Nonlinear massive gravity and cosmology

Shinji Mukohyama*
Kavli Institute for the Physics and Mathematics of the Universe
Todai Institutes for Advanced Study, University of Tokyo
5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan

Abstract

We review cosmology in the recently proposed nonlinear massive gravity, especially cosmological solutions and their stabilities.

1 Introduction

The concept of the mass has been central in many areas of physics. Gravitation is not an exception, and it is one of the simplest but yet unanswered questions whether the graviton, a spin-2 particle that mediates gravity, can have a non-vanishing mass or not. This question is relevant not only from a theoretical but also from a phenomenological viewpoint, since a nonzero graviton mass may lead to late-time acceleration of the universe and thus may be considered as an alternative to dark energy.

Recently Refs.[1, 2] proposed the first example of a fully nonlinear massive gravity theory, where the so called Boulware-Deser (BD) ghost [3], which had been one of the major obstacles against a stable nonlinear gravity theory with a non-vanishing graviton mass, is removed by construction. Due to the theoretical and phenomenological motivations mentioned above, this theory has been attracting significant interest. The purpose of this paper is to review cosmology in the nonlinear massive gravity, especially cosmological solutions and their stabilities.

2 Open FRW solution with Minkowski fiducial metric

In this section, we review the open FRW universe solution [4] in nonlinear massive gravity coupled to general matter content.

The covariant action for the gravity sector is constructed out of the four dimensional metric $g_{\mu\nu}$ and the four scalar fields ϕ^a $(a = 0, 1, 2, 3)$ called *Stückelberg fields*. The action respects the Poincare symmetry in the field space, i.e. invariance under the constant shift of each of ϕ^a and the Lorentz transformation mixing them:

$$\phi^a \to \phi^a + c^a, \quad \phi^a \to \Lambda^a_b \phi^b. \tag{1}$$

The following line element in the field space is invariant under these transformations.

$$\eta_{ab} d\phi^a d\phi^b = -(d\phi^0)^2 + \delta_{ij} d\phi^i d\phi^j. \tag{2}$$

Indeed, this is the unique geometrical quantity in the field space of ϕ^a. Thus the action can depend on ϕ^a only through the spacetime tensor

$$f_{\mu\nu} \equiv \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b. \tag{3}$$

* **e-mail**: shinji.mukohyama@ipmu.jp

In this language, general covariance is spontaneously broken by the vacuum expectation value (vev) of $f_{\mu\nu}$. By assumption, matter fields propagate on the physical metric $g_{\mu\nu}$, but are not coupled to $f_{\mu\nu}$ directly. The tensor $f_{\mu\nu}$, constructed from the invariant line element in the field space, is often called a *fiducial metric*. On the other hand, the spacetime metric $g_{\mu\nu}$, on which matter fields propagate, is often called a *physical metric*.

The gravity action is the sum of the Einstein-Hilbert action (with the cosmological constant Λ) $I_{EH,\Lambda}$ for the physical metric $g_{\mu\nu}$ and the graviton mass term I_{mass} specified below. Adding the matter action I_{matter}, the total action is

$$I = I_{EH,\Lambda}[g_{\mu\nu}] + I_{mass}[g_{\mu\nu}, f_{\mu\nu}] + I_{matter}[g_{\mu\nu}, \sigma_I], \tag{4}$$

where

$$\begin{aligned} I_{EH,\Lambda}[g_{\mu\nu}] &= \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}(R-2\Lambda), & (5)\\ I_{mass}[g_{\mu\nu}, f_{\mu\nu}] &= M_{Pl}^2 m_g^2 \int d^4x\sqrt{-g}\,(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4), & (6)\end{aligned}$$

and $\{\sigma_I\}$ $(I = 1, 2, \cdots)$ represent matter fields. Demanding the absence of ghost at least in the decoupling limit [2], each contribution in the mass term I_{mass} is constructed as

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \\ \mathcal{L}_3 &= \frac{1}{6}\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right), \\ \mathcal{L}_4 &= \frac{1}{24}\left([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]\right), \end{aligned} \tag{7}$$

where the square brackets denote trace operation and

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_{\ \nu}. \tag{8}$$

The square-root in this expression is the positive definite matrix defined through

$$\left(\sqrt{g^{-1}f}\right)^\mu_{\ \rho}\left(\sqrt{g^{-1}f}\right)^\rho_{\ \nu} = f^\mu_{\ \nu}\ (\equiv g^{\mu\rho}f_{\rho\nu}). \tag{9}$$

As already stated above, a vev of the tensor $f_{\mu\nu}$ breaks general covariance spontaneously. Thus, in order to find FRW cosmological solutions in this theory, we should adopt an ansatz in which not only $g_{\mu\nu}$ but also $f_{\mu\nu}$ respects the symmetry of the FRW universes [4]. Since the tensor $f_{\mu\nu}$ is the pullback of the Minkowski metric in the field space to the physical spacetime, construction of such an ansatz is equivalent to finding a flat, closed, or open FRW coordinate system for the Minkowski line element. It is well known that the Minkowski line element does not admit a closed FRW chart but allows an open FRW chart. For this reason, in order to find open FRW solutions [4], we first perform the field redefinition from ϕ^a to new fields φ^a so that $f_{\mu\nu}$ written in terms of φ^a manifestly has the symmetry of open FRW universes as

$$f_{\mu\nu} = -n^2(\varphi^0)\partial_\mu\varphi^0\partial_\nu\varphi^0 + \alpha^2(\varphi^0)\Omega_{ij}(\varphi^k)\partial_\mu\varphi^i\partial_\nu\varphi^j, \tag{10}$$

where $i, j = 1, 2, 3$, and

$$\Omega_{ij}(\varphi^k) = \delta_{ij} + \frac{K\delta_{il}\delta_{jm}\varphi^l\varphi^m}{1 - K\delta_{lm}\varphi^l\varphi^m} \tag{11}$$

is the metric of the maximally symmetric space with the curvature constant K (< 0). Concretely, this is achieved by

$$\phi^0 = f(\varphi^0)\sqrt{1 - K\delta_{ij}\varphi^i\varphi^j}, \qquad \phi^i = \sqrt{-K}f(\varphi^0)\varphi^i, \tag{12}$$

and

$$n(\varphi^0) = |\dot{f}(\varphi^0)|, \qquad \alpha(\varphi^0) = \sqrt{-K}|f(\varphi^0)|, \tag{13}$$

where f is a function to be determined and $\dot{f}$ represents its derivative. We then adopt the "unitary gauge"

$$\varphi^0 = t, \quad \varphi^i = x^i, \tag{14}$$

so that

$$f_{\mu\nu}dx^\mu dx^\nu = -(\dot{f}(t))^2\, dt^2 + |K|\,(f(t))^2\, \Omega_{ij}(x^k)dx^i dx^j\,. \tag{15}$$

This is nothing but the Minkowski line element in the open chart. For the physical metric, we adopt the open FRW ansatz

$$ds^2 = -N(t)^2 dt^2 + a(t)^2\, \Omega_{ij}(x^k)dx^i dx^j\,. \tag{16}$$

Hereafter, we assume that $N > 0$ and $a > 0$, without loss of generality.

The background action now yields, up to boundary terms,

$$I = M_{Pl}^2 \int dt\, d^3x N a^3 \sqrt{\Omega}\left(L_{EH}[N,a] + m_g^2 L_{mass}[N,a,f]\right) + I_{matter}[N,a,\sigma_I]\,, \tag{17}$$

consisting of the Einstein-Hilbert part

$$L_{EH} = \frac{3\,K}{a^2} - \frac{3\,\dot{a}^2}{a^2\,N^2}\,, \tag{18}$$

and the contribution from the mass term

$$\begin{aligned} L_{mass} \;=\; & \left(1-\frac{\sqrt{-K}|f|}{a}\right)\left[6+4\,\alpha_3+\alpha_4-\frac{\sqrt{-K}|f|}{a}\,(3+5\,\alpha_3+2\,\alpha_4)-\frac{K\,|f|^2}{a^2}\,(\alpha_3+\alpha_4)\right] \\ & + sgn(\dot{f}/f)\frac{|f|\,\dot{a}}{N\,a} \\ & \times\left[3\,(3+3\,\alpha_3+\alpha_4)-\frac{3\,\sqrt{-K}|f|}{a}\,(1+2\,\alpha_3+\alpha_4)-\frac{K\,|f|^2}{a^2}\,(\alpha_3+\alpha_4)\right]. \end{aligned} \tag{19}$$

Hereafter, an overdot represents derivative w.r.t. the time t.

Varying the action (17) with respect to f yields the following constraint

$$\begin{aligned} & \left[H - sgn(\dot{f}/f)\frac{\sqrt{-K}}{a}\right] \\ & \times\left[3+3\,\alpha_3+\alpha_4-\frac{2\,\sqrt{-K}\,|f|}{a}\,(1+2\,\alpha_3+\alpha_4)-\frac{K\,|f|^2}{a^2}\,(\alpha_3+\alpha_4)\right] = 0\,, \end{aligned} \tag{20}$$

where the Hubble expansion rate of the physical metric is defined as

$$H \equiv \frac{\dot{a}}{N\,a}\,. \tag{21}$$

Out of the three solutions of the constraint (20), the trivial solution $\dot{a} = sgn(\dot{f}/f)\sqrt{-K}\,N$ corresponds to the Minkowski spacetime in open chart. The remaining two branches of solutions are given by [4]

$$\alpha(t) = X_\pm a(t)\,, \qquad X_\pm \equiv \frac{1+2\,\alpha_3+\alpha_4 \pm \sqrt{1+\alpha_3+\alpha_3^2-\alpha_4}}{\alpha_3+\alpha_4}\;(>0)\,, \tag{22}$$

and describe FRW cosmologies with $K < 0$.[1] In the present paper we will focus only on these nontrivial cosmological solutions.

