Relativistic particles in a strong magnetic field and dense matter

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Abstract

A problem of electron motion in dense matter and magnetic field is considered. It is shown that the electron energies are quantized in case of immovable medium and constant homogenous magnetic field. Obtained exact solutions are examined for the particular case of the electron motion in a rotating neutron star with account for matter and magnetic field effects. We argue that all of these considerations can be useful for astrophysical applications, in particular for description of electrons and neutrinos motion in different environments.

Introduction. The studies of particle interactions in high energy physics as well as solving problems of charged particles motion in electromagnetic fields of terrestrial experimental devices, in astrophysics and cosmology base on using of different methods. The most effective tool is associated with using of exact solutions of quantum field equations of motion. They were first applied in quantum electrodynamics for studies of motion and radiation of the electron in a magnetic field, i.e. the synchrotron radiation (see, for instance, [1]), and also for studies of the electrodynamics and weak interaction in different configurations of external electromagnetic fields [2]. This method is based on the Furry representation [3] in quantum electrodynamics. Recently it has been shown, that the method of exact solutions can be also applied for the problem of neutrinos and electron motion in presence of dense matter (see [4] for a review on this topic). In [5, 6] the exact solution for the modified Dirac equation for a neutrino moving in matter was derived and discussed in details. In [7] the corresponding exact solution for an electron moving in matter was obtained. Analogous problem for electron in magnetized matter was solved in [8]. The problem of neutrino propagation in transversally moving matter was first solved in [9], and in [10] we considered neutrino propagation in a rotating matter accounting for the effect of nonzero neutrino mass.

Note that a study of the neutrino dispersion relations, neutrino mass generation and for derivation of the neutrino oscillation probabilities in matter also use the modified effective Dirac equations for a neutrino interacting with various background environments within different models (see [11] in details). In [12] different processes with neutrino in the presence of matter were studied on the same basis. In [13] the modified Dirac equations with anomalous vector and pseudovector interactions were considered within a framework for treatment of low-energy effects of spontaneous CPT violation and Lorentz breaking.

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In this paper we develop the method of exact solutions for the problem of charged leptons propagating in matter and strong magnetic fields. This review is based on our recent paper [8].

Modified Dirac Equation for electron moving in matter and magnetic field. We consider an electron propagating in immovable medium composed of neutrons and homogenous and constant magnetic field. This can be regarded as the first approach to modelling of an electron propagation inside a rotating neutron star. For distinctness we consider here the case of an electron, whereas generalization for other charged particles is just straightforward. The modified Dirac equation for the electron wave function exactly accounting for the electron interaction with matter and magnetic field [5] (see also [6]):

$$\Big\{\gamma_\mu(p^\mu + e_0 A^\mu) + \frac{1}{2}\gamma_\mu(1-4\sin^2\theta_W + \gamma^5)f^\mu - m\Big\}\Psi(x) = 0, \tag{1}$$

where e_0 is a module of the electron charge. This is the most general form of the equation for the electron wave function in which the effective potential $V_\mu = \frac{1}{2}(1-4\sin^2\theta_W+\gamma^5)f_\mu$ includes the neutral current interaction of the electron with the background particles, and which can also account for effects of matter motion and polarization.

Note, that in general case it is not a trivial task to find solutions of this equation. For the electromagnetic field and effective matter potential we obtain

$$A^\mu = (0, -\frac{yB}{2}, \frac{xB}{2}, 0), \quad f^\mu = -Gn(1,0,0,0), \tag{2}$$

where $G = \frac{G_F}{\sqrt{2}}$, n is matter number density. The Hamiltonian form of equation (1) is

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(x) = \hat{H}\Psi(x), \tag{3}$$

$$\hat{H} = -\gamma^5\boldsymbol{\sigma}(\boldsymbol{p} + e_0\boldsymbol{A}) + m\gamma^0 + \frac{1}{2}(1-4\sin^2\theta_W + \gamma^5)Gn \tag{4}$$

where $\boldsymbol{A} = (-\frac{yB}{2}, \frac{xB}{2}, 0)$ and the relation $\gamma^0\boldsymbol{\gamma} = -\gamma^5\boldsymbol{\sigma}$ was used. This form of the Hamiltonian makes quite transparent the solution describing spin properties of the electron.

