Oscillatory regime in multidimensional Gauss-Bonnet cosmology.

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Abstract

We examine a flat $(N+1)$-dimensional universe with Gauss-Bonnet gravity filled in with a homogeneous magnetic field and take an interest in the behavior of cosmological solutions near the singularity.

1. Overview.

In the sixties of the last century Belinskii, Khalatnikov and Lifshitz (BKL) ascertained that Kasner solution being a general solution for a vacuum Bianchi I Universe becomes unstable in the case of metric with positive spatial curvature (belonging to Bianchi IX class) and is replaced by a complicated sequence of transient "Kasner epochs" — BKL oscillations [1]. Later it was found that some classes of an anisotropic matter can induce similar type of cosmological behavior even in flat Bianchi I case. This can be shown for a magnetic field by LeBlanc [2] and for a general vector field by Kirillov et al [3].

At present, different modifications of General Relativity are widely used in cosmology. Therefore it is interesting to clear up does the oscillatory BKL-like regime exist in cosmological models with modified gravity. In this work we deal with Lovelock gravity; specifically, we consider Gauss-Bonnet term. Lovelock gravity gives corrections to GR only in higher-dimensional space-time; besides, cosmology with Gauss-Bonnet term in the (4+1)-dimensional case has some pathological features; so, in what follows we shall examine $(N+1)$-dimensional space-time with $N \geqslant 5$.

2. Model and basic equations.

We start from the action[1]

$$S = -\frac{1}{16\pi}\int d^{N+1}x\Big\{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 + F_{\alpha\beta}F^{\alpha\beta}\Big\}\,, \tag{1}$$

where $R, R_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, F_{\alpha\beta}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann tensor and the Faraday tensor respectively. The gravitational equations are given by:

$$2RR^{\mu}_{\nu} - 4R^{\mu}_{\gamma}R^{\gamma}_{\nu} - 4R^{\alpha\beta}R^{\mu}_{\cdot\,\alpha\nu\beta} + 2R^{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}\delta^{\mu}_{\nu}\Big(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\Big) =$$

$$= \frac{1}{4\pi}\left(F_{\nu\gamma}F^{\gamma\mu} + \frac{1}{4}\delta^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{2}$$

It is convenient to use the following metric parametrization:

$$ds^2 = dt^2 - \sum_k \mathrm{e}^{2a_k(t)}\,dx_k^2 \tag{3}$$

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[1]Here and after Greek indices run from 0 to N and Latin indices from 1 to N.

Thereupon equations (2) take the form:

$$F_{\alpha\gamma}F^{\gamma\beta}=0,\;\; \alpha\neq\beta \tag{4}$$

$$\sum_{i<j<k<l}\dot{a}_i\dot{a}_j\dot{a}_k\dot{a}_l=-\frac{1}{12}\sum_{i,j}F_{ij}^2e^{-2(a_i+a_j)} \tag{5}$$

$$\sum_{i\neq n}\left(\ddot{a}_i+\dot{a}_i^2\right)\sum_{\substack{j<k\\ j,k\neq i,n}}\dot{a}_j\dot{a}_k+3\sum_{\substack{i<j<k<l\\ i,j,k,l\neq n}}\dot{a}_i\dot{a}_j\dot{a}_k\dot{a}_l=\frac{1}{4}\left[\sum_{k\neq n}F_{nk}^2e^{-2(a_k+a_n)}-\sum_{\substack{i<j\\ i,j\neq n}}F_{ij}^2e^{-2(a_i+a_j)}\right] \tag{6}$$

The dot denotes the derivative with respect to t. From (4) it follows that some of the components of the Faraday tensor must be equal to zero identically; the number χ of non-zero components equals to $N/2$ for an even dimensions and $(N-1)/2$ for an odd dimensions.

3. Vacuum solution.

N. Deruelle [4] examined vacuum models with $N=4,5$ and revealed a Kasner-like cosmological solution:

$$ds^2=-dt^2+C_k^2t^{2p_k}dx_k^2 \tag{7}$$

where C_k are arbitrary constants and parameters p_k obey the relations:

$$\sum_n p_n=3,\qquad \sum_{i<j<l<m}p_ip_jp_lp_m=0 \tag{8}$$

A. Toporensky and P.Tretyakov [5] verified this solution for $N=6,7$; S.Pavluchenko [6] generalized this solution for all N and also to the Lovelock case.

