. The canonical coordinates q will be also denoted by φ when used as arguments of the density matrix kernel $\langle\varphi|\,\hat\rho\,|\varphi'\rangle = \rho(\varphi, \varphi')$.

The justification for (2.1) as the density matrix of a *microcanonical* ensemble in spatially closed cosmology was put forward in [6] based on the analogy with an unconstrained system having a conserved Hamiltonian $\hat H$. The microcanonical state with a fixed energy E for such a system is given by the density matrix $\hat\rho \sim \delta(\hat H - E)$. A major distinction of (2.1) from this case is that spatially closed cosmology does not have freely specifiable constants of motion like the energy or other global charges. Rather it has as constants of motion the Hamiltonian and momentum constraints H_μ, all having a particular value — zero. Therefore, the expression (2.1) can be considered as the analogue of equipartition – a natural candidate for the microcanonical quantum state of the *closed* Universe.

The definition (2.1) has, of course, a very formal nature because it is very incomplete in view of non-commutativity of the constraints $\hat H_\mu$, infinite dimensional (and continuous) nature of the space of indices μ and the phase space, etc. However, at the semiclassical level (within the perturbation loop expansion) the kernel of this projector can be written down as a Faddeev-Popov gauge-fixed path integral of the canonically quantized gravity theory [14, 15, 6]

$$\rho(\varphi_+, \varphi_-) = e^{\varGamma} \int\limits_{q(t_\pm) = \varphi_\pm} D[\,q, p, N\,]\, \exp\left[i \int_{t_-}^{t_+} dt\,(p\,\dot q - N^\mu H_\mu)\right]. \tag{2.3}$$

Here N^μ are the Lagrange multipliers dual to the constraints – lapse and shift functions $N^\mu = (N(\mathbf{x}), N^a(\mathbf{x}))$, and the functional integration runs over the histories interpolating between the configurations $\varphi_\pm$ which are the arguments of the density matrix kernel. The range of integration over

[1]Since a periodic function has only an even number of roots within its period, we will call the case of their lowest nonvanishing number, $2k = 2$, the *single-node* one. This is the case of the CFT driven cosmology whose statistical sum as a function of the primordial cosmological constant is dominated by the countable set of instantons having k oscillations, $k = 1, 2, ...$, of the cosmological scale factor $a(\tau)$ during the Euclidean time period [5, 6] – the so-called garlands which carry the multi-node zero mode $g(\tau) \propto \dot a$.

N^μ is of course real because this integration over the Lagrange multipliers is designed in order to generate delta functions of constraints. The Hamiltonian action in the exponential is the integral over the coordinate time t which is just the ordering parameter ranging between arbitrary initial and final values $t_\pm$, the result being entirely independent of their choice.[2] The integration measure $D[\,q,p,N\,]$, of course, includes the Faddeev-Popov gauge-fixing procedure which renders the whole integral gauge independent.

After integration over canonical momenta the path integral above takes the Lagrangian form of the integral over the configuration space coordinates q and the lapse and shift functions N^μ. Taken together they comprise the full set of the spacetime metric with the Lorentzian signature $g^L_{\mu\nu}$ and matter fields ϕ , $g^L_{\mu\nu}dx^\mu dx^\nu=-N_L^2dt^2+g_{ab}(dx^a+N^a dt)(dx^b+N^b dt)$, in terms of which the Lagrangian form of the classical action reads as $S[\,g^L_{\mu\nu},\phi\,]$. One more notational step consists in the observation that this Lorentzian metric can be viewed as the Euclidean metric $g_{\mu\nu}$ with the imaginary value of the Euclidean lapse function N,

$$g_{\mu\nu}dx^\mu dx^\nu=N^2d\tau^2+g_{ab}(dx^a+N^a d\tau)\,(dx^b+N^b d\tau), \tag{2.4}$$

$$N=iN_L, \tag{2.5}$$

so that the Euclidean theory action is related to the original Lorentzian action $S_L[\,g^L_{\mu\nu},\phi\,]$ by a typical equation $iS_L[\,g^L_{\mu\nu},\phi\,]=-S[\,g_{\mu\nu},\phi\,]$. Note that the analytic continuation from the Lorentzian to the Euclidean picture takes place in the complex plane of the lapse function rather than in the complex plane of time (time variable is the same in both pictures $\tau=t$), though of course it is equivalent to the usual Wick rotation.

With these notations the density matrix (2.3) takes the form of the Euclidean quantum gravity path integral

$$\rho(\,\varphi_+,\varphi_-\,)=e^{\varGamma}\int\limits_{q(t_\pm)=\,\varphi_\pm}D[\,g_{\mu\nu},\phi\,]\,e^{-S[\,g_{\mu\nu},\phi\,]}. \tag{2.6}$$

However, in view of (2.5) the range of integration over the Euclidean lapse N belongs to the imaginary axis $-i\infty<N<i\infty$.

The topology of spacetime configurations which are integrated over in (2.6) is $S^3\times R^1$ as depicted on the left part of Fig.1. This topology of the spacetime bulk interpolating between the hypersurfaces Σ and Σ' reflects the mixed nature of the density matrix and establishes entanglement correlations between φ and φ' [5].

The normalization factor $\exp\varGamma$ in (2.6) follows from the density matrix normalization $\mathrm{tr}\hat\rho=1$ and determines the main object of interest – the statistical sum of the model. The trace operation implies integration over the diagonal elements of the density matrix, so that the statistical sum takes the form of the path integral

$$e^{-\varGamma}=\int\limits_{\rm periodic}D[\,g_{\mu\nu},\phi\,]\;e^{-S[\,g_{\mu\nu},\phi\,]} \tag{2.7}$$

over periodic configurations whose spacetime topology $S^3\times S^1$ follows from the identification of the boundary surfaces Σ and Σ'. This leads to to the topology $S^1\times S^3$ depicted on the right part of Fig.1.