[1]Note that $X_\pm$ are positive by definition since $\alpha(t) > 0$ and we assumed $a(t) > 0$. If we instead assumed $a(t) < 0$ then the corresponding solutions would be $\alpha(t) = -X_\pm a(t)$ with the same $X_\pm$ and we would conclude $X_\pm > 0$ again. The essential reason for the positivity of $X_\pm$ is that the square-root in (8) is the positive one.

Using the above constraint and varying the action (17) with respect to N and a, we obtain the remaining background equations

$$\begin{aligned} 3\,H^2 + \frac{3\,K}{a^2} &= \Lambda_\pm + \frac{1}{M_{Pl}^2}\rho\,, \\ -\frac{2\dot{H}}{N} + \frac{2\,K}{a^2} &= \frac{1}{M_{Pl}^2}(\rho + P), \end{aligned} \tag{23}$$

where ρ and P are the energy density and the pressure of matter fields calculated from I_{matter}, and

$$\Lambda_\pm \equiv -\frac{m_g^2}{(\alpha_3+\alpha_4)^2}\left[(1+\alpha_3)\left(2+\alpha_3+2\,\alpha_3^2-3\,\alpha_4\right) \pm 2\,\left(1+\alpha_3+\alpha_3^2-\alpha_4\right)^{3/2}\right]. \tag{24}$$

Thus, for the cosmological solutions (22), the contribution from the graviton mass term I_{mass} at the background level mimics a cosmological constant with the value $\Lambda_\pm$.

For $\alpha_4 = (3+2\,\alpha_3+3\,\alpha_3^2)/4$ and $\pm(1+\alpha_3) > 0$, the effective cosmological constant $\Lambda_\pm$ vanishes, and the background solution reduces to the open FRW universe solution of GR. On the other hand, both $X_\pm$ and $\Lambda_\pm$ diverge for $\alpha_4 = -\alpha_3$ and $\pm(1+\alpha_3) > 0$. In Figure 1, we show the sign of $\Lambda_\pm$ in the (α_3, α_4) space. Note that $X_\pm$ are restricted to be positive by definition, as explained in footnote 1.

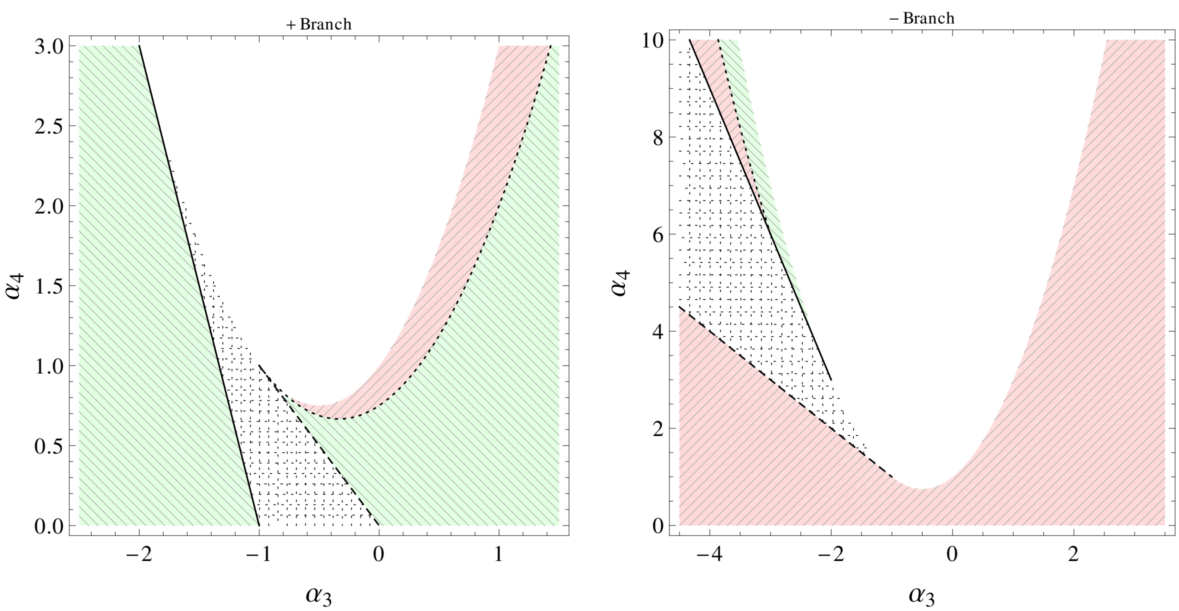

Figure 1: Sign of the effective cosmological constant $\Lambda_\pm$ in the positive (left panel) and negative (right panel) branches. In the red (green) region with $+45°$ $(-45°)$ lines, $\Lambda_\pm$ is positive (negative). The white region and the dotted squared region correspond to $1+\alpha_3+\alpha_3^2-\alpha_4 < 0$ and $X_\pm < 0$, respectively, and are excluded since the cosmological solutions (22) do not exist there. Along the dotted black line (defining the boundary between the red and green regions), $\Lambda_\pm = 0$ and the background solution reduces to the GR one. The solid line corresponds to $X_\pm = 0$ and thus defines one of the boundaries between the allowed (red or green) and excluded (dotted squared) regions. Along the dashed line, both $X_\pm$ and $\Lambda_\pm$ diverge, and it defines another boundary between the allowed (red or green) and excluded (dotted squared) regions.

3 FRW solutions with general fiducial metric

In appendix of [5], the open FRW solution was generalized to closed/flat/open FRW solutions by considering a fiducial metric of the general FRW type. In this section we review those general FRW solutions.

In this and next sections we consider the graviton mass term $I_{mass}[g_{\mu\nu}, f_{\mu\nu}]$ defined by (6)-(11), but with an arbitrary value of K and arbitrary functions $n(\varphi^0)$ and $\alpha(\varphi^0)$. We shall develop a formalism to analyze perturbations of this generalized system around flat ($K = 0$), closed ($K > 0$) and open ($K < 0$) FRW universes.

For the background we adopt the physical metric of the FRW form (16) with (14), but with general K, $n(\varphi^0)$ and $\alpha(\varphi^0)$. Without loss of generality, we assume that $N > 0$, $n > 0$, $a > 0$ and $\alpha > 0$ at least in the vicinity of the time of interest, where N and a are the lapse function and the scale factor of the background FRW physical metric. (Otherwise, we consider $|N|$, $|n|$, $|a|$ and $|\alpha|$ and rename them as N, n, a and α.)

We now derive the background equation of motion for the Stückelberg fields φ^a by expanding the graviton mass term I_{mass} up to the linear order in π^a $(= \delta\varphi^a)$ without variation of the physical metric $g_{\mu\nu}$. We define perturbations π^a of φ^a by

$$\varphi^a = x^a + \pi^a + O(\epsilon^2), \tag{25}$$

where ϵ is a small number counting the order of perturbative expansion: $\pi^a = O(\epsilon)$. By substituting this to the definition of the fiducial metric

$$f_{\mu\nu} = \bar{f}_{ab}(\varphi^c)\partial_\mu\varphi^a\partial_\nu\varphi^b, \tag{26}$$

where

$$\bar{f}_{00}(\varphi^c) = -n^2(\varphi^0), \quad \bar{f}_{0i}(\varphi^c) = \bar{f}_{i0}(\varphi^c) = 0, \quad \bar{f}_{ij}(\varphi^c) = \alpha^2(\varphi^0)\Omega_{ij}(\varphi^k), \tag{27}$$

we obtain

$$f_{\mu\nu} = \bar{f}_{\mu\nu}(x^\rho) + \mathcal{L}_\pi\bar{f}_{\mu\nu}(x^\rho) + O(\epsilon^2). \tag{28}$$

Here, $\mathcal{L}_\pi$ represents the Lie derivative along π^μ. Actually, this formula is not restricted to (27) but holds for any $\bar{f}_{ab}(\varphi^c)$.