4. Stability of the Kasner-like regime.

We considered the effect of the magnetic field as a perturbation. According to this solutions of the equations (5),(6) can be regarded as a disturbed solutions relative to vacuum solutions (7)-(8). As a consequence one should study stability of Kasner-like solutions. For this purpose it is convenient to use special variable τ introduced by the relation

$$d\tau=e^{-\frac{1}{3}\sum\limits_j a_j(t)}dt \tag{9}$$

Equations (5),(6) then take the form:

$$\sum_{i<j<k<l}a_i'a_j'a_k'a_l'=-\frac{1}{12}e^{\frac{4}{3}\sum\limits_j a_j}\sum_{i=\overline{1,\chi}}F_{2i-1,\,2i}^2e^{-2(a_{2i-1}+a_{2i})} \tag{10}$$

$$\sum_{\substack{i<j<k\\ i,j,k\neq 2n-1\,(2n)}}\left(a_i'a_j'a_k'\right)'=\frac{1}{6}e^{\frac{4}{3}\sum\limits_j a_j}\left[F_{2n-1,\,2n}^2e^{-2(a_{2n-1}+a_{2n})}-2\sum_{\substack{j=\overline{1,\chi}\\ j\neq n}}F_{2j-1,\,2j}^2e^{-2(a_{2j-1}+a_{2j})}\right] \tag{11}$$

The dash denotes the derivative with respect to τ. Power-law behavior of the components of the metric tensor with respect to t corresponds to a linear behavior with respect to τ:

$$g_{kk}(t)=C_kt^{2p_k}\quad\longrightarrow\quad g_{kk}(\tau)=p_k\tau+\widetilde{C}_k,\quad C_k,\widetilde{C}_k=\text{const}, \tag{12}$$

parameters p_k obey the relations (8). If a disturbed solution remains close to a Kasner-like solution beginning with some moment of time and approaches to the latter as $\tau\to\pm\infty$ we call the respective Kasner-like solution (asymptotically) stable.

Equations (8) specify $(N-2)$-dimensional surface in the space of parameters p_k; all parameters lying on that surface correspond to Kasner-like solutions. Our analysis shown that those of the parameters on this surface that obey the inequalities

$$p_{2n-1}+p_{2n}>2,\;\; n=\overline{1,\chi} \tag{13}$$

determine Kasner-like solutions which are stable for $\tau \to +\infty$ (future stable solutions). Analogously, parameters on the described surface that subject to the inequalities

$$p_{2n-1} + p_{2n} < 2, \; n = \overline{1, \chi} \tag{14}$$

determine Kasner-like solutions which are stable for $\tau \to -\infty$ (past stable solutions). This result is well confirmed by numerical calculations. Figure 1 shows past stable solution (a) and future stable solution (b) for $N = 5$; in the case (b) we used variable t instead of τ, because due to (9) derivatives $a'_k(\tau)$ grows up too rapidly and that makes impossible to carry out evaluations correctly.

5. Oscillatory regime.

We found numerically solutions of the equations (10),(11) which are far from any Kasner-like solutions over all time of evolution; these solutions has explicit oscillatory structure. On the Figure 2 oscillatory solution for the case of flat $(5+1)$-dimensional Gauss-Bonnet cosmological model with a homogenous magnetic field and BKL-oscillations are shown for comparison. These regimes differ greatly. For instant, the former has no Kasner epochs or something like this at all; in that case two of five functions oscillate, and the others grows up monotonically but very slowly. We checked this result for $N = 5, 6, 7, 8$ dimensions and wide range of initial data and got the same picture.

We have been able to obtain an appropriate analytical approximation of oscillatory solution. Let us assume that $a_1(\tau), a_2(\tau)$ are oscillatory functions; detailed analysis of numerical solutions makes it possible to think that

$$a'_3(\tau) \approx a'_4(\tau), \quad a'_5(\tau) \approx 0 \tag{15}$$

with a good accuracy for a large enough τ. Due to this assumption equations (10),(11) are simplified significantly. Then it is easy to show that

$$a_{1,2}(\tau) = \pm C_0 \tau^2 + \tau \left[C_1 J_{\frac{2}{3}} \left(\frac{2\xi\tau^{3/2}}{3} \right) + C_2 Y_{\frac{2}{3}} \left(\frac{2\xi\tau^{3/2}}{3} \right) \right] + D_{1,2}, \tag{16}$$

$$a_3(\tau) = \eta\sqrt{\tau} + D_3, \quad a_4(\tau) = \eta\sqrt{\tau} + D_4, \quad a_5(\tau) = D_5 \tag{17}$$

where $J_{\frac{2}{3}}$ and $Y_{\frac{2}{3}}$ are the Bessel functions of the first and second kind respectively; $C_0, C_1, C_2, D_1, \ldots, D_5, \eta$ are constant. Figure 3 illustrates comparison of numerical solution and analytical solution (16)-(17).

6. Conclusions.

1. Treatment the effect of the magnetic field as a perturbation lead us to the conclusion that there are two classes of vacuum Kasner-like solutions which are stable relative to this perturbation.

2. We revealed oscillatory regime which differ essentially from the well-known BKL oscillations and found its analytical approximation.

