Nonlocal gravity, Schwinger-Keldysh technique, dark energy and all that

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Abstract

We present a class of generally covariant ghost-free nonlocal gravity models generating a stable Einstein space background with an arbitrary value of the effective cosmological constant and having a good general relativistic limit in an asymptotically flat spacetime. The Euclidean version of the Schwinger-Keldysh formalism underlying the construction of the causal effective equations for the quantum expectation value of the metric field in these models is discussed. Though for cosmological (not asymptotically flat) problems they fail to have a general relativistic limit, indirectly this limit can be recovered in a special conformal frame nonlocally related to their original metric variables.

1 Introduction.

One of the main challenges of modern physics is the problem of dark energy (DE) – the mechanism which is supposed to explain observable cosmic acceleration [1]. Numerous efforts aimed to reconcile evidences for this phenomenon with gravity theory ([2, 3, 4, 5], etc.) suffer from the fine tuning problem associated with the hierarchy of the cosmic acceleration scale vs the fundamental Planck scale. Though this problem served as a motivation to go beyond the simplest appropriate modification of general relativity (GR) – explicit cosmological term, in this or that way it is creeping into almost all models of DE. Most of them in fact look as a sophisticated way to incorporate into their action in addition to the Planck scale the horizon scale (whether it is a graviton mass of massive gravity [5], multi-dimensional Planck mass in braneworld theories or the DGP scale in brane induced gravity models [4], etc.).

Somewhat separately stands the Deser-Woodard model of nonlocal cosmology with the nonlocal action of the form [6]

$$S \sim \int dx\, g^{1/2}\, R\, f\left(\frac{1}{\Box}R\right), \tag{1}$$

with a rather generic function $f(X)$ of the dimensionless nonlocal argument $X \equiv \Box^{-1}R$. Recently this model had a series of sound applications in the theory of the early and present Universe [7, 8, 9]. However, tuning the predictions in this model to observation suggests a concrete choice of the function $f(X)$ close to a hyperbolic function, which of course sounds very contrived to match this theory with some fundamental particle field model.

To circumvent this fine tuning difficulty one could adopt another, perhaps more promising, line of reasoning. If *a concrete fixed scale* incorporated in the model is not satisfactory, then one could look for a model that admits cosmic acceleration scenario with *an arbitrary scale*. Then its concrete value compatible with observations should arise dynamically by the analogue of symmetry breaking to be considered separately. Even this very unassuming approach is

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full of difficulties, because modified gravity models featuring this property (like unimodular gravity [10], $f(R)$-gravity [3], Lorentz breaking theory [11], etc.) generally violate some of its conventional symmetries and have additional degrees of freedom which might lead to ghost instabilities and make the theory inconsistent. This problem is central to numerous attempts to modify Einstein theory especially under the requirement to preserve its general covariance, and it will be a central question of this paper.

Here we discuss a recently suggested nonlocal infrared modification of the metric sector of the theory [12, 13], which is likely to implement the above approach. It is based on the realization of the old idea of a scale-dependent gravitational coupling – nonlocal Newton constant [14, 15, 16] – and amounts to the construction of the class of diffeomorphism invariant, ghost-free models and generating the de Sitter (dS) or anti-de Sitter (AdS) background with an *arbitrary value* of its effective cosmological constant Λ. In addition to fine-tuning argumentation of the above type, the driving force of this approach is the understanding of the fact that, to resolve such issues of DE as cosmic coincidence problem, this scale cannot be encoded in the fundamental or effective action of the theory (like, for instance, explicit Λ-term, massive graviton or $R + R^2/\Lambda$ models [17]), but rather should arise dynamically by the analogue of symmetry breaking (see, for example, [18]). Interestingly, as a bonus for the construction of the ghost-free cosmic acceleration we also get in our model a new mechanism of dark matter (DM) simulation.

Very briefly, this is the following alternative of the Deser-Woodard model

$$\int dx\, g^{1/2}\, R\, f\left(\frac{1}{\Box}R\right) \;\;\Rightarrow\;\; \int dx\, g^{1/2}\, R^{\mu\nu}\, f(\Box, R_{....})\, R_{\mu\nu} \tag{2}$$

with some operator function $f(\Box, R_{....})$ (involving also a nonlocal and nonlinear dependence on the curvature). This function will be made concrete and justified first on the flat-space background (Sect.2) and then extended to generic Einstein space solutions under the requirement of their stability (Sect.4). In Sect.3 we discuss the treatment of nonlocality within the Euclidean version of the Schwinger-Keldysh formalism [19, 20] for the quantum expectation values of the metric field, when the action of the form (8) is considered as the quantum effective action of the underlying fundamental quantum gravity theory. Sect. 5 is devoted to the problem of crossover to the general relativistic limit and recovery of the latter in the new conformal frame of the theory. Concluding Sect.6 contains the summary of results and their possible ramifications within critical gravity models [21] and black hole thermodynamics.