Using the formula (28), $f_{\mu\nu}$ is expanded up to the linear order as

$$\begin{aligned}
f_{00} &= -n^2\left[1 + \frac{2}{n}\partial_t(n\pi^0) + O(\epsilon^2)\right], \\
f_{0i} &= f_{i0} = \alpha n\left[-\frac{n}{\alpha}D_i\pi^0 + \frac{\alpha}{n}\dot{\pi}_i + O(\epsilon^2)\right], \\
f_{ij} &= \alpha^2\left[(1 + 2nH_f\pi^0)\Omega_{ij} + D_i\pi_j + D_j\pi_i + O(\epsilon^2)\right],
\end{aligned} \tag{29}$$

where an overdot represents differentiation with respect to the time t and

$$H_f \equiv \frac{\dot{\alpha}}{n\alpha}. \tag{30}$$

With the unperturbed physical metric

$$g_{00} = -N^2, \quad g_{0i} = g_{i0} = 0, \quad g_{ij} = a^2\Omega_{ij}, \tag{31}$$

this leads to the following expansion for $f^\mu{}_\nu$ $(\equiv g^{\mu\rho}f_{\rho\nu})$.

$$\begin{aligned}
f^0{}_0 &= \frac{n^2}{N^2}\left[1 + \frac{2}{n}\partial_t(n\pi^0) + O(\epsilon^2)\right], \\
f^0{}_i &= -\frac{\alpha n}{N^2}\left[-\frac{n}{\alpha}D_i\pi^0 + \frac{\alpha}{n}\dot{\pi}_i + O(\epsilon^2)\right], \\
f^i{}_0 &= \frac{\alpha n}{a^2}\left[-\frac{n}{\alpha}D^i\pi^0 + \frac{\alpha}{n}\dot{\pi}^i + O(\epsilon^2)\right], \\
f^i{}_j &= \frac{\alpha^2}{a^2}\left[(1 + 2nH_f\pi^0)\delta^i_j + D^i\pi_j + D_j\pi^i + O(\epsilon^2)\right].
\end{aligned} \tag{32}$$

Then, expanding the matrix square-root, $\mathcal{K}^\mu{}_\nu$ defined by (8)-(9) is expanded up to the linear order as

$$\mathcal{K}^\mu{}_\nu = \mathcal{K}^{(0)\mu}{}_\nu + \mathcal{K}^{(1)\mu}{}_\nu + O(\epsilon^2), \tag{33}$$

where

$$\mathcal{K}^{(0)0}{}_0 = 1 - \frac{n}{N}, \quad \mathcal{K}^{(0)0}{}_i = 0, \quad \mathcal{K}^{(0)i}{}_0 = 0, \quad \mathcal{K}^{(0)i}{}_j = \left(1 - \frac{\alpha}{a}\right)\delta^i{}_j, \tag{34}$$

and

$$\begin{aligned}
\mathcal{K}^{(1)0}{}_0 &= -\frac{1}{N}\partial_t(n\pi^0), \\
\mathcal{K}^{(1)0}{}_i &= \frac{na}{N^2(1+r)}\left[-\frac{n}{\alpha}D_i\pi^0 + \frac{\alpha}{n}\dot{\pi}_i\right], \\
\mathcal{K}^{(1)i}{}_0 &= -\frac{n}{a(1+r)}\left[-\frac{n}{\alpha}D^i\pi^0 + \frac{\alpha}{n}\dot{\pi}^i\right], \\
\mathcal{K}^{(1)i}{}_j &= -\frac{\alpha}{2a}\left[2nH_f\pi^0\delta^i_j + D^i\pi_j + D_j\pi^i\right].
\end{aligned} \tag{35}$$

Here, we have defined

$$r \equiv \frac{na}{N\alpha}. \tag{36}$$

It is now straightforward to expand the graviton mass term (6) up to the first order. The result is

$$I_{mass} = I^{(0)}_{mass} + M_{Pl}^2 m_g^2 \int d^4x N a^3 \sqrt{\Omega}\frac{3n}{a}(aH - \alpha H_f)J_\phi \pi^0 + O(\epsilon^2), \tag{37}$$

where the zero-th order part $I^{(0)}_{mass}$ does not depend on π^a and

$$J_\phi \equiv 3 - 2X + \alpha_3(1-X)(3-X) + \alpha_4(1-X)^2, \quad X \equiv \frac{\alpha}{a}. \tag{38}$$

Therefore, the background equation of motion for the Stückelberg fields is

$$(aH - \alpha H_f)J_\phi = 0, \tag{39}$$

where H_f is defined in (30) and

$$H \equiv \frac{\dot{a}}{Na}. \tag{40}$$

Setting $aH = \alpha H_f$ would not allow nontrivial cosmologies since this branch does not evade Higuchi bound [6] and thus linear perturbations around the corresponding solution [7, 8] include ghost in the cosmological history. Thus, we shall not consider this branch and restrict our attention to solutions of $J_\phi = 0$. This leads to $X = X_\pm$, where $X_\pm$ are given by (22).

Using $J_\phi = 0$ and varying the action (17) with respect to N and a, we obtain the remaining background equations. They agree with those obtained in the previous section, i.e. (23) with (24).

What is interesting is that the Friedmann equation, including the value of the effective cosmological constant induced by the graviton mass term, does not depend on the nature of the fiducial metric. In particular, even if the Hubble expansion rate of the fiducial metric is of the Planck scale, the induced cosmological constant remains to be of the order of the graviton mass squared.

4 Perturbations of homogeneous isotropic solutions

As shown in the previous section, for general cases with arbitrary K, $n(\varphi^0)$ and $\alpha(\varphi^0)$, the background equation of motion for the Stückelberg fields φ^a has three branches of solutions. One of them does not allow nontrivial cosmologies because of the Higuchi bound, and thus is not of our interest. The other two branches of solutions allow nontrivial cosmologies and are given by (22)-(24) even for general K, $n(\varphi^0)$ and $\alpha(\varphi^0)$. In this section we then consider perturbations of the physical metric and the Stückelberg fields around the FRW solutions in these nontrivial branches, following ref. [5].

4.1 Exponential map and Lie derivative

Since the fiducial metric $f_{\mu\nu}$ is defined as in (10) without referring to the physical metric $g_{\mu\nu}$, let us begin with perturbations of the Stückelberg fields φ^a. We define perturbations π^a of φ^a through the so-called exponential map. Actually, since the action will be expanded only up to the quadratic order, we can truncate the exponential map at the second order. We thus define π^a by

$$\varphi^a = x^a + \pi^a + \frac{1}{2}\pi^b\partial_b\pi^a + O(\epsilon^3), \tag{41}$$

or equivalently,

$$\pi^a = (\varphi^a - x^a) - \frac{1}{2}(\varphi^b - x^b)\partial_b(\varphi^a - x^a) + O(\epsilon^3). \tag{42}$$

Here, ϵ is a small number counting the order of perturbative expansion: $\pi^a = O(\epsilon)$ and $\varphi^a - x^a = O(\epsilon)$. By substituting the expansion (41) to the definition of the fiducial metric

$$f_{\mu\nu} = \bar{f}_{ab}(\varphi^c)\partial_\mu\varphi^a\partial_\nu\varphi^b, \tag{43}$$

where $\bar{f}_{ab}$ is defined in (27), we obtain

$$f_{\mu\nu} = \bar{f}_{\mu\nu}(x^\rho) + \mathcal{L}_\pi\bar{f}_{\mu\nu}(x^\rho) + \frac{1}{2}\left(\mathcal{L}_\pi\right)^2\bar{f}_{\mu\nu}(x^\rho) + O(\epsilon^3). \tag{44}$$

Here, $\mathcal{L}_\pi$ represents the Lie derivative along π^μ. Actually, this formula is not restricted to (27) but holds for any $\bar{f}_{ab}(\varphi^c)$.

4.2 Stückelberg fields and gauge invariant variables

We define perturbations ϕ, β_i and h_{ij} of the physical metric by

$$\begin{aligned} g_{00} &= -N^2(t)\left[1+2\phi\right], \\ g_{0i} &= N(t)a(t)\beta_i \\ g_{ij} &= a^2(t)\left[\Omega_{ij}(x^k) + h_{ij}\right]. \end{aligned} \tag{45}$$

We suppose that $\phi, \beta_i, h_{ij} = O(\epsilon)$.

Under the linear gauge transformation

$$x^\mu \to x^\mu + \xi^\mu, \quad (\xi^\mu = O(\epsilon)) \tag{46}$$

each variable transforms as

$$\begin{aligned} \pi^0 &\to \pi + \xi^0, \\ \pi_i &\to \pi_i + \xi_i, \\ \phi &\to \phi + \frac{1}{N}\partial_t(N\xi^0), \\ \beta_i &\to \beta_i - \frac{N}{a}D_i\xi^0 + \frac{a}{N}\dot{\xi}_i, \\ h_{ij} &\to h_{ij} + D_i\xi_j + D_j\xi_i + 2NH\xi^0\Omega_{ij}, \end{aligned} \tag{47}$$

where H is the Hubble expansion rate as defined in (21),

$$\pi_i \equiv \Omega_{ij}\pi^j, \quad \xi_i \equiv \Omega_{ij}\xi^j, \tag{48}$$

and D_i is the spatial covariant derivative compatible with Ω_{ij}.

We then define gauge invariant variables

$$\begin{aligned}
\phi^\pi &\equiv \phi - \frac{1}{N}\partial_t(N\pi^0), \\
\beta_i^\pi &\equiv \beta_i + \frac{N}{a}D_i\pi^0 - \frac{a}{N}\dot{\pi}_i, \\
h_{ij}^\pi &\equiv h_{ij} - D_i\pi_j - D_j\pi_i - 2NH\pi^0\Omega_{ij}.
\end{aligned} \tag{49}$$

For later convenience, let us decompose β_i^π and h_{ij}^π as

$$\begin{aligned}
\beta_i^\pi &= D_i\beta^\pi + S_i^\pi, \\
h_{ij}^\pi &= 2\psi^\pi\Omega_{ij} + \left(D_iD_j - \frac{1}{3}\Omega_{ij}\triangle\right)E^\pi + \frac{1}{2}(D_iF_j^\pi + D_jF_i^\pi) + \gamma_{ij},
\end{aligned} \tag{50}$$

where S_i^π and F_i^π are transverse, and γ_{ij} is transverse and traceless:

$$D^iS_i^\pi = D^iF_i^\pi = 0, \qquad D^i\gamma_{ij} = 0, \quad \Omega^{ij}\gamma_{ij} = 0, \tag{51}$$

and $D^i \equiv \Omega^{ij}D_j$.

