On critical dimension in spherical black brane phase transition

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Abstract

We study the Gregory-Laflamme instability of a large uniform black brane wrapping a two-sphere compactification manifold. This work continues the study [1], where the compactifications on p-torus were considered. The new features of the spherical case are the non-zero curvature of the compactification manifold and the absence of the rescaling symmetry due to a built-in stabilization mechanism. We calculate the order of the phase transition in dependence on the number d of extended dimensions using the Landau-Ginzburg approach. It is found that for $d > 11$ a uniform spherical black brane in microcanonical ensemble exhibits a smooth second order phase transition towards a stable branch of non-uniform black brane solutions.

1 Introduction

Uniform extended black branes in the presence of compact extra dimensions are unstable with respect to long wavelength perturbations if the black brane horizon size is substantially smaller than the size of the extra dimensions. This is known as the Gregory-Laflamme (GL) instability [2]. Therefore a large black brane should undergo a phase transition once its size becomes smaller than a certain critical size. By studying non-uniform perturbations on top of the critical black brane it is possible to find the order of this phase transition. It could be either a smooth second (or possibly higher) order transition when the black brane becomes slightly non-uniform in the compact dimension in a continuos way or a first order transition when below the critical size the black brane decays into some completely different solution. The final stage of the first order transition is in general unknown (see [3, 4, 5] for review). The first study of this kind was performed by Gubser for a five-dimensional black string on a single compact extra dimension in pure gravity in which case the transition is first order [6]. Recently this calculation was generalized by Sorkin to black strings in arbitrary number of extended dimensions. He found that the phase transition becomes second order in more than twelve extended dimensions [7]. It was also claimed that the critical number of dimensions, when there is a change in the phase transition order, depends on whether the phase transition happens at fixed black string mass (in microcanonical ensemble) or at fixed temperature (in canonical ensemble). In the latter case the transition becomes of the second order in more then eleven extended dimensions [8]. Later Kol and Sorkin considered the case with an arbitrary number d of extended dimensions and an arbitrary number p of the extra dimensions compactified on the torus $\mathbf{T}^p$ [1]. In the special case when the sizes of all p circles are equal they found that the phase transition order depends only on the number d of extended dimensions and not on p. This result is explained by the fact that it is thermodynamically preferable for the GL instability to develop only along

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one of the circles on the torus. Therefore the toric black brane with $p > 1$ behaves effectively like the $p = 1$ black string.

We determine the order of the phase transition for a black brane on a two-sphere compactification manifold. We use the spontaneously compactified $\mathcal{M}_d \times \mathbf{S}^2$ solution of the Einstein-Maxwell theory in $D = d + 2$ dimensions as the background geometry [9]. The presence of the Maxwell field is necessary in order to have a non-flat compactification manifold. Aside from having a non-gravitational matter field this case has two important features in comparison to the flat compactification set-ups studied earlier which affect the properties of the phase transition. First, the two-sphere is not a direct product of two flat compact dimensions, and thus it does not support a mode of instability analogous to the modes along a single circle on the torus. Instability on $\mathbf{S}^2$ inevitably feels the presence of both compact dimensions and in this respect is more similar to the mode on the torus when the inhomogeneities along both circles are excited with equal amplitude (this mode is referred to as the "diagonal" mode in [1]). The other important difference is that the size of the two-sphere is fixed by the parameters of the theory. Therefore there is no rescaling freedom which in case of the flat compactifications accounts for the fact that the size of the compact dimension can be set arbitrarily. In the terms of dimensional reduction, the radion field is stabilized and has a mass comparable to the inverse radius of the compact two-sphere. Because of the absence of an internal length scale for the flat compactification case, a set of thermodynamical quantities invariant under rescaling was introduced in [6] in order to study the phase transition. The results obtained in such a way correspond to the situation when the size of the compact extra dimension is held fixed and the radion is infinitely heavy. Since the critical black hole size is comparable to the radius of the two-sphere, the radion mass in our set-up is naturally of the same order as the phase transition temperature. Therefore one can expect the presence of the dynamical radion to play a non-trivial role in the phase transition. The presence of these features suggests that the phase transition order for the spherical black brane could be different from the case of the flat toric compactification.

2 Ginzburg-Landau method

In the Landau-Ginzburg approach in order to study the order of phase transition in canonical ensemble it is sufficient to know the local behaviour of the free energy of the system around the critical point. A detailed description of this method in the context of black hole phase transitions is given in the reference [1] which we follow closely in our calculation. One first computes the local expansion of the free energy as a function of the order parameter and the temperature which in our case plays the role of the parameter that controls the onset of the transition. The role of the order parameter is played by the amplitude λ of the inhomogeneous perturbations in the metric and vector field. It shows the degree of the deviation of the black brane from the uniform solution and is also our perturbative expansion parameter. In order to compute other thermodynamic characteristics it is useful to know the free energy as the function of the inverse temperature. We expand the free energy up to the fourth order in λ around critical point:[1]

$$F(\lambda;\beta) \simeq F_0(\beta) + \mathcal{A}\left(\frac{\delta\beta}{\beta_*}\right)\lambda^2 + \mathcal{C}\,\lambda^4\,, \tag{1}$$

where $F_0(\beta)$ is the free energy for the uniform unperturbed black brane, β_* is the inverse critical temperature, and $\delta\beta \equiv \beta - \beta_*$. The values of the coefficients $\mathcal{A}$ and $\mathcal{C}$ in the expansion (1) define the local thermodynamics completely. The phase transition occurs when the black brane becomes smaller than a certain critical size, which means that $\delta\beta$ becomes negative. This fixes the sign of $\mathcal{A}$ to be positive in order for the uniform phase $\lambda = 0$ to become an unstable