2 Flat-space background onset

Here we begin our search for a nonlocal modification of the Einstein theory within the concept of the effective scale-dependent gravitational constant. At a qualitative level this concept was introduced in [14] as an implementation of the idea that the effective cosmological term in modern cosmology is very small not because the vacuum energy of quantum fields is so small, but rather because it gravitates too little. This degravitation is possible if the effective gravitational coupling constant depends on the momentum scale and becomes small for fields nearly homogeneous at the horizon scale. Naive replacement of the Newton constant by a nonlocal operator suggested in [14] violates diffeomorphism invariance, but this procedure can be done covariantly due to the following observation [15]. The Einstein action in the vicinity of flat-space background can be rewritten in the form

$$S_E = \frac{M_P^2}{2}\int dx\, g^{1/2}\left(-R^{\mu\nu}\frac{1}{\Box}G_{\mu\nu} + \mathrm{O}[R^3_{\mu\nu}]\right), \tag{3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $1/\Box$ is the Green's function of the covariant d'Alembertian acting on a symmetric tensor[1]. This expression is nothing but a generally covariant version of the quadratic part of the Einstein action in metric perturbations $h_{\mu\nu}$ on a flat-space background. When rewritten in terms of the Ricci tensor $R_{\mu\nu} \sim \nabla\nabla h + O[h^2]$ this expression becomes nonlocal but preserves diffeomorphism invariance to all orders of its curvature expansion.

Thus, the idea of a nonlocal scale dependent Planck mass [14] can be realized as the replacement of M_P^2 by a nonlocal operator – a function $M^2(\Box)$ of $\Box$,

$$M_P^2 R^{\mu\nu} \frac{1}{\Box} G_{\mu\nu} \Rightarrow R^{\mu\nu} \frac{M^2(\Box)}{\Box} G_{\mu\nu}, \tag{4}$$

which would realize this idea at least within the lowest order of the covariant curvature expansion. This modification put forward in [14, 15] did not, however, find interesting applications because it has left unanswered a critical question – is this construction free of ghost instabilities for any nontrivial choice of $M^2(\Box)$? Here we try to fill up this omission and put some constraints on $M^2(\Box)$.

To begin with, if we adopt the above strategy, then the search for $M^2(\Box)$ should be encompassed by the correspondence principle. According to it nonlocal terms of the action should form a correction to the Einstein Lagrangian arising via the replacement

$$R \;\Rightarrow\; R + R^{\mu\nu} F(\Box) G_{\mu\nu}. \tag{5}$$

The nonlocal form factor of this correction $F(\Box)$ should be small in the GR domain, but it must considerably modify dynamics at the DE scale. Motivated by customary spectral representations for nonlocal quantities like

$$F(\Box) = \int dm^2\, \frac{\alpha(m^2)}{m^2 - \Box} \tag{6}$$

we might try the following ansatz,

$$F(\Box) = \frac{\alpha}{m^2 - \Box}, \tag{7}$$

corresponding to the situation when the spectral density $\alpha(m^2)$ is sharply peaked around some m^2. As we will see, for $m^2 \neq 0$ this immediately leads to a serious difficulty. Schematically the inverse propagator of the theory – the kernel of the quadratic part of the action in metric perturbations $h_{\mu\nu}$ – becomes $\sim -\Box + \alpha\Box^2/(m^2 - \Box)$ where the squared d'Alembertian $\Box^2$ follows from four derivatives contained in the term bilinear in curvatures. Then its physical modes are given by the two roots of this expression – the solutions of the corresponding quadratic equation $\Box = m_\pm^2$. In addition to the massless graviton with $m_-^2 = 0$ massive modes with $m_+^2 = O(m^2)$ appear and contribute a set of ghosts which cannot be eradicated by gauge transformations (for the latter were already expended on the cancelation of ghosts in the massless sector – longitudinal and trace components of $h_{\mu\nu}$).

Therefore, only the case of $m^2 = 0$ remains, and as a first step to the nonlocal gravity we will consider the action

$$S = \frac{M^2}{2} \int dx\, g^{1/2} \left(-R + \alpha\, R^{\mu\nu} \frac{1}{\Box} G_{\mu\nu} \right). \tag{8}$$

On the flat-space background this theory differs little from GR provided the dimensionless parameter α is small, $|\alpha| \ll 1$. The upper bound on $|\alpha|$ should follow from post-Newtonian corrections in this model. The additional effect of α is a small renormalization of the effective

[1] We use sign conventions for the Einstein action in the Euclidean signature spacetime and curvature tensor conventions, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \dots$.

Planck mass. Comparing the second term of (8) with (3) we have in the linearized theory the following relation

$$S = -\frac{M^2(1-\alpha)}{2}\int dx\, g^{1/2}R + \alpha O[h^3_{\mu\nu}], \tag{9}$$

which allows one to relate the constant M to M_P,

$$M^2 = \frac{M_P^2}{1-\alpha}. \tag{10}$$

3 Treatment of nonlocality: Schwinger-Keldysh technique vs Euclidean QFT

At this point we have to discuss the treatment of nonlocality in (8). In principle, handling fundamental theories with a nonlocal action is a sophisticated and very often an open issue, because their nonlocal equations of motion demand special care in setting boundary conditions. Contrary to local field theories subject to a clear Cauchy problem setup and local canonical commutation relations, nonlocal theories can have very ambiguous rules which are critical for physical predictions. In particular, the action (8) above requires specification of boundary conditions for the nonlocal Green's function $1/\Box$ which will necessarily violate causality in variational equations of motion for this action. Taken literally with any choice of boundary conditions for $1/\Box$, the action (8) effectively symmetrizes the kernel of this Green's function, so that nonlocal terms in equations of motion never have retarded nature and, therefore, break causality, as is easily seen from the equation

$$\frac{\delta S}{\delta g_{\mu\nu}(x)} \propto \nabla\nabla\int dy\,\Big[\,G(x,y)+G(y,x)\,\Big]\,R(y)+\dots\,. \tag{11}$$

To avoid these ambiguities and potential inconsistencies we will once and for all assume that our nonlocal action is not fundamental. Rather it is a certain approximation for the quantum effective action – the generating functional of one-particle irreducible diagrams – whose argument is the mean quantum field. This functional is necessarily nonlocal, and its nonlocality originates from quantum effects (by various mechanisms widely discussed in literature including [22]). In this case boundary conditions for nonlocal operations are uniquely fixed by the choice of the initial (and/or final) quantum state, and manifest breakdown of causality in variational equations for this action is harmless under a proper treatment of their nonlocal terms.