[2]The projector on the space of non-commuting constraints can be realized by integration over the relevant group. Integration over the canonically realized diffeomorphisms implicit in the gauge fixed integral over N^μ is just the analogue of this group integration. On the other hand, the chronological ordering generated by the path integration in (2.3) takes care of the operator ordering in (2.1). Since in closed cosmology there is no non-vanishing Hamiltonian, the history parameter t in (2.3) exclusively serves this operator-ordering role. This is a peculiarity of the theories with a parameterized time [15].

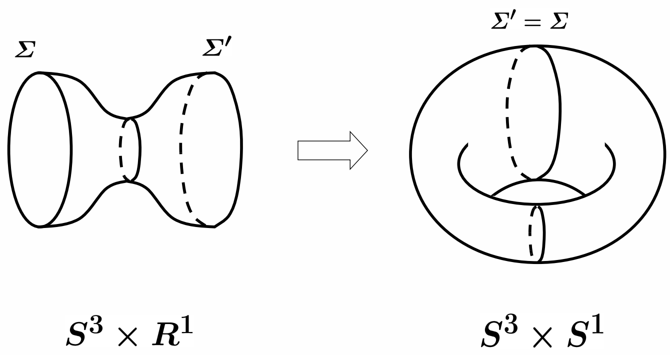

Figure 1: Transition from the density matrix to the statistical sum.

3. CFT driven cosmology and its thermal states

The actual calculation of the statistical sum can be based on decomposing the full set of fields, $[\,g_{\mu\nu}(x), \phi(x)\,] \to [\,a(\tau), N(\tau);\, \varPhi(x)\,]$, into the minisuperspace sector of the spatially closed Friedmann-Robertson-Walker (FRW) metric,

$$g^{FRW}_{\mu\nu}\, dx^\mu dx^\nu = N^2(\tau)\, d\tau^2 + a^2(\tau)\, d^2\Omega^{(3)}, \tag{3.1}$$

and all spatially inhomogeneous "matter" fields $\varPhi(x) = \varPhi(\tau, \mathbf{x})$, $\varPhi(x) = [\,\phi(x), \psi(x), A_\mu(x), h_{\mu\nu}(x), ...\,]$. Then the path integral can be cast into the form of an integral over a minisuperspace lapse function $N(\tau)$ and scale factor $a(\tau)$ of this metric,

$$e^{-\varGamma} = \int D[\,a, N\,]\; e^{-S_{\rm eff}[\,a,\,N\,]}, \tag{3.2}$$

$$e^{-S_{\rm eff}[\,a,\,N]} = \int D[\,\varPhi(x)\,]\, e^{-S[\,a,\,N;\,\varPhi(x)\,]}, \tag{3.3}$$

where $S_{\rm eff}[\,a,\,N\,]$ is the effective action of all these "matter" fields $\varPhi(x)$ (which include also the metric perturbations $h_{\mu\nu}(x)$) on the minisuperspace background of the FRW metric. The action $S[a, N; \varPhi(x)] \equiv S[\,g_{\mu\nu}, \phi\,]$ is the original Euclidean action rewritten in terms of the above minisuperspace decomposition.

This construction has a predictive power in the gravitational model with a matter sector dominated by a large number of free (linear) fields conformally coupled to gravity – conformal field theory (CFT)

$$S[\,g_{\mu\nu}, \phi\,] = -\frac{1}{16\pi G}\int d^4x\, g^{1/2}\,(R - 2\varLambda) + S_{CFT}[\,g_{\mu\nu}, \phi\,]. \tag{3.4}$$

The effective action in such a system is dominated by the quantum action of these conformal fields which simply outnumber the non-conformal fields (including the graviton). This quantum effective action, in its turn, is exactly calculable by the conformal transformation converting (3.1) into the static Einstein metric with a = const. It becomes the sum of the contribution of this conformal transformation [16, 17], determined by the well-known conformal anomaly of a quantum CFT in the external gravitational field [18] and the contribution of a static Einstein Universe – the combination of the Casimir energy [19] and free energy of a typical boson or fermion statistical sum. The temperature of the latter is given by the inverse of the Euclidean time period of the $S^1 \times S^3$ instanton, measured in units of the conformal time.

Namely, this effective action reads in units of the Planck mass $m_P = (3\pi/4G)^{1/2}$ [5]

$$S_{\rm eff}[\,a,N\,] = m_P^2 \int_{S^1} d\tau\, N \left\{ -aa'^2 - a + \frac{\Lambda}{3}a^3 + B\left(\frac{a'^2}{a} - \frac{a'^4}{6a}\right) + \frac{B}{2a} \right\} + F(\eta), \tag{3.5}$$

$$F(\eta) = \pm \sum_\omega \ln\left(1 \mp e^{-\omega\eta}\right), \quad \eta = \int_{S^1} \frac{d\tau N}{a}, \tag{3.6}$$

where $a' \equiv da/Nd\tau$. The first three terms in curly brackets of (3.5) represent the Einstein action with a primordial (but renormalized by quantum corrections) cosmological constant $\Lambda \equiv 3H^2$ (H is the corresponding Hubble constant). The terms proportional to the constant B correspond to the contribution of the conformal anomaly and the contribution of the vacuum (Casimir) energy ($B/2a$) of conformal fields on a static Einstein spacetime of the size a. Finally, $F(\eta)$ is the free energy of these fields – a typical boson or fermion sum over CFT field oscillators with energies ω on a unit 3-sphere, η playing the role of the inverse temperature — an overall circumference of the $S^1 \times S^3$ instanton in terms of the conformal time (3.6).