4.3 Graviton mass term

At the FRW background level, the graviton mass term acts as an effective cosmological constant $\Lambda_\pm$ shown in (24). The proof of this statement was presented in Sec. 3 for arbitrary K, $n(\varphi^0)$ and $\alpha(\varphi^0)$. Thus, calculations are expected to be simplified if we add $M_{Pl}^2\int d^4x\sqrt{-g}\Lambda_\pm$ to I_{mass} before performing perturbative expansion. For this reason, we define

$$\tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}] \equiv I_{mass}[g_{\mu\nu}, f_{\mu\nu}] + M_{Pl}^2\int d^4x\sqrt{-g}\,\Lambda_\pm, \tag{52}$$

and expand it instead of I_{mass} itself.

As shown explicitly in [5], upon using the background equation of motion for the Stückelberg fields but without using the background equation of motion for the physical metric, the graviton mass term can be expanded up to the quadratic order as

$$\begin{aligned}
\tilde{I}_{mass} &= \tilde{I}_{mass}^{(0)} + \tilde{I}_{mass}^{(2)}[h_{ij}^\pi] + O(\epsilon^3), \\
\tilde{I}_{mass}^{(2)}[h_{ij}^\pi] &= \frac{M_{Pl}^2}{8}\int d^4xNa^3\sqrt{\Omega}\,M_{GW}^2\left[(h^\pi)^2 - h_\pi^{ij}h_{ij}^\pi\right],
\end{aligned} \tag{53}$$

where the zero-th order part $\tilde{I}_{mass}^{(0)}$ is independent of perturbations,

$$\begin{aligned}
M_{GW}^2 &\equiv \pm(r-1)m_g^2X_\pm^2\sqrt{1+\alpha_3+\alpha_3^2-\alpha_4}, \\
r &\equiv \frac{na}{N\alpha} = \frac{1}{X_\pm}\frac{H}{H_f}, \quad H \equiv \frac{\dot{a}}{Na}, \quad H_f \equiv \frac{\dot{\alpha}}{n\alpha},
\end{aligned} \tag{54}$$

$X_\pm$ is given by (22), and

$$h^\pi \equiv \Omega^{ij}h_{ij}^\pi, \quad h_\pi^{ij} \equiv \Omega^{ik}\Omega^{jl}h_{kl}^\pi. \tag{55}$$

With the decomposition of h^π_{ij} in (50), the quadratic mass term is expanded as

$$\begin{aligned}\tilde{I}^{(2)}_{mass} &= M_{Pl}^2 \int d^4x N a^3 \sqrt{\Omega}\, M_{GW}^2 \\ &\times \left[3(\psi^\pi)^2 - \frac{1}{12} E^\pi \triangle(\triangle + 3K) E^\pi + \frac{1}{16} F^i_\pi (\triangle + 2K) F^\pi_i - \frac{1}{8}\gamma^{ij}\gamma_{ij} \right], \end{aligned} \tag{56}$$

where

$$F^i_\pi \equiv \Omega^{ij} F^\pi_j, \quad \gamma^{ik} \equiv \Omega^{jl}\Omega^{jl}\gamma_{kl}. \tag{57}$$

What is important here is that the quadratic part $\tilde{I}^{(2)}_{mass}$ is gauge-invariant and depends only on h^π_{ij}, or equivalently (ψ^π, E^π, F^π_i, γ_{ij}). In particular, it does not contribute to the equations of motion for ϕ and β_i.

We note that M_{GW}^2 vanishes or diverges for some special values of the parameters (α_3, α_4):

$$\begin{aligned} \alpha_4 = -3\,(1+\alpha_3), \quad \pm(\alpha_3+2) > 0 &\implies M_{GW}^2 = 0\,, \\ \alpha_4 = 1 + \alpha_3 + \alpha_3^2 \quad &\implies M_{GW}^2 = 0\,, \\ \alpha_4 \to -\alpha_3, \quad \pm(1+\alpha_3) > 0 &\implies |M_{GW}^2| \to \infty, \end{aligned} \tag{58}$$

where the $\pm$ signs are for the $\pm$ branches, respectively. In the following we suppose that the parameters (α_3, α_4) take generic values away from the special values shown in (58).

4.4 Matter perturbations and gauge-invariant variables

Let us divide matter fields σ_I ($I = 1, 2, \cdots$) into the background values $\sigma_I^{(0)}$ and perturbations as

$$\sigma_I = \sigma_I^{(0)} + \delta\sigma_I. \tag{59}$$

We suppose that $\{\sigma_I\}$ forms a set of mutually independent physical degrees of freedom. Otherwise, we consider a subset of the original $\{\sigma_I\}$ consisting of independent physical degrees of freedom and rename it as $\{\sigma_I\}$. We can construct gauge-invariant variables Q_I from $\delta\sigma_I$ and metric perturbations, without referring to the Stückelberg fields.

For illustrative purpose let us decompose β_i, h_{ij} and ξ_i as

$$\begin{aligned} \beta_i &= D_i\beta + S_i, \\ h_{ij} &= 2\psi\Omega_{ij} + \left(D_i D_j - \frac{1}{3}\Omega_{ij}\triangle \right) E + \frac{1}{2}(D_i F_j + D_j F_i) + \gamma_{ij}, \\ \xi_i &= D_i \xi + \xi_i^T, \end{aligned} \tag{60}$$

where S_i, F_i and ξ_i^T are transverse, and $\triangle$ is the Laplacian associated with Ω_{ij}:

$$D^i S_i = D^i F_i = D^i \xi_i^T = 0, \quad \triangle \equiv D^i D_i. \tag{61}$$

Under the gauge transformation (46), each component of the physical metric perturbation transforms as

$$\begin{aligned} \phi &\to \phi + \frac{1}{N}\partial_t (N\xi^0), \\ \beta &\to \beta - \frac{N}{a}\xi^0 + \frac{a}{N}\dot{\xi}, \\ \psi &\to \psi + NH\xi^0 + \frac{1}{3}\triangle\xi, \\ E &\to E + 2\xi, \\ S_i &\to S_i + \frac{a}{N}\dot{\xi}_i^T, \\ F_i &\to F_i + 2\xi_i^T, \\ \gamma_{ij} &\to \gamma_{ij}. \end{aligned} \tag{62}$$

Noting that the vector Z^μ defined by

$$Z^0 = -\frac{a}{N}\beta + \frac{a^2}{2N^2}\dot{E}, \quad Z^i = \frac{1}{2}\Omega^{ij}(D_j E + F_j) \tag{63}$$

transforms as

$$Z^\mu \to Z^\mu + \xi^\mu\,, \tag{64}$$

we can construct the following gauge-invariant variables out of matter perturbations and physical metric perturbations:

$$\begin{aligned}
Q_I &\equiv \delta\sigma_I - \mathcal{L}_Z \sigma_I^{(0)},\\
\Phi &\equiv \phi - \frac{1}{N}\partial_t(NZ^0),\\
\Psi &\equiv \psi - NHZ^0 - \frac{1}{6}\triangle E,\\
B_i &\equiv S_i - \frac{a}{2N}\dot{F}_i\,,
\end{aligned} \tag{65}$$

and γ_{ij} is gauge-invariant by itself. In the above, $\mathcal{L}_Z$ is the Lie derivative along Z^μ.

Those gauge-invariant variables defined here and in Subsection 4.2, i.e. $\{Q_I,\ \Phi,\ \Psi,\ B_i,\ \gamma_{ij},\ \phi^\pi,\ \beta^\pi,\ S_i^\pi,\ \psi^\pi,\ E^\pi,\ F_i^\pi\}$, are not independent. Indeed, it is easy to show that

$$\begin{aligned}
\phi^\pi &= \Phi + \frac{1}{N}\partial_t\left[\frac{1}{H}\left(\psi^\pi - \Psi - \frac{1}{6}\triangle E^\pi\right)\right],\\
\beta^\pi &= -\frac{1}{aH}\left(\psi^\pi - \Psi - \frac{1}{6}\triangle E^\pi\right) + \frac{a}{2N}\dot{E}^\pi,\\
S_i^\pi &= B_i + \frac{a}{2N}\dot{F}_i^\pi.
\end{aligned} \tag{66}$$

There are no more independent relations among gauge-invariant variables defined here and in Subsection 4.2.[2] Therefore, we have the following set of independent gauge-invariant variables.

$$\{Q_I,\ \Phi,\ \Psi,\ B_i,\ \gamma_{ij},\ \psi^\pi,\ E^\pi,\ F_i^\pi\}. \tag{67}$$

Based on their origins, we can divide this set of independent gauge-invariant variables into two categories as

$$\{Q_I,\ \Phi,\ \Psi,\ B_i,\ \gamma_{ij}\} \quad \text{and} \quad \{\psi^\pi,\ E^\pi,\ F_i^\pi\}. \tag{68}$$

The first category consists of those gauge-invariant variables that originate from the physical metric $g_{\mu\nu}$ and the matter fields $\{\sigma_I\}$. Thus, those in the first category already exist in GR coupled to the same matter content. On the other hand, those in the second category are physical degrees of freedom associated with the four Stückelberg fields φ^a.

4.5 Structure of total quadratic action

Let us now define

$$\tilde{I}[g_{\mu\nu},\sigma_I] \equiv I_{EH,\tilde{\Lambda}}[g_{\mu\nu}] + I_{matter}[g_{\mu\nu},\sigma_I], \quad \tilde{\Lambda} \equiv \Lambda + \Lambda_\pm, \tag{69}$$

so that

$$I = \tilde{I}[g_{\mu\nu},\sigma_I] + \tilde{I}_{mass}[g_{\mu\nu}, f_{\mu\nu}]. \tag{70}$$

Since $\tilde{I}_{mass}$ was already shown to be gauge-invariant up to the quadratic order, (70) implies that $\tilde{I}$ is also gauge-invariant up to that order. Thus the quadratic part $\tilde{I}^{(2)}$ of $\tilde{I}$ can be written in

[2] See also the sentence just after (59).

terms of gauge-invariant variables constructed solely from perturbations of the physical metric perturbations (ϕ, β_i, h_{ij}) and matter perturbations $\delta\sigma_I$, i.e. $\{Q_I,\ \Phi,\ \Psi,\ B_i,\ \gamma_{ij}\}$.