To begin with, this causality breakdown does not immediately signify inconsistency in the calculation of scattering amplitudes or in-out matrix elements. These amplitudes are determined by Feynman diagrammatic technique and do not have manifest retardation properties because they are not directly physically observable. Physically observable quantities like probabilities are bilinear combinations of scattering amplitudes and can always be represented as expectation values $\langle\, IN\,|\,\hat{\mathcal{O}}\,|\,IN\,\rangle$ of certain quantum operators $\hat{\mathcal{O}}$ in the initial quantum state $|\,IN\,\rangle$. For example, the probability of transition from this state to some final state $|\,\mathrm{fin}\,\rangle$, $P_{IN\to fin} = \langle\, IN\,|\,\mathrm{fin}\,\rangle\langle\,\mathrm{fin}\,|\,IN\,\rangle = \langle\, IN\,|\,\hat{P}_{\mathrm{fin}}|\,IN\,\rangle$, is an expectation value of the projector $\hat{P}_{\mathrm{fin}} \equiv |\,\mathrm{fin}\,\rangle\langle\,\mathrm{fin}\,|$ onto this final state. In contrast to in-out matrix elements these expectation values are subject to Schwinger-Keldysh diagrammatic technique [19] which guarantees causality of $\langle\, IN\,|\,\hat{\mathcal{O}}(x)\,|\,IN\,\rangle$. This property can be formulated as a retarded response of this average to the variation of the classical external source $J(y)$ coupled to the quantum fields in terms of which the observable $\hat{\mathcal{O}}(x)$ is built,

$$\frac{\delta\langle\, IN\,|\,\hat{\mathcal{O}}(x)\,|\,IN\,\rangle}{\delta J(y)} = 0, \quad x^0 < y^0. \tag{12}$$

This property is also not manifest and turns out to be the consequence of locality and unitarity of the original fundamental field theory (achieved via a complex set of cancellations between nonlocal terms with chronological and anti-chronological boundary conditions). However, there exists a class of problems for which a retarded nature of effective equations of motion explicitly follows from their quantum effective action calculated in the Euclidean spacetime [20]. This is a statement based on Schwinger-Keldysh technique [19] that for an appropriately defined initial quantum state $|IN\rangle$ the effective equations for the mean field $g_{\mu\nu} = \langle IN|\,\hat g_{\mu\nu}|IN\rangle$ originate from the *Euclidean* quantum effective action $\varGamma = \varGamma_{\rm Euclidean}[g_{\mu\nu}]$ by the following procedure [20][2]. Calculate the nonlocal $S_{\rm Euclidean}[g_{\mu\nu}]$ and its variational derivative. In the Euclidean signature spacetime nonlocal quantities, relevant Green's functions and their variations are generally uniquely determined by their trivial (zero) boundary conditions at infinity, so that this variational derivative is unambiguous in Euclidean theory. Then make a transition to the Lorentzian signature and impose the *retarded* boundary conditions on the resulting nonlocal operators,

$$\frac{\delta\varGamma_{\rm Euclidean}}{\delta g_{\mu\nu}(x)}\Bigg|^{\rm retarded}_{++++\;\Rightarrow\;-+++}=0. \qquad (13)$$

These equations are causal ($g_{\mu\nu}(x)$ depending only on the field behavior in the past of the point x in full accordance with Eq.(12)) and satisfy all local gauge and diffeomorphism symmetries encoded in the original $S_{\rm Euclidean}[g_{\mu\nu}]$.[3]

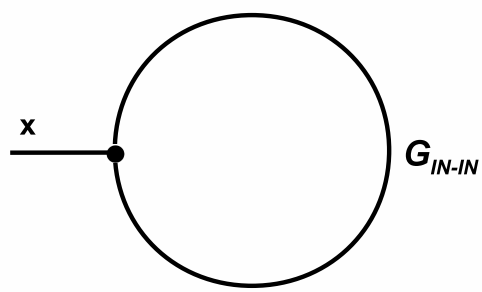

Figure 1: Tadpole diagram of the one-loop contribution to the IN-IN effective equation.

In the one-loop approximation the relation (13), that was proven to the first order of perturbation theory in [23] and to all orders in [20], originates as follows. The one-loop equation for the mean IN-IN field $g(x)$ contains the quantum contribution depicted by the tadpole diagram of Fig.1

$$\frac{\delta S}{\delta g(x)}+\frac{i}{2}\int dy\,dz\,\frac{\delta^3 S}{\delta g(x)\,\delta g(y)\,\delta g(z)}\,G_{IN\text{–}IN}(y,z)=0, \qquad (14)$$

$$G_{IN\text{–}IN}(x,y)=\langle IN\,|\,\hat g(x)\,\hat g(y)\,|\,IN\rangle, \qquad (15)$$

with the IN-IN Wightman Green's function $G_{IN\text{–}IN}(x,y)$ alternative to the conventional Feynman propagator. As was shown in [20] for the Poincare invariant vacuum state (associated with

[2]We formulate this statement directly for the case of gravity theory with the expectation value of the metric field operator $\hat g_{\mu\nu}(x)$, though it is valid in a much wider context of a generic local field theory [20].