The constant $B = 3\beta/4m_P^2$, is determined by the coefficient β of the topological Gauss-Bonnet invariant $E = R_{\mu\nu\alpha\gamma}^2 - 4R_{\mu\nu}^2 + R^2$ in the overall conformal anomaly

$$g_{\mu\nu}\frac{\delta S_{\rm eff}^{CFT}}{\delta g_{\mu\nu}} = \frac{1}{4(4\pi)^2} g^{1/2}\left(\alpha\Box R + \beta E + \gamma C_{\mu\nu\alpha\beta}^2\right), \tag{3.7}$$

which is always positive for any CFT particle content [18]. The UV ambiguous coefficient α was renormalized to zero by a local counterterm $\sim \alpha R^2$ to guarantee the absence of higher derivative terms in the action (3.5). This automatically gives the renormalized Casimir energy the value $m_P^2 B/2a = 3\beta/8a$ which universally expresses in terms of the same coefficient in the conformal anomaly [20].[3] The coefficient γ of the Weyl tensor squared term $C_{\mu\nu\alpha\beta}^2$ does not enter the expression (3.5) because $C_{\mu\nu\alpha\beta}$ identically vanishes for any FRW metric.

Semiclassically the statistical sum (3.2) is dominated by the solutions of the effective equation for the action (3.5), $\delta S_{\rm eff}/\delta N(\tau) = 0$. This is the modification of the Euclidean Friedmann equation, by the conformal anomaly, Casimir energy and radiation terms (the energy of the gas of thermally excited particles with the inverse temperature η given by (3.6)). As shown in [5, 6, 7] the solutions of this equation give rise to the set of periodic $S^3 \times S^1$ instantons with the oscillating scale factor – *garlands* that can be regarded as the thermal version of the Hartle-Hawking instantons. In these solutions the scale factor oscillates k times ($k = 1, 2, 3, ...$) between its maximum and minimum values $a_\pm = a(\tau_\pm)$, $a_- \leq a(\tau) \leq a_+$, so that the full period of the conformal time (3.6) is given by the $2k$-multiple of the integral between the two neighboring turning points of the scale factor history $a(\tau)$ at which $\dot{a}(\tau_\pm) = 0$. This value of η is finite and determines a finite effective temperature $T = 1/\eta$ as a function of G and Λ. This is the artifact of a microcanonical ensemble in cosmology [6] with only two freely specifiable dimensional parameters — the renormalized gravitational and renormalized cosmological constants.

These $S^3 \times S^1$ garland-type instantons exist only in the limited range of the cosmological constant $\Lambda = 3H^2$ [5], $0 < \Lambda_{\rm min} < \Lambda < \Lambda_{\rm max} = 3/2B$, where the spectrum of admissible values of Λ has a band structure. The countable ($k = 1, 2, 3, ...$) sequence of bands Δ_k of ever narrowing widths, each of them corresponding to k-fold instantons of the above type, with $k \to \infty$ accumulates at the upper bound of this range. Periodicity of all these instantons originates from the tracing operation signifying the transition from the density matrix to the statistical sum, which is depicted on Fig.1 for the one-folded case $k = 1$.

[3] This universality property follows from the fact that in a static Einstein Universe of the size a the Casimir energy of conformal fields is determined by the conformal anomaly coefficients and equals $(3\beta - \alpha/2)/8a$ [20].

4. One-loop reduced functional determinants

Quadratic part of the action (3.5) in perturbations δa and δN on the cosmological instanton backgrounds described above essentially simplifies in the new variables

$$\delta a = \frac{aa'}{g}\,\varphi, \quad \delta N = \frac{a'}{g}\,\varphi + na. \tag{4.1}$$

Here the function $g = g(\tau)$ – the zero mode introduced above – expresses in terms of the Hessian of the Lagrangian of the local part of the action (3.5), $\mathcal{L} = \mathcal{L}(a, a')$, with respect to the scale factor "velocity" $a' = da/Nd\tau$,

$$g = a'a\sqrt{|\mathcal{D}|}, \quad \mathcal{D} = \frac{\partial^2 \mathcal{L}}{\partial a' \partial a'}. \tag{4.2}$$

The most important property of these variables is that φ is canonically normalized, and that their quadratic action has a simple closed form which is universally parameterized by the single function $g(\tau)$,

$$S^{(2)}_{\rm eff}[\,\varphi, n\,] = \frac{1}{2}\,\varepsilon_{\mathcal{D}} \oint d\tau\,\Big\{\dot\varphi^2 + \frac{\ddot g}{g}\,\varphi^2 + \big(4n\dot g + 2\dot n g\big)\varphi + g^2 n^2\Big\}$$
$$+\frac{1}{2}\,\frac{d^2F}{d\eta^2}\Big(\oint d\tau\, n\Big)^2, \quad \varepsilon_{\mathcal{D}} = \frac{\mathcal{D}}{|\,\mathcal{D}\,|} = \pm 1. \tag{4.3}$$

Here we assume that the background lapse $N = 1$ and $\varphi' = \dot\varphi$, $g' = \dot g$, etc. [4]