Therefore, the total quadratic action has the following structure.

$$I^{(2)} = \tilde{I}^{(2)}[Q_I, \Phi, \Psi, B_i, \gamma_{ij}] + \tilde{I}^{(2)}_{mass}[\psi^\pi,\ E^\pi,\ F^\pi_i, \gamma_{ij}], \tag{71}$$

where the explicit form of $\tilde{I}^{(2)}_{mass}$ is shown in (56). As already stated, gauge-invariant variables listed in (67) are independent from each other.

Note that ψ^π, E^π and F^π_i do not have kinetic terms but have non-vanishing masses, provided that the parameters (α_3, α_4) take generic values away from the special values shown in (58). Thus, we can integrate them out: their equations of motion lead to

$$\psi^\pi = E^\pi = 0, \quad F^\pi_i = 0, \tag{72}$$

and then

$$I^{(2)} = \tilde{I}^{(2)}[Q_I, \Phi, \Psi, B_i, \gamma_{ij}] - \frac{M_{Pl}^2}{8}\int d^4x N a^3\sqrt{\Omega} M_{GW}^2 \gamma^{ij}\gamma_{ij}, \tag{73}$$

where M_{GW}^2 is given by (54). For scalar and vector modes, this quadratic action is exactly the same as that in GR with the matter content $\{\sigma_I\}$.

In Fierz-Pauli theory in de Sitter background, it has been known that the scalar mode among five degrees of freedom of massive spin-2 graviton becomes ghost unless $2H^2 \leq m_{FP}^2$, where H is the Hubble expansion rate and m_{FP} is the graviton mass [6]. This conclusion, called Higuchi bound, does not hold in the nontrivial cosmological branches of the nonlinear massive gravity. Indeed, as stated above, the scalar and vector modes have vanishing kinetic terms. This sharp contrast to the linear (Fierz-Pauli) massive gravity stems from a peculiar structure of the graviton mass term expanded up to the quadratic order in perturbations: it depends only on the (ij)-components of metric perturbations and thus are independent of (00) and $(0i)$-components. This Lorentz-violating structure is possible because the vev of $f_{\mu\nu}$ in the cosmological branches spontaneously breaks diffeomorphism invariance in a nontrivial way.

4.6 Gravitational waves with time-dependent mass

The total quadratic action for the tensor sector is

$$I^{(2)}_{tensor} = \frac{M_{Pl}^2}{8}\int d^4x N a^3\sqrt{\Omega}\left[\frac{1}{N^2}\dot{\gamma}^{ij}\dot{\gamma}_{ij} + \frac{1}{a^2}\gamma^{ij}(\triangle - 2K)\gamma_{ij} - M_{GW}^2\gamma^{ij}\gamma_{ij}\right], \tag{74}$$

provided that there is no tensor-type contribution from the quadratic part of I_{matter}. In this way the dispersion relation of gravitational waves is modified. The squared mass of gravitational waves M_{GW}^2 is given by (54) and is time-dependent.

If M_{GW}^2 is negative then long wavelength gravity waves exhibit linear instability. For generic values of parameters (α_3, α_4) away from the special values shown in (58), we see from the formula (54) that the sign of M_{GW}^2 is the same as the sign of the combination $\pm(r-1)m_g^2$, where $\pm$ signs correspond to $\pm$ branches, respectively.

5 Nonlinear instability of homogeneous, isotropic solutions

Although a massive spin-2 particle generically has 5 propagating degrees of freedom, we have seen in the previous section that the number of propagating gravity degrees of freedom around the FRW solutions is 2, same as in general relativity (GR). This is due to the vanishing of the kinetic terms for the expected additional degrees.

The goal of this section is to determine the fate of the extra degrees of freedom. Following ref. [9], we argue that all homogeneous and isotropic solutions in nonlinear massive gravity are unstable. For this purpose, we study the propagating modes on a Bianchi type–I manifold. We analyze their kinetic terms and dispersion relations as the background manifold approaches the homogeneous and isotropic limit. We show that in this limit, at least one ghost always exists and that its frequency tends to vanish for large scales, meaning that it cannot be integrated out from the low energy effective theory. This ghost mode is interpreted as a leading nonlinear perturbation around a homogeneous and isotropic background.

5.1 The model and the background

Since we are interested in the stability of the gravity sector only, it is sufficient to consider a vacuum configuration, with a cosmological constant Λ.

The physical metric is chosen to be the simplest anisotropic extension of FRW, namely, the axisymmetric Bianchi type–I metric

$$ds^2 = -N^2 dt^2 + a^2 (e^{4\sigma}\, dx^2 + e^{-2\sigma}\, \delta_{ij}\, dy^i\, dy^j)\,, \tag{75}$$

where N, a, and σ are functions of the time variable t. In the rest of the paper, Greek indices span the space-time coordinates, while the indices $i, j = 2, 3$ correspond to the coordinates on the y–z plane, with $y^2 = y$, $y^3 = z$. Since our goal is to obtain the stability conditions of this metric in the isotropic limit, the whole system in this limit needs to reduce to the general cosmological solutions given in [4, 5] and reviewed in Sec. 2 and 3. For this reason, we consider a fiducial metric to be in the flat FRW form,

$$f_{\mu\nu} = -n^2 \partial_\mu \phi^0 \partial_\nu \phi^0 + \alpha^2 (\partial_\mu \phi^1 \partial_\nu \phi^1 + \delta_{ij} \partial_\mu \phi^i \partial_\nu \phi^j), \tag{76}$$

where both n and α are functions of the time-Stückelberg field ϕ^0.

The equations of motion for the background can be calculated by varying the action with respect to the Stückelberg fields and the metric. As a result, we obtain three independent equations as

$$\begin{aligned} 3\left(H^2 - \Sigma^2\right) - \Lambda &= m_g^2 \left[-(3\,\gamma_1 - 3\,\gamma_2 + \gamma_3) + \gamma_1\,(2\,e^{\sigma} + e^{-2\sigma}) X \right. \\ &\quad \left. - \gamma_2 (e^{2\sigma} + 2\,e^{-\sigma})\, X^2 + \gamma_3\, X^3 \right], \\ \frac{3\dot{\Sigma}}{N} + 9 H \Sigma &= m_g^2 (e^{-2\sigma} - e^{\sigma}) X \left[\gamma_1 - \gamma_2 (e^{\sigma} + r) X + \gamma_3\, r e^{\sigma} X^2 \right], \end{aligned} \tag{77}$$

and

$$J_\phi^{(x)}\,(H + 2\Sigma - H_f\, e^{-2\sigma} X) + 2 J_\phi^{(y)}\,(H - \Sigma - H_f e^{\sigma} X) = 0\,, \tag{78}$$

where

$$\begin{aligned} J_\phi^{(x)} &\equiv \gamma_1 - 2\,\gamma_2\, e^{\sigma}\, X + \gamma_3\, e^{2\sigma}\, X^2\,, \\ J_\phi^{(y)} &\equiv \gamma_1 - \gamma_2\,(e^{-2\sigma} + e^{\sigma})\, X + \gamma_3\, e^{-\sigma}\, X^2\,, \end{aligned} \tag{79}$$

and

$$\begin{gathered} \gamma_1 \equiv 3 + 3\,\alpha_3 + \alpha_4\,,\ \gamma_2 \equiv 1 + 2\,\alpha_3 + \alpha_4\,,\ \gamma_3 \equiv \alpha_3 + \alpha_4 \\ H \equiv \dot{a}/(aN)\,,\ H_f \equiv \dot{\alpha}/(\alpha n)\,,\ \Sigma \equiv \dot{\sigma}/N\,, \\ X \equiv \alpha/a\,,\ r \equiv an/(\alpha N). \end{gathered} \tag{80}$$

We note that, in the isotropic limit (σ, $\Sigma \to 0$), we have $J_\phi^{(x)} = J_\phi^{(y)}$, so that the Stückelberg equation of motion, Eq. (78), at leading order, gives

$$\gamma_1 - 2\,\gamma_2\, X + \gamma_3\, X^2 \simeq 0\,, \tag{81}$$

that is $X \to$ constant, which corresponds to the FRW result found in [5] and reviewed in the previous sections. In the same limit, we can also see that $H \to$ constant, as expected.