[3]A similar treatment of a nonlocal action in [7] was very reservedly called the "integration by parts trick" needing justification from the Schwinger-Keldysh technique. However, this technique only provides the causality of effective equations, but does not guarantee the Euclidean-Lorentzian relation (13). The latter is based, among other things, on the choice of the $|IN\rangle$-state.

a plane wave decomposition of the IN-operators) the following relation holds

$$\frac{i}{2}\int dy\,dz\,\frac{\delta^3 S}{\delta g(x)\,\delta g(y)\,\delta g(z)}\,G_{IN\text{–}IN}(y,z) = \left.\frac{\delta \varGamma_E^{1-\rm loop}}{\delta g(x)}\right|^{\rm retarded}_{++++\Rightarrow-+++} \tag{16}$$

$$\varGamma_E^{1-\rm loop} = \frac{1}{2}\,{\rm Tr}\ln\frac{\delta^2 S_{\rm Euclidean}}{\delta g(x)\,\delta g(y)}. \tag{17}$$

This confirms the relation (13) with the full one-loop Euclidean effective action $\varGamma_{\rm Euclidean} = S_{\rm Euclidean} + \varGamma_E^{1-\rm loop}$.

We will assume that our model falls into the range of validity of this procedure, which implies a particular vacuum state $|{\rm in}\rangle$ and the one-loop approximation. The extension of this range is likely to include multi-loop orders and the $|{\rm in}\rangle$-state on the (A)dS background considered below, for which this state apparently coincides with the Euclidean Bunch-Davies vacuum.

At the heuristical level the justification for this extension follows from Fig.2 depicting the compact Euclidean spacetime used as a tool for constructing the Euclidean vacuum within a well-known no-boundary prescription [24]. Attaching a Euclidean space hemisphere to the Lorentzian de Sitter spacetime makes it *compact* instead of the original asymptotic de Sitter infinity. Thus it simulates by the path integral over regular field configurations on this spacetime the effect of the Euclidean de Sitter invariant vacuum. The role of spacetime *compactness* is very important here because it allows one to disregards possible surface terms originating from integrations by parts or using cyclic permutations under the functional traces in the Feynman diagrammatic technique for the effective action.

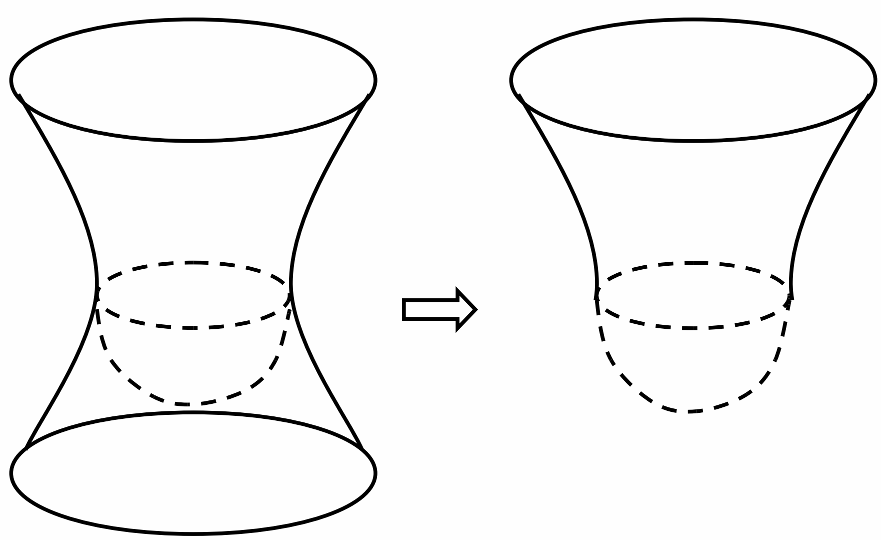

Figure 2: Euclidean de Sitter hemisphere denoted by dashed lines is used as a tool for constructing the Euclidean de Sitter invariant vacuum by the path integral over regular fields on the resulting compact spacetime.

Thus, the action (8) is understood as the Euclidean one (this explains our sign choice in the Einstein term) with zero boundary conditions for $1/\Box$ at infinity. It can be localized in terms of the auxiliary tensor field subject to the same Dirichlet boundary conditions[4], and in the resulting local representation directly applied to the FRW cosmology. This shows that close to a certain

[4]This field formally carries ghosts, but this does not indicate physical instability because it never exists as a

moment t_0 corresponding to the present epoch the model easily yields a (quasi) de Sitter point of the cosmological evolution [13], its Hubble factor $H = \dot{a}/a$ and the equation of state parameter $w = -1 - 2\dot{H}/3H^2$ satisfying the relations $w(t_0) = -1$, $\dot{w}(t_0) = O(1) \times H(t_0) < 0$, which make the model qualitatively compatible with the observable DE data. These preliminary estimates could have served as a starting point for a quantitative comparison with the DE scenario. However, a formal application of (8) to the FRW setup disregards nontrivial boundary conditions in cosmology. To see this, note that on the de Sitter background (which is a zeroth-order approximation for the cosmic acceleration scenario) the Ricci curvature $R_{\mu\nu} = \Lambda g_{\mu\nu}$ is covariantly constant, and the nonlocal part of (8) is divergent,

$$R^{\mu\nu}\frac{1}{\Box}G_{\mu\nu}\Big|_{(A)dS} = \infty, \tag{18}$$

because $g_{\mu\nu}$ is a zero eigenvector of $\Box$. This means that the action (8) should be modified to circumvent this difficulty.