Therefore, in the gauge $n' = 0$ the one-loop part of the statistical sum would have been determined by the Gaussian functional integral with the quadratic action

$$S[\,\varphi\,] = \frac{1}{2}\oint d\tau\, \varphi(\tau)\boldsymbol{F}\varphi(\tau) = \frac{1}{2}\oint d\tau\,\Big(\dot\varphi^2 + \frac{\ddot g}{g}\,\varphi^2\Big). \tag{4.4}$$

However, the operator $\boldsymbol{F}$ is degenerate, and its zero mode $g(\tau)$ arises as the generator of the global gauge invariance of the action (4.4) under the transformation with a constant ε,

$$\delta^\varepsilon \varphi = R(\tau)\,\varepsilon, \tag{4.5}$$
$$R(\tau) = \frac{g(\tau)}{||g||}, \quad ||g||^2 = \oint d\tau\, g^2(\tau). \tag{4.6}$$

In fact this is a residual gauge transformation which is not fixed by the gauge $\dot n = 0$. Therefore, this situation can be handled by the additional Faddeev-Popov gauge fixing procedure. It consists of imposing the additional gauge $\chi[\,\varphi\,] = 0$ and inserting in the path integral the relevant Faddeev-Popov factor. This gauge condition $\chi[\,\varphi\,]$ and the Faddeev-Popov ghost factor $Q/||g||$ can be chosen in the form

$$\chi[\,\varphi\,] = \oint d\tau\, k(\tau)\,\varphi(\tau), \quad \delta^\varepsilon \chi = \frac{Q}{||g||}\,\varepsilon, \tag{4.7}$$
$$Q \equiv \oint d\tau\, k(\tau)\, g(\tau), \tag{4.8}$$

where $k(\tau)$ is a gauge fixing function and the generator (4.6) is normalized to unity with respect to L^2 inner product on S^1. Thus, integration over φ takes the form of the Gaussian functional integral with the delta-function type gauge

$$({\rm Det}^*\boldsymbol{F})^{-1/2} = {\rm const} \times \int D\varphi\; \delta\Big(\chi[\,\varphi\,]\Big)\,\frac{Q}{||g||}\,\exp\Big\{-S[\,\varphi\,]\Big\}, \tag{4.9}$$

[4] This expression holds irrespective of the form of the Lagrangian $\mathcal{L}(a, a')$, and the source of this universality is, of course, the time-parametrization invariance of the action and the fact that $\mathcal{L}(a, a')$ does not contain higher order derivatives of a.

and serves as the definition of the restricted functional determinant of $\boldsymbol{F}$. This definition is in fact independent of the choice of gauge by the usual gauge independence mechanism for the Faddeev-Popov integral. In particular, enforcing the gauge $\chi = 0$ means that the field φ is functionally orthogonal to the gauge fixing function $k(\tau)$ in the L^2 metric on S^1, and the above definition is independent of the choice of this gauge fixing function.

The normalization of the generator (4.6) has the following explanation. In local gauge theories the Faddeev-Popov path integral is not invariant under arbitrary rescalings of gauge generators. Their normalization is always implicitly fixed by the requirement of locality and the unit coefficient of the time-derivative term in the gauge transformation of Lagrange multipliers, $R\varepsilon \sim 1 \times \dot{\varepsilon} + ...$, (which follows from the canonical quantization underlying the Hamiltonian version of the Faddeev-Popov path integral). For the global symmetry of (4.4) we do not have a counterpart in canonical formalism, and such a founding principle as canonical quantization does not seem to be available. Therefore, we choose this normalization with respect to L^2 unit norm corresponding to the canonical normalization of the variable φ in (4.4). From the viewpoint of the definition of $\mathrm{Det}^*\boldsymbol{F}$ as the product of operator eigenvalues, this corresponds to the omission of a zero eigenvalue of $\boldsymbol{F}$,

$$\mathrm{Det}^*\,\boldsymbol{F} = \prod_{\lambda\neq 0}\lambda, \quad \boldsymbol{F}\varphi_\lambda(\tau) = \lambda\,\varphi_\lambda(\tau), \quad \tau\in S^1. \tag{4.10}$$

This follows from the orthogonal decomposition of the integration variable $\varphi(\tau)$ in the series of eigenfunctions $\varphi_\lambda(\tau)$ satisfying $\oint d\tau\,\varphi_\lambda(\tau)\,\varphi_{\lambda'}(\tau) = \delta_{\lambda\lambda'}$.

Representing the delta function of the gauge condition in (4.9) via the integral over the Lagrangian multiplier π we get the Gaussian path integral over the periodic function $\varphi(\tau)$ and the numerical variable π,

$$\begin{aligned}(\mathrm{Det}^*_{\mathcal{D}}\boldsymbol{F})^{-1/2} &= \mathrm{const}\times Q\,||g||^{-1}\int D\varphi\,d\pi\,\exp\Big(-S_{\rm eff}[\,\varphi(\tau),\pi\,]\Big)\\ &= \mathrm{const}\times Q\,||g||^{-1}\Big(\mathrm{Det}\,\mathbb{F}\Big)^{-1/2}.\end{aligned} \tag{4.11}$$

Here $S_{\rm eff}[\,\varphi(\tau),\pi\,]$ is the effective action of these variables and $\mathbb{F}$ is the matrix valued Hessian of this action with respect to $\varPhi = (\varphi(\tau),\pi)$,

$$S_{\rm eff}[\,\varphi(\tau),\pi\,] = S[\,\varphi\,] - i\pi\oint d\tau\,k\varphi, \tag{4.12}$$

$$\mathbb{F} = \frac{\delta^2 S_{\rm eff}}{\delta\varPhi\,\delta\varPhi'} = \begin{bmatrix}\boldsymbol{F}\,\delta(\tau,\tau') & -ik(\tau)\\ & \\ -ik(\tau') & 0\end{bmatrix} \tag{4.13}$$

(note the position of time entries associated with the variables $\varPhi = (\varphi(\tau),\pi)$ and $\varPhi' = (\varphi(\tau'),\pi)$).