5.2 Even modes

Let us now consider the perturbations which transform as 2d scalars under a spatial rotation in the $y-z$ plane (also referred as even modes). Then, the perturbed metric for the even sector can be written as

$$\begin{aligned} ds^2 &= -N^2(1+2\Phi)dt^2 + 2aNdt[e^{2\sigma}\partial_x\chi dx + e^{-\sigma}\partial_i B dy^i] \\ &\quad + a^2 e^{4\sigma}(1+\psi)dx^2 + 2a^2 e^{\sigma}\partial_x\partial_i\beta dx dy^i \\ &\quad + a^2 e^{-2\sigma}[\delta_{ij}(1+\tau) + \partial_i\partial_j E]dy^i dy^j \, , \end{aligned} \tag{82}$$

while the even-type perturbations of Stückelberg fields read

$$\phi^0 = t + \pi^0 \, , \; \phi^1 = x + \partial_x \pi^1 \, , \; \phi^i = y^i + \partial^i \pi \, . \tag{83}$$

We can then define gauge invariant combinations as follows

$$\begin{aligned} \hat{\Phi} &= \Phi - \frac{1}{2N}\partial_t\left(\frac{\tau}{H-\Sigma}\right) , \\ \hat{\chi} &= \chi + \frac{\tau\, e^{-2\sigma}}{2a(H-\Sigma)} - \frac{ae^{2\sigma}}{N}\partial_t\left[e^{-3\sigma}\left(\beta - \frac{e^{-3\sigma}}{2}E\right)\right] , \\ \hat{B} &= B + \frac{e^{\sigma}}{2\,a\,(H-\Sigma)}\tau - \frac{a\,e^{-\sigma}}{2\,N}\dot{E} \, , \\ \hat{\psi} &= \psi - \frac{H+2\,\Sigma}{H-\Sigma}\tau - e^{-3\,\sigma}\,\partial_x^2\left(2\,\beta - e^{-3\,\sigma}\,E\right) , \\ \hat{\tau}_\pi &= \pi^0 - \frac{\tau}{2\,N\,(H-\Sigma)} , \\ \hat{\beta}_\pi &= \pi^1 - e^{-3\,\sigma}\left(\beta - \frac{e^{-3\,\sigma}}{2}\,E\right) , \\ \hat{E}_\pi &= \pi - \frac{1}{2}\,E \, . \end{aligned} \tag{84}$$

The first four definitions do not refer to the Stückelberg perturbations and are thus already present in GR. However, the additional three degrees arise from the breaking of general coordinate invariance by the non zero expectation value of the Stückelberg fields.

In order to find the behavior of the perturbations, we proceed as usual by expanding the action at second order in the perturbation fields, then by employing the Fourier plane-wave decompositions, as in $\exp[i(k_L x + k_i y^i)]$. The degrees of freedom arising from the $g_{0\mu}$ perturbations, namely $\hat{\Phi}$, $\hat{B}$ and $\hat{\chi}$, are nondynamical, thus can be integrated out. Furthermore, the kinetic term for the $\hat{\tau}_\pi$ is proportional to the background equations of motion, so that this degree of freedom is also nondynamical. We interpret this field as the would-be BD ghost, which is eliminated in this theory by construction.

In the massless theory (i.e. GR), using the constraint equations also removes the degrees $\hat{\beta}_\pi$, $\hat{E}_\pi$, leaving only $\hat{\psi}$ in the action, which becomes one of two gravity wave polarizations in the isotropic limit. However, in our case, due to the nonzero mass of the graviton, these two degrees of freedom are dynamical, in general.

Thus, the Lagrangian for even-type perturbations in vacuum has three physical propagating modes, $\mathcal{V}_a$, $(a = 1, 2, 3)$. Assuming small deviation from FRW, with $|\sigma| \ll 1$ and $|\Sigma/H| \ll 1$, we study the kinetic matrix $\mathcal{K}_{ab}$

$$S^{(2)}_{\rm even} \ni \frac{M_p^2}{2}\int N\,dt\,dk_L\,d^2k_T\,a^3\left(\frac{\dot{\mathcal{V}}^\star_a}{N}\mathcal{K}_{ab}\frac{\dot{\mathcal{V}}_b}{N}\right) . \tag{85}$$

Thanks to the 2d rotational symmetry on the y–z plane, the action depends on $k_T \equiv \sqrt{k_2^2 + k_3^2}$, instead of the individual components. The eigenvalues of $\mathcal{K}_{ab}$, at leading order in small anisotropy expansion, are

$$\kappa_1 \simeq \frac{p_T^4}{8\,p^4}\,,\ \kappa_2 \simeq -\frac{2a^4 M_{\rm GW}^2 p_L^2}{1-r^2}\sigma\,,\ \kappa_3 \simeq -\frac{p_T^2}{2\,p_L^2}\,\kappa_2\,, \tag{86}$$

where we defined $M_{\rm GW}^2 \equiv m_g^2(1-r)X^2(\gamma_2 - \gamma_3 X)$, and introduced the physical momenta

$$p_L \equiv \tfrac{k_L}{ae^{2\sigma}} \simeq \tfrac{k_L}{a}\,,\ p_T \equiv \tfrac{k_T}{ae^{-\sigma}} \simeq \tfrac{k_T}{a}\,,\ p^2 \equiv p_L^2 + p_T^2\,. \tag{87}$$

The kinetic term κ_1 which is the only eigenvalue that does not vanish in isotropic limit, corresponds to one of the gravity wave polarizations in FRW. Once small but non-vanishing anisotropy is introduced, two additional even modes acquire nonzero kinetic terms at quadratic order. More importantly, from (86), we see that κ_2 and κ_3 have opposite signs, regardless of the parameters of the theory. Thus, we conclude that in the isotropic limit, one of the new degrees is always a ghost. Assuming that $\sigma(1-r) > 0$ (which turns out to be the condition for stability in the odd sector, as we show later), the ghost mode is associated with the eigenvalue $\kappa_2 < 0$.

We conclude the discussion of the even modes by presenting their dispersion relations. We first make a field redefinition into new field basis fields $\mathcal{W}_a$ defined such that the kinetic action can be written as

$$S^{(2)}_{\rm even} \ni \frac{1}{2}\int N\,dt\,dk_L d^2k_T a^3 \left(\frac{\dot{\mathcal{W}}_a^\star}{N}\eta_{ab}\frac{\dot{\mathcal{W}}_b}{N}\right)\,, \tag{88}$$

where $\eta_{ab} = {\rm diag}(1,-1,1)$. The mass spectrum can be determined either by studying the equation for the frequency-discriminant, or equivalently, by performing a Lorentz transformation to diagonalize the frequency matrix. Eventually, we find

$$\begin{aligned}
\omega_1^2 &\simeq p^2 + M_{\rm GW}^2\,,\\
\omega_2^2 &\simeq -\frac{1-r^2}{24\sigma}\left[\sqrt{(10p^2+p_T^2)^2 - 8p_L^2 p_T^2} - (2p^2+3p_T^2)\right]\,,\\
\omega_3^2 &\simeq -\omega_2^2 + \frac{1-r^2}{12\sigma}\,(2p^2+3p_T^2)\,,
\end{aligned} \tag{89}$$

with $\omega_2^2\omega_3^2 < 0$ in general, and $\omega_2^2 < 0$ by assuming $\sigma(1-r) > 0$. We note that the dispersion relation corresponding to the ghost, ω_2^2, becomes smaller at larger scales. Therefore, at sufficiently large scales, this mode cannot be integrated out from the low energy effective theory. This feature makes the FRW background unstable for massive gravity. As a consequence, the homogeneous and isotropic cosmology cannot be accommodated in the nonlinear massive gravity theory.

5.3 Odd modes

Let us now discuss the odd sector (i.e. the divergence-less part of the modes which transform as 2d vectors under a rotation in the $y-z$ plane). The perturbed metric we consider is

$$\begin{aligned}
ds^2 &= -N^2dt^2 + 2ae^{-\sigma}Nv_i dt dy^i + 2a^2e^{\sigma}\partial_x\lambda_i dx dy^i\\
&\quad + a^2e^{4\sigma}dx^2 + a^2e^{-2\sigma}(\delta_{ij} + \partial_{(i}h_{j)})dy^i dy^j\,,
\end{aligned} \tag{90}$$

where $\partial_{(i}h_{j)} \equiv (\partial_i h_j + \partial_i h_j)/2$ and $\partial^i v_i = \partial^i\lambda_i = \partial^i h_i = 0$. For the Stückelberg fields, we consider instead

$$\phi^0 = t\,,\ \phi^1 = x\,,\ \phi^i = y^i + \pi^i\,, \tag{91}$$

where $\partial_i \pi^i = 0$. Since the vectors are defined on the 2d y–z plane, the transverse condition can be used to reduce each of these vectors to a single degree of freedom

$$v_i = \epsilon_i^{\ j} \partial_j v, \;\; \lambda_i = \epsilon_i^{\ j} \partial_j \lambda, \;\; h_i = \epsilon_i^{\ j} \partial_j h, \;\; \pi_i = \epsilon_i^{\ j} \partial_j \pi_{\rm odd},$$

where $\epsilon_i^{\ j}$ is a unit anti-symmetric tensor with $\epsilon_2^{\ 3} = -\epsilon_3^{\ 2} = 1$. Also for the odd modes we can introduce gauge invariant combinations as follows

$$\begin{aligned} \hat{v} &= v - \frac{a\, e^{-2\,\sigma}}{2\,N}\, \dot{h}\,, \\ \hat{\lambda} &= \lambda - \frac{e^{-3\,\sigma}}{2}\, h\,, \\ \hat{h}_\pi &= \pi_{\rm odd} - \frac{1}{2}\, h\,. \end{aligned} \tag{92}$$

Using these fields, the second-order resulting action depends on the three perturbations ($\hat{v}$, $\hat{\lambda}$, $\hat{h}_\pi$). Among these, $\hat{v}$ does not have any time derivatives and can be removed by solving its own constraint equation. In General Relativity, this operation also removes $\hat{h}_\pi$ and the final action can be written in terms of $\hat{\lambda}$ only. However, in this nonlinear theory of massive gravity, we expect the field $\hat{h}_\pi$ to remain in the action as an extra degree of freedom coming from the Stückelberg sector.