4 Nonlocal gravity with a stable (A)dS background: DE and DM

We will regulate the action (8) by adding to the generally covariant $\Box$ the matrix-valued potential term built of a generic combination of tensor structures linear in the curvature,

$$S = \frac{M^2}{2}\int dx\, g^{1/2}\left(-R + \alpha\, R^{\mu\nu}\frac{1}{\Box + \hat{P}}\,G_{\mu\nu}\right), \tag{19}$$

$$\hat{P} \equiv P_{\alpha\beta}^{\;\;\mu\nu} = aR_{(\alpha\;\;\beta)}^{\;(\mu\;\;\nu)} + b\big(g_{\alpha\beta}R^{\mu\nu} + g^{\mu\nu}R_{\alpha\beta}\big) + cR_{(\alpha}^{(\mu}\delta_{\beta)}^{\nu)} + dR\, g_{\alpha\beta}g^{\mu\nu} + eR\delta_{\alpha\beta}^{\mu\nu}. \tag{20}$$

Here we use the condensed notation for the Green's function of the operator $\Box + \hat{P} \equiv \Box\,\delta_{\alpha\beta}^{\;\;\mu\nu} + P_{\alpha\beta}^{\;\;\mu\nu}$, acting on a symmetric tensor field as

$$\frac{1}{\Box + \hat{P}}\,G_{\mu\nu} \equiv \Big[\frac{1}{\Box + \hat{P}}\Big]_{\mu\nu}^{\alpha\beta} G_{\alpha\beta} \tag{21}$$

and a, b, c, d and e represent arbitrary parameters to be restricted by the requirement of a stable (A)dS solution in the model. Of course, such a modification of the original action (8) leaves its linear approximation on a flat background intact, because it deals with $O[h^3_{\mu\nu}]$-terms.

Now the Green's function $1/(\Box + \hat{P})$ acting on the Einstein and Ricci tensors in (19) is well defined even for the (A)dS background with the covariantly constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and $R_{\alpha\mu\beta\nu} = \frac{\Lambda}{3}(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu})$, for which

$$P_{\alpha\beta}^{\;\;\mu\nu} = \frac{A + 4B}{4}\,\Lambda g_{\alpha\beta}g^{\mu\nu} - C\Lambda\left(\delta_{\alpha\beta}^{\mu\nu} - \frac{1}{4}\,g_{\alpha\beta}g^{\mu\nu}\right), \tag{22}$$

$$A = a + 4\,b + c, \quad B = b + 4\,d + e, \tag{23}$$

$$C = \frac{a}{3} - c - 4e, \tag{24}$$

so that $\hat{P}\,g_{\mu\nu} \equiv P_{\mu\nu}^{\;\;\alpha\beta}\,g_{\alpha\beta} = (A + 4B)\,\Lambda g_{\mu\nu}$.

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free field in the external lines of Feynman graphse [13]. The actual particle content of the theory is determined in terms of the original metric field $g_{\mu\nu}$ and indeed turns out to be ghost-free on the flat-space background, because the quadratic part of the action coincides with the Einstein's one. A similar mechanism excluding ghosts by boundary conditions was recently used in the conformal gravity model of [25].

The properties of this model are as follows. First, under a certain restriction on parameters of $\hat{P}$ the model (19) has (A)dS solution with an *arbitrary* value of the cosmological constant Λ. Since

$$\frac{\delta S}{\delta g_{\mu\nu}}\Big|_{\rm (A)dS} = -\frac{1}{2}M^2\Lambda\left(1+\frac{\alpha}{A+4B}\right)g^{\mu\nu}g^{1/2}, \tag{25}$$

the equation of motion holds with an arbitrary value of Λ when

$$\alpha = -A - 4B. \tag{26}$$

This property of the model was generalized in [26] to generic Einstin space backgrounds with

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}R = 0. \tag{27}$$

Note that the existence of the Einstein space solution with an arbitrary Λ is neither the result of the local Weyl invariance of the theory, nor even its global scale invariance. Rather this is a corollary of the relation (26) which, in particular, guarantees the vanishing on-shell value of the action $S\,|_{\rm (A)dS} = 0$. Thus, this solution is another vacuum – a direct analogue of the flat-space one.

Another remarkable consequence of Eq.(26) is the stability of the Einstein space against ghost and tachyon excitations. The check of this property is based on the calculation of the quadratic part of the action $S_{(2)}$ on this background in the DeWitt gauge,

$$\chi^\mu \equiv \nabla_\nu h^{\mu\nu} - \frac{1}{2}\,\nabla^\mu h = 0. \tag{28}$$