[1]We checked that the odd terms in λ do not appear in the expansion of the black brane free energy.

extremum of the free energy for $\delta\beta < 0$. In the case when $\mathcal{C}$ is positive there is a minimum of the free energy for $\delta\beta < 0$ located at

$$\lambda_*^2 \equiv -\frac{\mathcal{A}}{2\mathcal{C}}\left(\frac{\delta\beta}{\beta_*}\right) . \tag{2}$$

The presence of the non-trivial minimum in the vicinity of the uniform phase signals a smooth second order phase transition towards the slightly non-uniform phase with $\lambda = \lambda_*$. The difference in free energies between the non-uniform and uniform black branes is of the fourth order in perturbative expansion parameter λ and is given by

$$F_*(\beta) - F_0(\beta) \simeq -\frac{\mathcal{A}^2}{4\,\mathcal{C}}\left(\frac{\delta\beta}{\beta_*}\right)^2 = -\,\mathcal{C}\,\lambda_*^4 . \tag{3}$$

The entropy $S(\lambda; M)$ can be computed by performing a Legendre transform of $\beta F(\lambda;\beta)$ with respect to β and is given by

$$S(\lambda; M) \simeq S_0(M) - \beta_* \frac{\mathcal{A}}{(d-3)}\left(\frac{\delta M}{M_*}\right)\lambda^2 - \beta_*\left(\mathcal{C} - \frac{\mathcal{A}^2}{2(d-2)(d-3)}\right)\lambda^4 . \tag{4}$$

The entropy as a function of mass determines the behaviour of the system in microcanonical ensemble. In full analogy with the canonical ensemble case the order of the phase transition in the microcanonical ensemble is determined by the sign of the coefficient in front of the fourth order in λ term.

The free energy of the black brane as a function of the metric and the vector field potential is given by the Euclidean action of the Einstein-Maxwell theory evaluated on the corresponding solution. The configuration space is spanned by the Euclidean solutions for $g_{\mu\nu}$ and V_μ that asymptote to this reference geometry. In our case the reference geometry is $\mathbf{S}^1_\beta \times \mathbb{R}^{d-1} \times \mathbf{S}^2_a$ with the Euclidean time period given by the inverse temperature β and a fixed radius a of the external two-sphere.

The non-uniform solution can be found perturbatively by expanding the equations of motion and field deviations above the background in the powers of perturbative parameter λ. The parameter λ also plays the role of the order parameter in the free energy expansion (1). In order to compute the free energy up to the fourth order in λ it is sufficient to find the solution up to the second order. The first order perturbation is nothing else but the static inhomogeneous Gregory-Laflamme mode, and the second order perturbation corresponds to the back-reaction of the black brane on the GL mode. In practice one expands the free energy in powers of metric and matter fields perturbations and then plugs back the solution for them. For determining the order of the phase transition one is interested in the $\mathcal{O}(\lambda^4)$ term in the free energy.

3 Results

We have found the solution for the metric and the vector field and computed the coefficients in the free energy expansion (1). By changing the parameters of numerical integration we found the change in the values of $\mathcal{C}$ to be less than a percent, which thus can serve as the estimate of the numerical error. The values of $\mathcal{C}$ for a various number of extended dimensions d are listed in table 1. Note the change of sign of the quartic coefficient for $d > 9$. For a thermodynamically stable system it would mean that the phase transition for $d > 9$ is of the second order in canonical ensemble.

It is instructive to compare the obtained behaviour of $\mathcal{C}$ for a black brane on a two-sphere with the case of a flat compactification on the square two-torus $\mathbf{T}^2$ considered by Kol and Sorkin in [1]. In the latter case there are two independent inhomogeneous modes corresponding to the two circles of $\mathbf{T}^2$. In order to study the free energy one can consider two limiting cases: when only a single mode along one of the two circles is excited, or when both of the modes are excited with equal amplitude, the so-called "diagonal" mode [1]. The quartic coefficient $\mathcal{C}$ in all three cases is presented in dependence of the number of extended dimensions d in figure 1. We see that the single direction mode on a torus has lower free energy and thus thermodynamically favourable. Due to this fact the toric black branes during the phase transition effectively behave like black strings, with only the mode along a single circle being excited. In contrast, on the spherical black brane there is only one mode, and its dependence on the number of extended dimensions d is different from the modes on $\mathbf{T}^2$. The change of the sign of the coefficient $\mathcal{C}$ for the spherical black brane happens between $d = 9$ and 10.