The dependence of this determinant on $g(\tau)$ and $k(\tau)$ can be found from its variation with respect to these functions. From (4.11) we have $\delta\ln\Big(\mathrm{Det}^*\,\boldsymbol{F}\Big) = -2\,\delta\ln Q + 2\,\delta\ln||g|| + \mathrm{Tr}\,\Big(\delta\mathbb{F}\,\mathbb{G}\Big)$, where $\mathbb{G}$ is the Green's function of $\mathbb{F}$, $\mathbb{F}\,\mathbb{G} = \mathbb{I}$ and Tr is the functional trace of any matrix with the block-structure of (4.13). The block structure of the matrix Green's function $\mathbb{G}$ has the form

$$\mathbb{G} = \begin{bmatrix} G(\tau,\tau') & \dfrac{ig(\tau)}{Q}\\ \dfrac{ig(\tau')}{Q} & 0\end{bmatrix}, \tag{4.14}$$

where the Green's function $G(\tau,\tau')$ in the diagonal block satisfies the system of equations

$$\boldsymbol{F}\,G(\tau,\tau') = \delta(\tau,\tau') - \frac{k(\tau)\,g(\tau')}{Q}, \tag{4.15}$$

$$\oint d\tau\,g(\tau)\,G(\tau,\tau') = 0, \tag{4.16}$$

which uniquely fix it. The second equation imposes the needed gauge, whereas the right hand side of the first equation implies that $G(\tau,\tau')$ is the inverse of the operator F on the subspace orthogonal to its zero mode.

The trace of the functional block-structure matrix corresponding to the variation of $g(\tau)$ reads

$$\mathrm{Tr}\Big(\delta_g\mathbb{F}\,\mathbb{G}\Big) = \mathrm{Tr}\Big(\delta\boldsymbol{F}\,G(\tau,\tau')\Big) \equiv \oint d\tau\,\delta\boldsymbol{F}\,G(\tau,\tau')\Big|_{\tau'=\tau}. \tag{4.17}$$

A similar variation of the gauge-fixing function gives a vanishing answer $\delta_k \ln\big(\mathrm{Det}^*\boldsymbol{F}\big) = 0$ as, of course, it should be in view of the gauge independent nature of the Faddeev-Popov path integral.[5] This guarantees the uniqueness of the definition of the reduced determinant $\mathrm{Det}^*\boldsymbol{F}$.

5. The monodromy method

Let the circle range of the time variable S^1 having of the circumference T be parameterized by τ

$$\tau_0 < \tau < \tau_0 + T, \tag{5.1}$$

with the points τ_0 and τ_0+T being identified. This range can be infinitely extended to the whole axis $-\infty < \tau < \infty$, multiple covering of S^1, on which the function $g(\tau)$ and, consequently, the operator $\boldsymbol{F}$ are periodic with the period T,

$$g(\tau+T) = g(\tau). \tag{5.2}$$

In the multi-node case, motivated by the above applications in cosmology, the periodic function $g(\tau)$ is oscillating and has within its period $2k$ simple roots

$$\tau_0 < \tau_1 < \tau_2 < ...\tau_{2k} = \tau_0 + T, \tag{5.3}$$

$$g(\tau_i) = 0, \quad \dot g(\tau_i) \neq 0, \tag{5.4}$$

$$\ddot g(\tau_i) = 0. \tag{5.5}$$

For simplicity we assume that one of them coincides with the final (or starting) point of this period. A significant assumption is that the second order derivative of this function at its roots is vanishing, which will be important for analyticity properties of our formalism.

Another important property of the operator $\boldsymbol{F}$ is its Wronskian relation. For any two functions φ_1 and φ_2 this operator determines their Wronskian $W[\varphi_1,\varphi_2] \equiv \varphi_1\dot\varphi_2 - \dot\varphi_1\varphi_2$ which enters the relation

$$\int_{\tau_-}^{\tau_+} d\tau\,\varphi_1 \overrightarrow{\boldsymbol{F}}\varphi_2 = \int_{\tau_-}^{\tau_+} d\tau\,\varphi_1 \overleftarrow{\boldsymbol{F}}\varphi_2 - W[\varphi_1,\varphi_2]\,\Big|_{\tau_-}^{\tau_+}. \tag{5.6}$$

Arrows here denote the direction of action of the operator $\boldsymbol{F}$, i. e. $\varphi_1\overleftarrow{\boldsymbol{F}} = (\boldsymbol{F}\varphi_1)$, and the Wronskians appear as total derivative terms generated by integration by parts of the derivatives in $\boldsymbol{F}$. When both φ_1 and φ_2 satisfy a homogeneous equation with the operator $\boldsymbol{F}$, their Wronskian turns out to be constant. Also the vanishing Wronskian implies linear dependence of these solutions.

[5] In the works involving the treatment of soliton and instanton zero modes it is implicitly assumed that the gauge-fixing function coincides with the zero mode itself, $k(\tau) = g(\tau)$, which considerably simplifies the formalism, but makes it less flexible.