Curious fact is that in this gauge it depends only on the traceless part of $h_{\mu\nu}$, $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}h$, and is very simple [26]

$$S_{(2)}\Big|_{E_{\mu\nu}=0} = -\frac{M_{\rm eff}^2}{2}\int d^4x\,g^{1/2}\big(D_1\bar{h}^{\mu\nu}\big)\,\frac{1}{D_2}\,\big(D_1\bar{h}_{\mu\nu}\big). \tag{29}$$

$$M_{\rm eff}^2 = M^2\,\frac{A^2-\alpha^2}{\alpha} \tag{30}$$

$$D_1 \equiv \Box + 2\,\hat{W} - \frac{1}{6}\,R\,\hat{1}, \tag{31}$$

$$D_2 \equiv \Box + a\,\hat{W} - \frac{C}{4}\,R\,\hat{1}, \tag{32}$$

so that the equality of two differential operators $D_1 = D_2$ guarantees a local nature of S_2. Thus the demand of unitarity leads to the additional constraints on the parameters of the model

$$a = 2, \tag{33}$$

$$C \equiv \frac{a}{3} - c - 4e = \frac{2}{3}, \tag{34}$$

and the positivity requirement for $G_{\rm eff} \equiv 1/8\pi M_{\rm eff}^2$ (the condition (33) is not necessary on maximally symmetric background with $\hat{W} = 0$ and, thus, was derived in [26] in the course of generalizing the model of [12, 13] to generic Einstein spaces). This requirement selects the range of the parameter B, $B < -\alpha/2$ and $B > 0$ for a positive α, and even more interesting compact range of B for a negative α,

$$0 < B < -\frac{\alpha}{2}, \quad \alpha < 0. \tag{35}$$

Free propagating modes of the theory satisfy the linearized equations obtained from the Euclidean action by the recipe (13). For a particular case of the Einstein space – the (A)dS background – they read

$$\left(-\Box+\frac{2}{3}\mathit{\Lambda}\right)h_{\mu\nu}+\frac{1}{2}\,g_{\mu\nu}\left(\Box+\frac{2}{3}\mathit{\Lambda}\right)h$$
$$+\frac{1}{2}\,g_{\mu\nu}R_{(1)}+2\,\nabla_{(\mu}\mathit{\Phi}_{\nu)}-g_{\mu\nu}\nabla_\alpha\mathit{\Phi}^\alpha=0, \qquad (36)$$
$$R_{(1)}\equiv\nabla_\mu\nabla_\nu h^{\mu\nu}-\Box h-\mathit{\Lambda}h, \qquad (37)$$
$$\mathit{\Phi}^\mu\equiv\chi^\mu-\frac{1}{2}\,\nabla^\mu\frac{1}{\Box+2\mathit{\Lambda}}\Big|_{\rm ret}R_{(1)}. \qquad (38)$$

The trace of this equation according to (13) gives the homogeneous equation for $R_{(1)}$ with a retarded nonlocality, which is equivalent to the local initial Cauchy problem $\Box R_{(1)}=0$ with zero initial data in the remote past [13]. Therefore $R_{(1)}(x)\equiv 0$, whence in the DeWitt gauge $\chi^\mu=0$ we have $(\Box+2\Lambda)\,h=0$. Then the residual gauge transformations $\Delta^f h_{\mu\nu}=2\nabla_{(\mu}f_{\nu)}$ with the parameter f_μ satisfying the equation $(\Box+\Lambda)f_\mu=0$ can be used to select two polarizations – non-ghost physical modes. In particular, the boundary conditions for h can be nullified, so that h identically vanishes and makes in view of the DeWitt gauge the propagating free modes transverse and traceless as in the Einstein theory with the Λ-term.

In the presence of matter sources with a stress tensor $T_{\mu\nu}$ of a compact support the causal effective equations for retarded gravitational potentials become local in the DeWitt gauge,

$$\left(-\Box+\frac{2}{3}\Lambda\right)h_{\mu\nu}+\frac{1}{2}\nabla_\mu\nabla_\nu h-\frac{\Lambda}{6}g_{\mu\nu}h=\frac{2}{M_{\rm eff}^2}T_{\mu\nu}. \qquad (39)$$

Modulo the gauge transformation their solution takes the following form – the result of a careful commutation of covariant derivatives with $(-\Box+\frac{2}{3}\Lambda)^{-1}$,

$$h_{\mu\nu}=\frac{8\pi G_{\rm eff}}{-\Box+\frac{2}{3}\Lambda}\left(T_{\mu\nu}+g_{\mu\nu}\frac{\Box-2\Lambda}{\Box+2\Lambda}\frac{\Lambda}{3\Box}T\right). \qquad (40)$$

The tensor structure here differs from the GR analog $T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T$, which for non-relativistic sources gives $O(1)$ correction. What is much more interesting, it yields an unexpected bonus in the form of the dark matter simulation – $1/|\alpha|$-amplification of the gravitational attraction due to the replacement of the Newton gravitational constant G_N by $G_{\rm eff}\sim G_N/|\alpha|$ with $|\alpha|\ll 1$. This is possible when $|B|\sim\alpha$ for a positive α and necessarily happens in the case (35) of a negative α, because the factor $\alpha/8B(2B+\alpha)\geq 1/|\alpha|$ and

$$G_{\rm eff}\geq\frac{1-\alpha}{|\alpha|}\,G_N\gg G_N. \qquad (41)$$

On the other hand, with $|B|\sim\sqrt{\alpha}$ in the case of a positive α both Newton and effective gravitational coupling constants can be of the same order of magnitude, $G_N/G_{\rm eff}=O(1)$ even for $\alpha\ll 1$, which together with (41) leaves a large window for a possible strength of DM attraction relative to GR behavior.

The theory (19) seems to have two phases. For short distances corresponding to the range of wavelengths with $\nabla\nabla\sim\Box\gg R$ this is a GR phase on the zero curvature background with small $O(\alpha)\times R/\Box$ corrections of higher orders in spacetime curvature (collectively denoted by R). This regime would apply to galactic, Solar system and other small scale phenomena and is likely to pass all general relativistic tests for a sufficiently small α.