After a further field redefinition,

$$\mathcal{Q}_1 \equiv -e^{3\,\sigma}\,\hat{\lambda}\,, \qquad \mathcal{Q}_2 \equiv \frac{2\,e^{3\,\sigma}\,p_L^2}{p^2}\,\hat{\lambda} - 2\,\hat{h}_\pi\,, \tag{93}$$

the quadratic action, for small anisotropy, takes the following form

$$\begin{aligned} S^{(2)}_{\rm odd} \simeq \frac{M_{\rm Pl}^2}{2} \int N\,dt\,dk_L d^2k_T a^3 \Bigg[& K_{11} \frac{|\dot{\mathcal{Q}}_1|^2}{N^2} - \Omega_{11}^2\, |\mathcal{Q}_1|^2 \\ & + K_{22}\, \frac{|\dot{\mathcal{Q}}_2|^2}{N^2} - \Omega_{22}^2\, |\mathcal{Q}_2|^2 \Bigg]\,, \end{aligned} \tag{94}$$

where the two modes decouple at leading order in the small anisotropy expansion, with coefficients

$$\begin{aligned} K_{11} &= \frac{a^4\,p_L^2\,p_T^4}{2\,p^2}\,, \quad K_{22} = \frac{a^4\,p_T^2\,M_{GW}^2}{4\,(1-r^2)}\,\sigma\,, \\ \frac{\Omega_{11}^2}{K_{11}} &= p^2 + M_{\rm GW}^2\,, \quad \frac{\Omega_{22}^2}{K_{22}} = c_{\rm odd}^2\,p^2\,, \end{aligned} \tag{95}$$

and $c_{\rm odd}^2 = (1-r^2)/(2\sigma)$. Thus, at leading order, we identify the mode $\mathcal{Q}_1$ with one of the gravity wave polarizations in the FRW background [5]. The extra degree of freedom $\mathcal{Q}_2$ is massless and has sound speed $c_{\rm odd}$. In order for this mode to be stable, we require the kinetic term for $\mathcal{Q}_2$ to be positive, that is

$$(1-r)\,\sigma > 0\,. \tag{96}$$

In this case, also $c_{\rm odd}^2$ becomes positive, and the odd mode $\mathcal{Q}_2$ is, in general, free from ghost instabilities.

6 Anisotropic FRW solutions

As reviewed in Sec. 2 and 3, the nonlinear massive gravity allows self-accelerating open Friedmann-Robertson-Walker (FRW) solutions with the Minkowski fiducial metric [4] as well as flat/closed/open

FRW solutions with general FRW fiducial metric [5]. Unlike the other branch of solutions [7, 8], we have seen in Sec. 4 that these backgrounds evade the Higuchi bound [6] and thus are free from ghost at the linearized level even when the expansion rate is significantly higher than the graviton mass. This is because there are only two propagating modes on these backgrounds. However, as we have seen in Sec. 5, these constructions exhibit a ghost instability at nonlinear order in perturbations [9]. This is a consequence of the FRW symmetries; in order to obtain a stable solution, some of these symmetries need to be broken.

An inhomogeneous background solution was obtained in [10], where the observable universe is approximately FRW for a horizon size smaller than the Compton length of graviton. Similar solutions with inhomogeneities in the Stückelberg sector, meaning that the physical metric and the fiducial metric do not have common isometries acting transitively, were found in [11]. Note that those inhomogeneous solutions cannot be isotropic everywhere since isotropy at every point implies homogeneity [12]. Note also that cosmological perturbations can in principle probe inhomogeneities in the Stückelberg sector. For example, generic spherically-symmetric solutions are isotropic only when they are observed from the center of the universe.

The goal of this section is, following ref. [13], to introduce an alternative option, where the assumption of isotropy is dropped but homogeneity, i.e. the cosmological principle, is kept. In a region with relatively large anisotropy, we find an attractor solution. On the attractor, the physical metric is still isotropic, and the background geometry is of FRW type. Hence, the thermal history of the standard cosmology can be accommodated in this class of solutions. However, the Stückelberg field configuration is anisotropic, which may lead to effects at the level of the perturbations, suppressed by smallness of the graviton mass.

6.1 The action and background

We consider a simple description of the universe at present time. We assume that the late-time acceleration is sourced by a cosmological constant Λ, as well as the contribution from the graviton mass. (Setting $\Lambda = 0$ corresponds to self-accelerating solutions as in the example shown in Fig.2.) For this purpose, the vacuum configuration is sufficient.

For the physical metric, we adopt the axisymmetric Bianchi type–I metric (75), which is the simplest anisotropic extension of FRW ansatz. As for the fiducial metric, we assume the flat FRW form as (76). This includes a de Sitter fiducial as a special case, with $H_f \equiv \dot{\alpha}/\alpha n =$ constant.

Varying the Stückelberg fields around the background value $\phi^a = x^a + \pi^a$, the variation of the mass term up to first order gives the equation of motion (78). The expansion rate for the fiducial metric H_f is related to the invariants of the field space metric, and is independent of the choice of the background values of ϕ^a. Thus, Eq.(78) can be interpreted as an algebraic equation for α (or equivalently for X), instead of a differential equation. Varying the action with respect to $g_{\mu\nu}$, the field equations for the physical metric are obtained as (77). Additionally, there is also an equation for $\dot{H}$, which can be recovered by combining Eq.(78) with Eq.(77).

6.2 Fixed Points

We consider a de Sitter fiducial metric ($H_f = const.$) and seek solutions with $\dot{H} = \Sigma = \dot{X} = 0$. The constancy of X allows us to express H as $H = H_f\, X\, r$. In this setup, the independent equations become

$$\begin{aligned} 3\lambda - (3\gamma_1 - 3\gamma_2 + \gamma_3) + \gamma_1(2e^{\sigma} + e^{-2\sigma})X & \\ - \left[\gamma_2(2e^{-\sigma} + e^{2\sigma}) + 3\, r^2\mu^{-2}\right] X^2 + \gamma_3 X^3 &= 0, \quad (97)\\ (e^{\sigma} - 1)\left[\gamma_1 - \gamma_2(r + e^{\sigma})X + \gamma_3 e^{\sigma} r X^2\right] &= 0, \quad (98)\\ \gamma_1(3r - 2e^{\sigma} - e^{-2\sigma}) - 2\gamma_2\left[(2e^{\sigma} + e^{-2\sigma})r - (e^{2\sigma} + 2e^{-\sigma})\right] X & \\ + \gamma_3\left[(e^{2\sigma} + 2e^{-\sigma})r - 3\right] X^2 &= 0, \quad (99) \end{aligned}$$

where $\lambda \equiv \frac{\Lambda}{3m_g^2}$ and $\mu \equiv \frac{m_g}{H_f}$ are dimensionless parameters.

For $\sigma = 0$, the set of equations is reduced to that for isotropic configurations, which was already investigated in [4, 5] and reviewed in Sec. 2 and 3. Assuming $\sigma \neq 0$ and using (98), Eq.(99) can be rewritten as

$$(\gamma_1 - \gamma_2 X e^{\sigma})(e^{\sigma} - r)(re^{2\sigma} - 1) = 0\,. \tag{100}$$

Considering (100) as an algebraic equation for e^{σ}, there are three solutions:

$$e^{\sigma} = \left\{ \frac{\gamma_1}{\gamma_2\, X}\,,\ r\,,\ r^{-1/2} \right\}\,. \tag{101}$$

We now consider each solution separately.

<u>*Case I.*</u> $e^{\sigma} = \frac{\gamma_1}{\gamma_2\, X}$. Using this solution in Eq.(98) gives $X = \gamma_1/\gamma_2$, implying $\sigma = 0$. In other words, this solution is isotropic and thus is not of our interest.

<u>*Case II.*</u> $e^{\sigma} = r$. In this case, Eq.(98) gives

$$(r-1)\left[\gamma_1 - 2\,\gamma_2\, r\, X + \gamma_3\,(r\,X)^2\right] = 0\,. \tag{102}$$

This equations have two solutions; The first solution is $r = 1$, and leads to isotropy $\sigma = 0$ which is not our interest. The second solution gives $rX = (\gamma_2 \pm \sqrt{\gamma_2^2 - \gamma_1\gamma_3})/\gamma_3$, which reduces Eq.(97) to a nontrivial constraint between the parameters of the theory. Since this case requires a fine-tuning of a parameter, it is not of our interest either.