5.1. Monodromy for the multi-node case

Let us now consider the solution of the homogeneous equation $\psi(\tau)$ normalized by a unit value of its Wronskian with g

$$\boldsymbol{F}\psi(\tau) = 0, \quad W[g,\psi] = 1. \tag{5.7}$$

Together with $g(\tau)$ this solution forms a set of linearly independent basis functions of $\boldsymbol{F}$. However, in contrast to $g(\tau)$ the basis function is not periodic, because we assume that the operator (1.1) has only one periodic zero mode smoothly defined on a circle (5.1). On the other hand, when considered on the full axis of τ, due to periodicity of $g(\tau)$ this operator is also periodic $\boldsymbol{F}(\tau+T) = \boldsymbol{F}(\tau)$. Therefore $\psi(\tau+T)$ is also a solution of the equation $\boldsymbol{F}(\tau)\psi(\tau+T) = 0$, and consequently it can be decomposed into a linear combination of the original two basis functions with constant coefficients

$$\psi(\tau+T) = \psi(\tau) + \Delta\, g(\tau). \tag{5.8}$$

The unit coefficient in the first term follows from the conservation in time of the Wronskian of any two solutions of the equation (5.7), periodicity of $g(\tau)$ and an obvious fact that $W[g,g] = 0$ and

$$1 = W[g(\tau+T), \psi(\tau+T)] = W[g(\tau), \psi(\tau) + \Delta\, g(\tau)]. \tag{5.9}$$

The coefficient Δ in the second term of (5.8) is nontrivial – this is the *monodromy* of $\psi(\tau)$ which will play a central role in the construction of the determinant.

The function $\psi(\tau)$ can be composed of the set of functions $\varPsi_i(\tau)$ defined by (1.3) on various segments of τ-range connecting the pairs of neighboring roots of $g(\tau)$[6]

$$\varPsi_i(\tau) = g(\tau)\int_{\tau_i^*}^{\tau}\frac{dy}{g^2(y)}, \qquad \tau_{i-1} < \tau, \tau_i^* < \tau_i\ , \quad i = 1, ..., 2k \tag{5.10}$$

Here τ_i^* are the auxiliary points arbitrarily chosen in the same segments, and all these solutions are normalized by the unit Wronskian with $g(\tau)$, $W[g,\varPsi_i] = 1$. The main property of these functions $\varPsi_i(\tau)$ is that each of them is defined in the i-th segment of the full period of τ where the integral (5.10) is convergent because the roots of $g(\tau)$ do not occur in the integration range. Its limits are well defined also at the boundaries of this segment,

$$\varPsi_i(\tau_{i-1}) = -\frac{1}{\dot g(\tau_{i-1})}, \quad \varPsi_i(\tau_i) = -\frac{1}{\dot g(\tau_i)}, \tag{5.11}$$

because the factor $g(\tau)$ tending to zero compensates for the divergence of the integral at $\tau \to \tau_i - 0$ and $\tau \to \tau_{i-1} + 0$.

For an arbitrary choice of auxiliary points τ_i^* in (5.10) the composite function

$$\psi(\tau) = \varPsi_i(\tau), \qquad \tau_{i-1} \leq \tau \leq \tau_i, \tag{5.12}$$

will be continuous in view of (5.11), but the continuity of its derivative will generally be broken, because generally the equality $\dot\varPsi_i(\tau_i - 0) = \dot\varPsi_{i+1}(\tau_i + 0)$ is not satisfied. However, this equality for $i = 1, 2, ..., 2k-1$ can be enforced by a special choice of these auxiliary points τ_i^*, becoming the equation for their determination. The solution for τ_i^* is unique, always exists and belongs to the corresponding segment $\tau_{i-1} < \tau_i^* < \tau_i$.[7] On the other hand, the continuity of the derivative of $\psi(\tau)$ cannot be attained at all roots of $g(\tau)$, $i = 1, 2, ...2k$, because it would correspond to the existence of

[6] For the extended range the missing $\varPsi_0(\tau)$ can be defined by identifying τ_{-1} with $\tau_{2k-1} - T$ and choosing some τ_0^* in $\tau_{-1} < \tau_0^* < \tau_0$.

[7] Indeed, the quantity $d\dot\varPsi_i(\tau_i)/d\tau_i^* = -\dot g(\tau_i)/g^2(\tau_i^*)$ is a sign definite function of τ_i^* nowhere vanishing on this segment, its absolute value quadratically divergent to ∞ at its boundaries. This in its turn means that $\dot\varPsi_i(\tau_i)$ is a monotonic function of τ_i^* which also ranges between $-\infty$ and $+\infty$ and therefore guarantees the unique solution for τ_i^* on this segment.

the second zero mode periodic on the circle, which is ruled out by construction. Therefore, the second basis function of $\boldsymbol{F}$ is not periodic on the circle, but in view of periodicity of the operator it satisfies the fundamental monodromy property (5.8). In the next subsection we construct the periodic Green's function of $\boldsymbol{F}$ in terms of this monodromy parameter Δ, whereas here we give in a closed form the analytic expression for Δ as a functional of $g(\tau)$.