Another phase of the theory should correspond to the infrared wavelengths range $\nabla\nabla\ll R$ in which a stable (A)dS background exists and the modified gravitational potential of matter sources is given by Eq.(40). This equation is valid for the perturbation range $|\delta R^\mu_\nu| \sim |\nabla\nabla h^\mu_\nu| \ll \Lambda$ and $|h^\mu_\nu| \ll 1$ equivalent to very small matter densities $|T^\mu_\nu| \ll M_P^2 \Lambda$ characteristic of galaxy, galaxy cluster and horizon scales for which DE and DM modification of gravity theory becomes important. Thus, nonlocal gravity is expected to interpolate between GR theory and its strongly coupled infrared modification which is likely to generate a stable ghost-free stage of cosmic acceleration and, perhaps, even simulate the DM effect on rotation curves.

5 The problem of the GR phase

Unfortunately, however, the tensor structure $T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$ in (40) does not arise in the short distance limit $\nabla\nabla \sim \Box \gg R$ – rather we have $T_{\mu\nu}$ without the trace term. This means that the theory does not really have a GR phase for any however small value of α even despite the $O(\alpha)$ modification of the original Einstein theory (19). In fact, this follows from the equivalent representation for its Euclidean action with the critical value (26) of the parameter α [27]

$$S = -\frac{M_{\rm eff}^2}{2} \int dx\, g^{1/2}\, E^{\mu\nu} \frac{1}{\Box + \hat{P}}\, E_{\mu\nu}. \tag{42}$$

in which the original Einstein term gets canceled in virtue of the identity

$$\frac{1}{\Box - \frac{\alpha}{4} R}\, R = -\frac{4}{\alpha}. \tag{43}$$

This identity is valid in any smooth compact Euclidean spacetime without a boundary – the Euclidean Schwinger-Keldysh framework appropriate for a cosmological setup (cf. discussion above in Sect.3).

The representation (42) quadratic in $E_{\mu\nu}$ immediately implies the existence of Einstein space solutions with an arbitrary value of Λ. However, in the UV limit $\nabla\nabla \gg R$ its variational derivative

$$\frac{\delta S}{\delta g_{\mu\nu}} \simeq \frac{M_{\rm eff}^2}{2}\, g^{1/2} \left(R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu \frac{1}{\Box} R \right) + O[\,E^2\,] \tag{44}$$

remains nonlocal and differs from the general relativistic expression even for $\alpha \to 0$. In particular, in the approximation linear in the curvatures matter sources are coupled to gravity according to

$$R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu \frac{1}{\Box} R + O[\,R^2\,] = \frac{1}{M_{\rm eff}^2}\, T_{\mu\nu}. \tag{45}$$

The local Ricci scalar term of the Einstein tensor is replaced here with the nonlocal expression which guarantees in this approximation the stress tensor conservation, but in contrast to anticipations of [13] does not provide the GR phase of the theory.

The absence of the GR phase might seem paradoxical because the original action (19) obviously reduces to the Einstein one in the limit $\alpha \to 0$. The explanation of this paradox consists in the observation that the transition from (19) to the new representation (42) is based on the identity (43) which is not analytic both in α and in the curvature. The source of this property is the constant zero mode of the scalar operator $\Box$ on compact Euclidean spacetimes without a boundary. On such manifolds the left hand side of (43) is not well defined for $\alpha = 0$. The equivalence of the actions (19) and (42) was obtained only on this class of Euclidean manifolds. The latter, in turn, were motivated in Sect.3 by extending the duality between the Schwinger-Keldysh technique and Euclidean QFT [20] to the cosmological (quasi-de Sitter) context.

In contrast to this class of manifolds, the representations (19) and (42) are not equivalent in asymptotically flat (AF) spacetime because Eq. (43) does not apply there. First, with zero boundary conditions at infinity the scalar $\Box$ does not have zero modes. Second, due to the natural AF falloff conditions, $R(x) \sim 1/|x|^4$ and $(1/\Box)\delta(x-y) \sim 1/|x-y|^2$, integration by parts in the chain of identical transformations leading to (43) gives a finite surface term at infinity $|x-y| \to \infty$. This leads to an alternative equation

$$\frac{1}{\Box - \frac{\alpha}{4}R} R \Big|_{\rm AF} = O\,[\,R\,] \tag{46}$$

with a nontrivial right hand side analytic in α and tending to zero for a vanishing scalar curvature. This explains why the model (19) on AF background has a good GR limit with nonlinear curvature corrections controlled by a small α [15, 13].[5]

This undermines the utility of the model (19) as a possible solution of the dark energy problem and simulation of dark matter phenomenon advocated in [13]. Absence of the GR limit for $\alpha \to 0$ and for short distance regime $\nabla\nabla \gg R$ becomes a critical drawback of this model[6] caused by its infrared behavior – presence of a constant zero mode on a compact spacetime. Possible solution of this problem could be a reformulation of the nonlocal action by projecting out this zero mode from the definition of the Green's function in (19) (see [29] for the technique of such a truncation).