<u>*Case III.*</u> $r = e^{-2\,\sigma}$. With this solution, Eq.(98) is reduced to

$$\gamma_1 e^{\sigma} - \gamma_2(e^{2\sigma} + e^{-\sigma})X + \gamma_3 X^2 = 0. \tag{103}$$

while Eq.(97) becomes

$$\begin{aligned}&(3\lambda - 3\gamma_1 + 3\gamma_2 - \gamma_3) + \gamma_1(e^{-2\sigma} + 2e^{\sigma})X \\ &\quad -[\gamma_2(2e^{-\sigma} + e^{2\sigma}) + 3\,e^{-4\sigma}\,\mu^{-2}]X^2 + \gamma_3 X^3 = 0.\end{aligned} \tag{104}$$

Combining these two equations, we obtain an expression linear in X,

$$X = \frac{3\gamma_1 + [\gamma_1\gamma_2 - \gamma_3^2 + 3\gamma_3(\gamma_2 - \gamma_1 + \lambda)]\mu^2 e^{3\sigma}}{(e^{\sigma} + e^{-2\sigma})\,[3\gamma_2 + (\gamma_2^2 - \gamma_1\gamma_3)]\mu^2 e^{3\sigma}}. \tag{105}$$

and an equation which only depends on σ

$$c_0 + c_1 e^{3\sigma} + c_2 e^{6\sigma} + c_3 e^{9\sigma} = 0, \tag{106}$$

where

$$\begin{aligned}
c_0 &= 3\gamma_2\left(\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1\gamma_2 - \gamma_2\gamma_3 + 3\gamma_2\lambda\right)\mu^2 - 9\gamma_1^2\,,\\
c_1 &= (\gamma_2^2 - \gamma_1\gamma_3)\left[-6(3\gamma_1 - 3\gamma_2 + \gamma_3 - 3\lambda) + \left(\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1\gamma_2 - \gamma_2\gamma_3 + 3\gamma_2\lambda\right)\mu^2\right]\mu^2\,,\\
c_2 &= \left[3\gamma_2 + (2\gamma_2^2 - 2\gamma_1\gamma_3)\mu^2\right]\left(\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1\gamma_2 - \gamma_2\gamma_3 + 3\gamma_2\lambda\right)\mu^2\\
&\quad -(\gamma_1\gamma_2 - 3\gamma_1\gamma_3 + 3\gamma_2\gamma_3 - \gamma_3^2 + 3\gamma_3\lambda)^2\mu^4\,,\\
c_3 &= (\gamma_2^2 - \gamma_1\gamma_3)\left(\gamma_1^2 + 3\gamma_2^2 - 3\gamma_1\gamma_2 - \gamma_2\gamma_3 + 3\gamma_2\lambda\right)\mu^4\,,
\end{aligned} \tag{107}$$

For a given set of parameters $(\alpha_3, \alpha_4, \lambda, \mu)$, one can solve the cubic equation (106) for $e^{3\sigma}$ and then use (105) to calculate the corresponding value of X.

6.3 Local Stability

We now introduce homogeneous perturbations around the fixed point described in the third case above.

$$\begin{aligned} H &= H_f[r_0 X_0 + \epsilon\, h_1(t) + O(\epsilon^2)], \\ \sigma &= \sigma_0 + \epsilon\, \sigma_1(t) + O(\epsilon^2), \\ X &= X_0 + \epsilon\, X_1(t) + O(\epsilon^2), \end{aligned} \tag{108}$$

where $(X_0,\ \sigma_0,\ r_0 = e^{-2\sigma_0})$ is the background representing the fixed point. Adopting this expansion, we calculate the equations of motion up to $\mathcal{O}(\epsilon)$. At linear order, σ_1 can be decoupled from the remaining $\mathcal{O}(\epsilon)$ quantities, and a second-order evolution equation is obtained as

$$\frac{d^2\sigma_1}{d\tau^2} + 3X_0 e^{-2\sigma}\frac{d\sigma_1}{d\tau} + M^2\sigma_1 = 0, \tag{109}$$

where

$$\begin{aligned} M^2 &= \frac{X_0^2\mu^2 e^{-4\sigma_0}}{2}\left(\frac{d_1\,(3\,d_1 - d_2)(6 + d_1\,\mu^2)}{2\,d_2 - {d_1}^2\,\mu^2}\right), \\ d_1 &\equiv (e^{3\,\sigma_0} - 1)\,\left[\gamma_2 - \gamma_3 e^{\sigma_0}\,X_0\right], \\ d_2 &\equiv (e^{3\,\sigma_0} - 1)\,\left[\gamma_2(3 + 2\,e^{3\,\sigma_0}) - 5\,\gamma_3 e^{\sigma_0}\,X_0\right], \end{aligned} \tag{110}$$

while the dimensionless time coordinate τ is defined by $d\tau = H_f N dt$. The fixed point is locally stable if

$$M^2 > 0. \tag{111}$$

6.4 Global Stability

To study the global stability of the fixed point, we consider an example with

$$\lambda = 0\,, \quad \mu = 20\,, \quad \alpha_3 = -1/20\,, \quad \alpha_4 = 1\,, \tag{112}$$

for which the local stability condition (111) is satisfied. For this parameter set, there is only one set of real solutions to the equations of motion (97)-(99)

$$X \simeq 4\,, \quad e^{\sigma} \simeq \tfrac{1}{2}\,, \quad r \simeq 4\,. \tag{113}$$

In order to determine the phase flow, we first reduce the system of equations. Using Eq.(78), the first of Eq.(77) and their time derivatives, we can express X, H and their derivatives in terms of σ and Σ. Since these equations are nonlinear, there are always more than one solution. For the parameter set (112), we find that there are three branches of solutions which give $X > 0$ and $H > 0$. For each branch, we use this solution with the second of Eq.(77). As a result we obtain, for a set of (σ, Σ), the corresponding set of $(\dot{\sigma}, \dot{\Sigma})$ pair. Out of the three branches, only one contains an attractor. The flow corresponding to this branch is shown in Fig.2. The flow proceeds towards the fixed point we found in Eq.(113).

7 Summary and discussions

As reviewed in Sec. 2 and 3, the recently proposed nonlinear massive gravity [1] allows self-accelerating open Friedmann-Robertson-Walker (FRW) solutions with the Minkowski fiducial metric [4] as well as flat/closed/open FRW solutions with general FRW fiducial metric [5]. Unlike the other branch of solutions [7, 8], we have seen in Sec. 4 that these backgrounds evade the Higuchi bound [6] and thus are free from ghost at the linearized level even when the

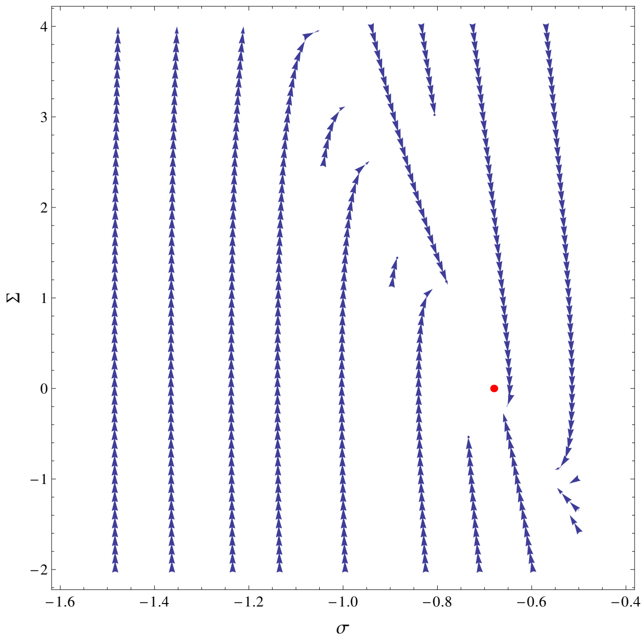

Figure 2: The phase flow for $(\sigma,\,\Sigma)$ for parameters $(\alpha_3,\,\alpha_4,\,\lambda,\,\mu)$ =$(-0.05,\,1,\,0,\,20)$. The flow is directed toward the red dot at $(\sigma,\,\Sigma)=(0.5,\,0)$, which is the fixed point obtained by solving Eqs.(97)-(99).

expansion rate is significantly higher than the graviton mass. This is because there are only two propagating modes on these backgrounds at the linearized level.

However, as we have seen in Sec. 5, these constructions exhibit a ghost instability at nonlinear order in perturbations [9]. Our conclusion about ghost instability is far more general than it appears [3], despite simplicity of the analysis presented above. This is because, whenever a quadratic kinetic term vanishes, the leading kinetic term is generically cubic and thus can easily become negative, signaling the existence of ghost at the nonlinear level. Moreover, the other type of homogeneous and isotropic solutions (in the non-self-accelerating branch) suffer from ghost instability already at the linearized level [7], as expected from classical work of Higuchi [6]. Therefore, all homogeneous and isotropic backgrounds, as well as most (if not all) of known spherically-symmetric inhomogeneous solutions, are unstable [9].

Thus, although the theory admits homogeneous and isotropic solutions, these suffer from an unavoidable nonlinear ghost [9] or a linear ghost [7]. Since this is a consequence of the FRW symmetries, either homogeneity or isotropy needs to be broken in order to obtain a stable solution.

In Sec. 6 we thus explored regions with relatively large anisotropy for homogeneous attractor solutions. The classification of fixed points revealed the existence of a single anisotropic attractor. The local and global stability analyses indicate that, a universe with a sufficiently large anisotropy at the onset of the late-time accelerated expansion should flow to this point.

A very interesting implication is that the scale factors corresponding to the two directions differ only by a constant normalization, thus the expansion rate is completely isotropic. In general relativity, such a solution will be identical to an isotropic universe, up to a coordinate redefinition. Conversely, in nonlinear massive gravity, such a redefinition cannot remove the anisotropy completely; it is merely shifted from the physical metric to the fiducial metric. Although the background metric is of FRW type, the signature from the anisotropy will be imprinted on the spectrum of cosmological perturbations. The statistical anisotropy signal is

[3]Partially massless gravity [14] may evade our conclusion but it is a different theory. Also, nonlinear completion is not known.

expected to be suppressed by smallness of the graviton mass m_g. The type of anisotropy, i.e. statistical anisotropy for perturbations without anisotropic background expansion, is totally new. For example, none of the anisotropic inflation scenarios [15] has this type of anisotropy. Detailed analysis of perturbations and comparison with observational data are worthwhile pursuing. As the first step, a preliminary analysis of perturbations indicates that the anisotropic attractor solutions found in this paper are free from ghost for a range of parameters [16].

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