From the definition of the monodromy parameter (5.8) it follows that in the limit $\tau \to \tau_0$, $\tau + T \to \tau_{2k}$,

$$\Delta = \frac{\dot\psi(\tau_{2k}) - \dot\psi(\tau_0)}{\dot g(\tau_0)} = \left(\frac{\dot\psi(\tau_{2k})}{\dot g(\tau_{2k})} - \frac{\dot\psi(\tau_{2k-1})}{\dot g(\tau_{2k-1})}\right) + \ldots + \left(\frac{\dot\psi(\tau_1)}{\dot g(\tau_1)} - \frac{\dot\psi(\tau_0)}{\dot g(\tau_0)}\right), \tag{5.13}$$

where we took into account that $\dot g(\tau_{2k}) = \dot g(\tau_0)$. Then the monodromy reads as the additive sum of contributions of pairs of neighboring roots of $g(\tau)$ [21],

$$\Delta = \sum_{i=1}^{2k} \varDelta_i, \quad \varDelta_i = -\Big(\varPsi_i(\tau_i)\,\dot\varPsi_i(\tau_i) - \varPsi_i(\tau_{i-1})\,\dot\varPsi_i(\tau_{i-1})\Big). \tag{5.14}$$

Because of (5.5) the functions $\varPsi_i(\tau)$ are differentiable in these limits, and all the quantities which enter the algorithm (5.14) are well defined. In particular, for any such time segment $[\tau_{i-1}, \tau_i] \equiv [\tau_-, \tau_+]$ the derivatives of $\varPsi(\tau)$ at its boundaries are given by the convergent integral

$$\dot\varPsi(\tau_\pm) = \int\limits_{\tau^*}^{\tau_\pm} dy\, \frac{\dot g(\tau_\pm) - \dot g(y)}{g^2(y)} + \frac{1}{g(\tau^*)}. \tag{5.15}$$

Note that the integrand here is finite at $y \to \tau_\pm$ because of $\ddot g(\tau_\pm) = 0$. These properties of $\varPsi_i(\tau)$ guarantee that the obtained result is independent of the choice of the auxiliary point τ_i^* for each Δ_i, and the monodromy (5.14) is uniquely defined.

It is important that unlike in the construction of the function $\psi(\tau)$ which has to be smooth on S^1 at all roots τ_i except τ_0 (the property that was attained above by a special choice of the auxiliary points τ_i^*), the derivatives of neighboring functions $\varPsi_i(\tau)$ in (5.14) should not necessarily be matched at these junction points. This is because the partial contributions Δ_i to the overall monodromy Δ are individually independent of τ_i^*, $d\varDelta_i/d\tau_i^* = 0$, which can be easily verified by using a simple relation $d\dot\varPsi_i(\tau)/d\tau_i^* = -\dot g(\tau)/g^2(\tau_i^*)$. Thus, the monodromy is uniquely defined and independent of the choice of the auxiliary points τ_i^* necessarily entering the definition of functions $\varPsi_i(\tau)$ in Eq.(5.10).

5.2. Periodic Green's function and the variation of the determinant

For the calculation of the variation (4.17) above we need the Green's function of the problem (4.15)-(4.16) which should be periodic on the circle (5.1). To achieve this property we will slightly extend the circle domain to the left of the point τ_0, $\tau_0 - \varepsilon < \tau < \tau_0 + T$, $\varepsilon > 0$, with an arbitrarily small positive ε and demand that the monodromy of $G(\tau, \tau')$ is vanishing for this small ε-range of τ near τ_0

$$G(\tau + T, \tau') - G(\tau, \tau') = 0, \quad \tau_0 - \varepsilon < \tau < \tau_0. \tag{5.16}$$

The ansatz for $G(\tau, \tau')$ can be as usual built with the aid of two linearly independent basis functions of the operator. One basis function coincides with the periodic zero mode $g(\tau)$ and another one is given by the function $\psi(\tau)$ built above. Thus it can be represented as a sum of the particular solution of the inhomogeneous equation (4.15) and the bilinear combination of $g(\tau)$ and $\psi(\tau)$ with the coefficients providing the periodicity property (5.16). As shown in [21] it reads

$$G(\tau, \tau') = G_F(\tau, \tau') + \frac{1}{Q}\,\Omega(\tau, \tau') + \alpha\, H_{\psi\psi}(\tau, \tau') + \beta\, H_{\psi g}(\tau, \tau') + \gamma\, H_{gg}(\tau, \tau'), \tag{5.17}$$

where

$$G_F(\tau,\tau') \equiv \frac{1}{2}\big(g(\tau)\psi(\tau') - \psi(\tau)g(\tau')\big)\,\theta(\tau-\tau') \;+\; \frac{1}{2}\big(\psi(\tau)g(\tau') - g(\tau)\psi(\tau')\big)\,\theta(\tau'-\tau)\,, \quad (5.18)$$

$$\Omega(\tau,\tau') \;\equiv\; \omega(\tau)\,g(\tau') + g(\tau)\,\omega(\tau')\,, \quad (5.19)$$

$$H_{\psi\psi}(\tau,\tau') \;\equiv\; \psi(\tau)\psi(\tau')\,, \quad (5.20)$$

$$H_{\psi g}(\tau,\tau') \;\equiv\; \psi(\tau)\,g(\tau') + g(\tau)\,\psi(\tau')\,, \quad (5.21)$$

$$H_{gg}(\tau,\tau') \;\equiv\; g(\tau)\,g(\tau')\,, \quad (5.22)$$

and the function $\omega(\tau)$ is defined by

$$\omega(\tau) = \psi(\tau)\int\limits_{\tau_*}^{\tau} dy\, g(y)k(y) - g(\tau)\int\limits_{\tau_*}^{\tau} dy\,\psi(y)k(y) \quad (5.23)$$

with an arbitrary τ_*. The first term of (5.17) generates the delta-function in the right hand side of the equation (4.15), the second term $\Omega(\tau,\tau')/Q$ gives $-k(\tau)\,g(\tau')/Q$, while $H_{\psi\psi}(\tau,\tau')$, $H_{\psi g}(\tau,\tau')$ and $H_{gg}(\tau,\tau')$ represent symmetric solutions of the homogeneous equation with coefficients α, β and γ which are fixed by the periodicity condition and the gauge condition (4.16). With their knowledge the calculation of the variational term (4.17) is based on integration by parts and a systematic use of the Wronskian relation (5.6) together with equations for g and ψ. The result of this calculation is presented in much detail in [21] and reads $\delta\ln(\mathrm{Det}_*\boldsymbol{F}) = 2\,\delta\ln||g|| + \delta\ln\Delta$. It finally gives the explicit answer for $\mathrm{Det}_*\boldsymbol{F}$

$$\mathrm{Det}^*\boldsymbol{F} = C(T)\times\Delta\oint d\tau\, g^2(\tau). \quad (5.24)$$