Another possible way to circumvent this difficulty can be based on the conformal transformation to a new metric

$$\tilde g_{\mu\nu}[\,g\,] = e^{2\sigma[\,g\,]}\, g_{\mu\nu}, \tag{47}$$

which is assumed to be physical (that is directly coupled to matter) in contrast to the original metric $g_{\mu\nu}$ playing the auxiliary role. With the conformal factor function

$$\sigma[\,g\,] \simeq \frac{1}{4}\frac{1}{\Box}R, \tag{48}$$

which is small in the UV limit, $\sigma \ll 1$, but has large second order derivatives[7], $\nabla\nabla\sigma \sim R$, one can express the covariant Einstein tensor of the new metric $\tilde G_{\mu\nu}$ in terms of the original metric as

$$\begin{aligned}\tilde G_{\mu\nu} &= G_{\mu\nu} + 2\big(g_{\mu\nu}\Box\sigma - \nabla_\mu\nabla_\nu\sigma\big) + g_{\mu\nu}\sigma_\alpha^2 + 2\sigma_\mu\sigma_\nu\\ &= R_{\mu\nu} - \frac{1}{2}\,\nabla_\mu\nabla_\nu\frac{1}{\Box}R + O\left[\Big(\nabla\frac{1}{\Box}R\Big)^2\right], \quad \sigma_\mu \equiv \nabla_\mu\sigma. \end{aligned}\tag{49}$$

We see that $\tilde G_{\mu\nu}$ in this limit in fact reproduces the left hand side of (45). Therefore, if we couple matter to the new metric $\tilde g_{\mu\nu}$ in the total action as

$$S_{\rm total}[\,g,\phi\,] = S[\,g\,] + S_{\rm matter}[\phi, \tilde g[\,g\,]], \tag{50}$$

[5]Basic example of a physically nontrivial Einstein space is the Schwarzchild-de Sitter background. A priori it can also generate surface terms in (43), because its metric is not smooth simultaneously at the black hole and cosmological horizons and has a conical singularity [28]. However, one can show that for any type of boundary conditions at this singularity the relevant surface term vanishes and leaves Eq.(43) intact. A similar issue remains open in the case of the Schwarzchild-AdS background for which the operator $\hat D$ with $R<0$ is not guaranteed to be free of zero modes and does not provide a well defined representation (42) [26]. We are grateful to S. Solodukhin for a discussion of this point.

[6]In [26] this was interpreted as the phase transition between the $R = 4\Lambda > 0$ and $R = 0$ phases – the absence of crossover between these phases. We see that in fact this transition has a topological nature.

[7]Note that this expression is assumed to hold only in the formal UV limit of $\nabla\nabla \gg R$, so that the zero mode of $\Box$ should not invalidate it.

then for $\tilde{g}_{\mu\nu}$ in the short distance limit we will recover the usual Einstein equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\,\tilde{g}_{\mu\nu}\tilde{R} = \frac{1}{M_{\rm eff}^2}\,\tilde{T}_{\mu\nu}, \quad \tilde{T}_{\mu\nu} = \frac{2}{\tilde{g}^{1/2}}\,\tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta}\,\frac{\delta S_{\rm matter}}{\delta\tilde{g}_{\alpha\beta}} \tag{51}$$

where $\tilde{T}_{\mu\nu}$ is a matter stress tensor in the frame of the $\tilde{g}_{\mu\nu}$-metric. When deriving this equation we took into account smallness of σ and $\delta\sigma/\delta g_{\mu\nu} = O(\sigma)$ in the short distance limit $\nabla\nabla \gg R$. Thus we get a GR phase in the conformally related frame of the theory. Unfortunately, however, the magnitude of corrections to the GR behavior is no longer controlled by a small parameter α, which makes application of this idea to realistic cosmology somewhat questionable.

6 Conclusions

Thus we have a class of generally covariant nonlocal gravity models which have a general relativistic limit on an asymptotically flat background and also possess stable Einstein space solutions with an arbitrary value of their cosmological constant. Their nonlocal action was formulated in the Euclidean signature spacetime and is understood as an approximation to the quantum effective action originating from fundamental quantum gravity theory. In the framework of the Euclidean version of the Schwinger-Keldysh formalism [20] for quantum expectation values we derived from this action the *causal* effective equations of motion for mean value of the metric field in the physical Lorentzian-signature spacetime. Thus we have shown that the (A)dS background of the theory carries as free propagating modes massless gravitons having two polarizations identical to those of the Einstein theory with a cosmological term. We also obtained linearized gravitational potentials of compact matter sources and showed that in the long distance (A)dS phase their effective gravitational coupling $G_{\rm eff}$ can be essentially different from the Newton gravitational constant G_N of the short-distance GR phase. When $G_{\rm eff} \gg G_N$ the (A)dS phase can be regarded as a strongly coupled infrared modification of Einstein theory not only describing the dark energy mechanism of cosmic acceleration but also simulating the dark matter phenomenon by enhanced gravitational attraction at long distances.

Unfortunately, in contrast to AF spacetimes this model fails to have a general relativistic limit in the cosmological problems for the mean metric field, treated within the Euclidean version of the Schwinger-Keldysh formalism. The short-distance GR limit can be attained in a special conformal frame (physical metric minimally coupled to matter) nonlocally related to the original one. This limit, however, cannot be controlled by smallness of the parameter α that was initially designed in [13] to moderate the effect of nonlocal corrections to the Einstein theory.

Thus, direct cosmological applications of this class of models (19) for the accelerating Universe are not likely to be available now and deserve further development. However, these models might be interesting as a nonlocal generalization of critical gravity theories [21] – holographic duals of the logarithmic conformal models [30] – and in black hole thermodynamics. In particular, as advocated in [26], they have Schwarzschild-de Sitter black hole solutions with zero entropy analogous to the zero entropy and energy black holes of [21]. All this makes the class of nonlocal gravity models open for interesting future implications.

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