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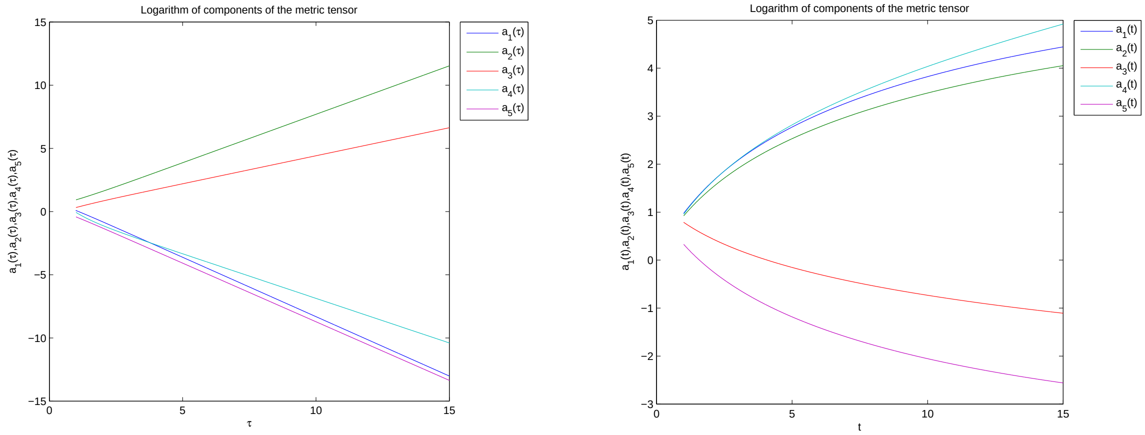
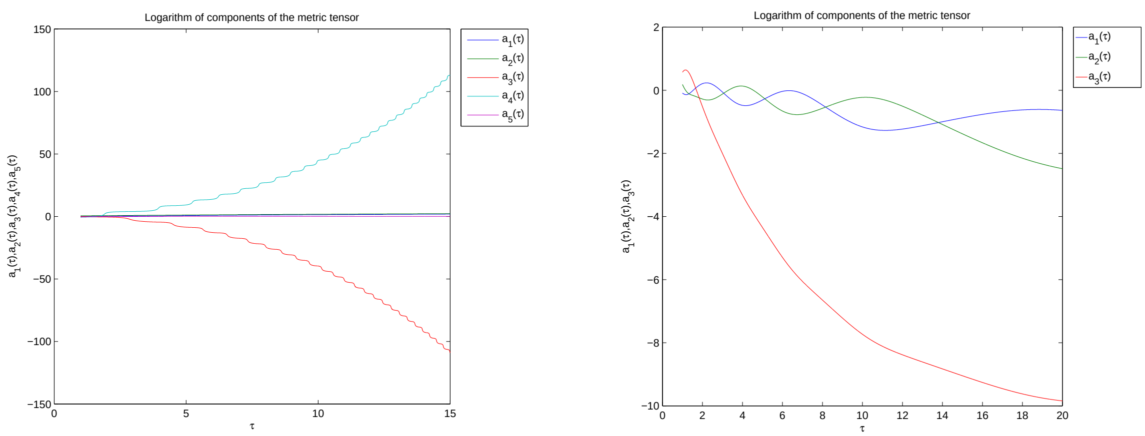


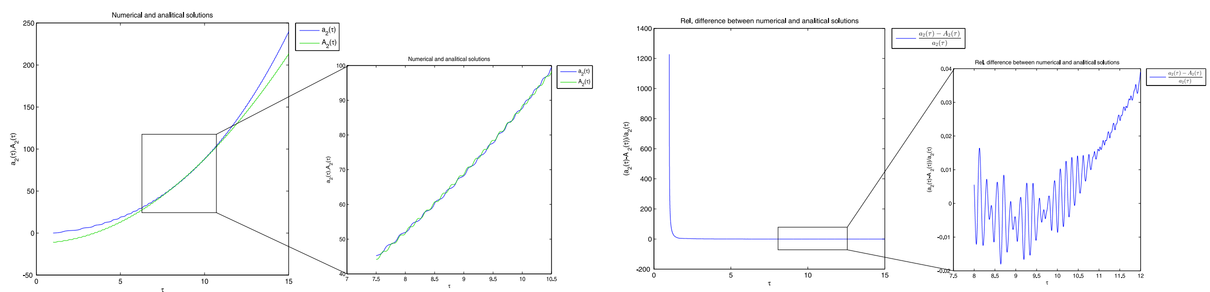
a) $p_{2n-1} + p_{2n} < 2, \ n = 1, 2$ b) $p_{2n-1} + p_{2n} > 2, \ n = 1, 2$

Figure 1: Stability of the Kasner-like regime.



a) Gauss-Bonnet+magnetic field b) BKL-oscillations

Figure 2: Oscillatory regime.



a) b)

Figure 3: a) Comparison of numerical (a_2) and analytical (A_2) solutions. b) Relative difference of numerical (a_2) and analytical (A_2) solutions.

$$A_2(\tau) = -\tau^2 + 0.7\tau J_{\frac{2}{3}}\left(10\tau^{\frac{3}{2}}\right) + 0.7\tau Y_{\frac{2}{3}}\left(10\tau^{\frac{3}{2}}\right) + 15.2.$$