In fact, this is the McKane-Tarlie formula (Eq.(5.2) of [10]) obtained by the regularization and contour integration methods [10, 12, 13] based on the earlier work of Forman [23] for a generic second order differential operator. We have reproduced this formula by the variational method for functional determinants. Beyond this, the structure of the operator (1.1) makes the problem exactly solvable and gives the monodromy (5.8) in quadratures as an explicit functional of $g(\tau)$.

Final comment concerns the overall normalization in (5.24). The formula of McKane-Tarlie [10, 12, 13] or the variational method which we use below, in principle, give only the ratio of determinants for two different operators with different functions g, whereas each determinant contains an infinite numerical factor generated by UV divergent product of eigenvalues of $\boldsymbol{F}$. This factor is independent of $g(\tau)$ but depends on the UV regularization and can be a function of the period T – the only remaining free parameter of the problem. This coefficient function can be determined by the zeta-function method [22] for a particular case of the constant function $g(\tau) = c$ corresponding to the operator $\boldsymbol{F} = -d^2/d\tau^2$ with the explicit spectrum of eigenfunctions and respective eigenvalues

$$\begin{aligned} &\varphi_0 = 1, \quad \lambda_0 = 0,\\ &\varphi_{1\,n}(\tau) = \sin\big(2\pi n\tau/T\big),\; \varphi_{2\,n}(\tau) = \cos\big(2\pi n\tau/T\big),\\ &\lambda_n = \big(2\pi n/T\big)^2, \quad n = 1,2,\dots\,. \end{aligned} \quad (5.25)$$

The logarithm of the corresponding restricted determinant – the product of all nonvanishing eigenvalues regularized by zeta-function method – equals

$$\ln\mathrm{Det}_*\left(-\frac{d^2}{d\tau^2}\right) = 4\ln\left(\frac{2\pi}{T}\right)\zeta_R(0) - 4\zeta_R'(0) = 2\ln T. \quad (5.26)$$

Here $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function having the following particular value $\zeta_R(0) = -\frac{1}{2}$ and the value of its derivative $\zeta_R'(0) = -\frac{1}{2}\ln 2\pi$. On the other hand, the basis functions and the monodromy Δ for this operator read $g(\tau) = c$, $\oint d\tau\, g^2 = c^2\, T$, $\psi(\tau) = (\tau - \tau_*)/c$, $\Delta = T/c^2$. Therefore according to (5.24) $\mathrm{Det}_*\big(-d^2/d\tau^2\big) = C(T)\, T^2$, and the comparison with (5.26) gives the T-independent result

$$C(T) = 1. \tag{5.27}$$

6. Conclusions

Thus we see that the above combination of methods gives exhaustive answer (5.24), (5.27), for the reduced functional determinant of the operator (1.1) having a multi-node zero mode in the periodic boundary value problem. This determinant expresses in terms of the monodromy of its basis function, which is obtained in quadratures as a sum of contributions (5.14) of time segments connecting neighboring pairs of the zero mode roots within the period range. Few words are in order here just to reiterate our special interest in this particular problem, briefly mentioned in Introduction.

The operator $\boldsymbol{F}$ determines the one-loop statistical sum for the microcanonical ensemble in cosmology generated by a conformal field theory [5, 6, 3]. This ensemble realizes the concept of cosmological initial conditions by generalizing the notion of the no-boundary wavefunction of the Universe to the level of a special quasi-thermal state which is dominated by instantons with an oscillating cosmological scale factor $a(\tau)$ of their Euclidean FRW metric. These oscillations result in the multi-node nature of the zero mode $g(\tau) \sim \dot{a}(\tau)$ of $\boldsymbol{F}$, which itself arises as the residual conformal Killing symmetry of the FRW background. This, in particular, explains the motivation for the gauge-fixing treatment of the zero mode considered above.

As was mentioned above, a very attractive feature of the cosmological microcanonical ensemble is that in the case of the CFT driven cosmology it suggests a possible solution of the cosmological constant problem – the restriction of the range of the primordial Λ by a new quantum gravity scale, its value being encoded in the conformal anomaly of the underlying CFT [5, 6]. Moreover, as suggested in [7], these microcanonical initial conditions admit inflationary scenario in the early Universe and can provide a thermal input in the red tilt of the COBE part of the CMB spectrum. This tilt can be additional or, perhaps, even alternative to the conventional red tilt generated from primordial vacuum fluctuations of [9]. This makes the hypothesis of microcanonical initial conditions in quantum cosmology not only feasible, but also observationally verifiable, perhaps, in a foreseeable future. Also, the statistical sum of this ensemble is likely to predict interesting phase transitions for multi-node cosmological instantons [24] which makes physics of this model very rich and interesting. The results and methods presented above seem indispensable for a further progress in these intriguing issues.

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