The behaviour of the black brane in microcanonical ensemble is determined by the sign of the coefficient σ_2 in the difference of the entropy between the non-uniform and uniform black branes. In order to find σ_2 the quadratic coefficient $\mathcal{A}$ should be determined from the first variation of the free energ with respect to the temperature, which at the leading order in λ can be expressed using only the solution with $\delta\beta = 0$. The resulting values of σ_2 in dependence on the number of extended dimensions d are given in the table 1 and presented in comparison to the case of the toric black brane in figure 2. For $d > 11$ the non-uniform black brane has larger entropy than the uniform one, and the phase transition in microcanonical becomes of the second order. We note that in microcanonical ensemble the critical number of extended dimensions for the spherical black brane coincides with the one for the diagonal inhomogeneous mode of the toric black brane.

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d	4	5	6	7	8	9	10	11	12	13
$\mathcal{A}$	0.544	1.73	3.59	6.15	9.39	13.3	17.9	23.2	29.2	35.8
$\mathcal{C}$	-0.114	-0.384	-0.785	-1.20	-1.41	-1.10	0.14	2.84	7.62	15.2
σ_2	-0.188	-0.317	-0.441	-0.536	-0.576	-0.535	-0.389	-0.112	0.322	0.938

Table 1: The coefficients $\mathcal{A}$ and $\mathcal{C}$ in the free energy expansion (1) and the entropy variation σ_2 for a different number of extended dimensions d. The change of the sign of σ_2 between $d = 11$ and 12 indicates that the phase transition in microcanonical ensemble becomes of the second order for $d > 11$. The analogous change in the free energy behaviour happens between $d = 9$ and 10.

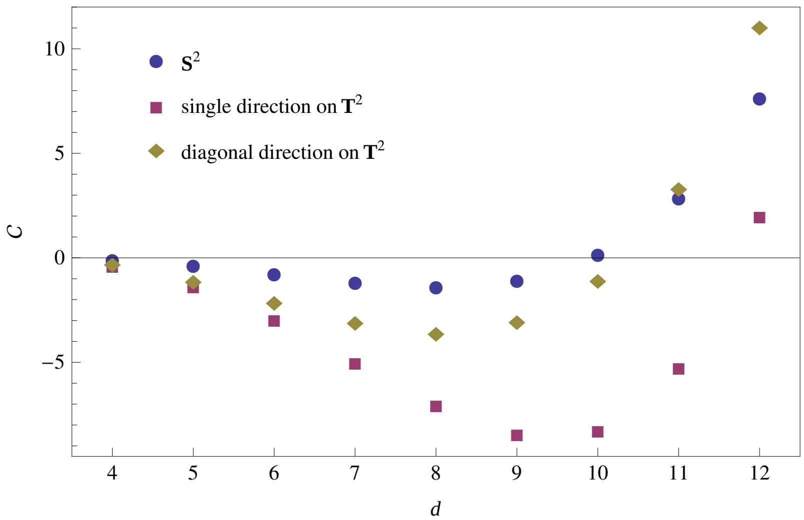

Figure 1: The quartic coefficient $\mathcal{C}$ in the free energy expansion (1) for the spherical black brane (circles) in a various number of extended dimensions d in comparison to the single direction (squares) and "diagonal" (dimonds) modes of the black brane on a square two-torus from [1].

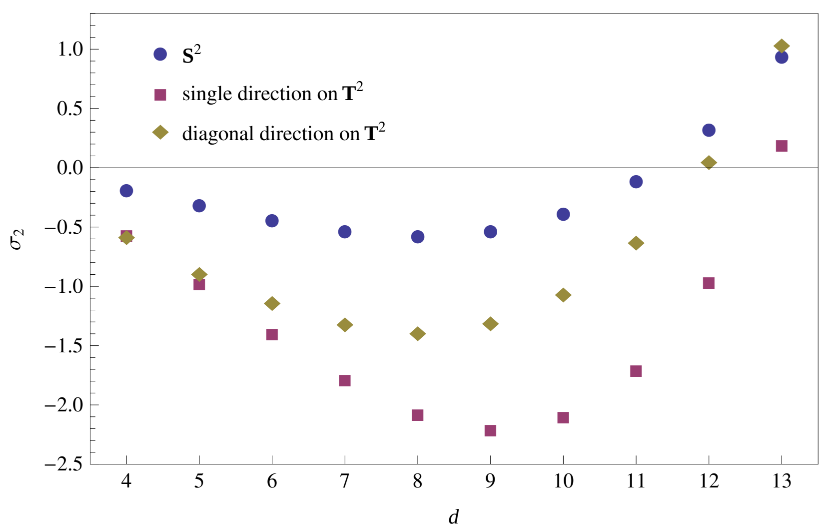

Figure 2: The coefficient σ_2 of the entropy difference between the non-uniform and uniform spherical black branes (diamonds) in comparison to the single direction (circles) and "diagonal" (squares) modes of the toric black brane from [1].