Spin operator. Note that the longitudinal polarization operator commutes with the Hamiltonian, $[\hat{T}^0, \hat{H}] = 0$. Therefore for any of its eigenvectors the Hamiltonian can be presented in the following form

$$\hat{H} = -\gamma^5 T^0 + m\gamma^0 + \frac{1}{2}(1-4\sin^2\theta_W + \gamma^5)Gn, \tag{5}$$

where T^0 is one of the eigenvalues of the spin operator $\hat{T}^0$. Note that in the presence of the matter potential proportional to Gn the transverse polarization operator does not commute with the Hamiltonian. This is a consequence of γ^5 presence in (4).

Energy spectrum of electron in matter and magnetic field. To find the electron energy spectrum p_0 in the matter and constant magnetic field, $\hat{H}\Psi = p_0\Psi$, we use the chiral representation of the γ-matrices and solve the equation

$$\begin{vmatrix} -mT^0 + Gn - \tilde{p}_0 & m \\ m & mT^0 - \tilde{p}_0 \end{vmatrix} = 0, \tag{6}$$

where $\tilde{p}_0 = p_0 + 2Gn\sin^2\theta_W$. The solutions can be written in the form

$$p_0 = \frac{Gn}{2} - 2Gn\sin^2\theta_W + \varepsilon\sqrt{(mT^0 - \frac{Gn}{2})^2 + m^2}, \tag{7}$$

where $\varepsilon = \pm 1$ is the "sign" of the energy.

It is significant to note an interesting feature of the electron energy spectrum in the magnetized matter following from (7). It is well known, that the energy spectrum of the electron in the magnetic field is degenerated in respect of spin quantum number (each electron Landau energy level in the magnetic field corresponds to both spin orientations). The presence of the matter (of any non-vanishing density $n \neq 0$) removes the spin degeneracy. This phenomenon can be attributed to the parity violation in weak interactions. But the infinite degeneracy of the Landau levels associated with quantum number p_2 is still preserved.

Let us emphasize one important relation between p_0 and T_0 that immediately follows from the spectrum (7):

$$(p_0 - \frac{Gn}{2} + 2Gn \sin^2 \theta_W)^2 = (mT^0 - \frac{Gn}{2})^2 + m^2, \tag{8}$$

where T^0 is one of the eigenvalues of the spin operator $\hat{T}^0$. Note that this formula can be also obtained by using the concept of $*$-spin introduced in [14].

The electron wave functions. In the paper [8] it has been shown that the solution of the equation (3) due to symmetries can be sought in the form

$$\Psi(t, x, y, z) = \mathrm{e}^{-\mathrm{i}p_0 t + \mathrm{i}p_3 z} \begin{pmatrix} \chi_1(r)\mathrm{e}^{\mathrm{i}(l-1)\phi} \\ \mathrm{i}\chi_2(r)\mathrm{e}^{\mathrm{i}l\phi} \\ \chi_3(r)\mathrm{e}^{\mathrm{i}(l-1)\phi} \\ \mathrm{i}\chi_4(r)\mathrm{e}^{\mathrm{i}l\phi} \end{pmatrix}, \tag{9}$$

where r and ϕ are polar coordinates. This form of solution also based on the fact that the operator of the total angular momentum $J_z = L_z + S_z$, where $L_z = -i\frac{\partial}{\partial\phi}$, $S_z = \frac{1}{2}\sigma_3$, commutes with the Hamiltonian. The solutions are the eigenvectors of the total momentum operator J_z with the corresponding eigenvalues $l - \frac{1}{2}$. After substitution of (9) into (3)-(4) we get the following system for $\chi_i(r)$:

$$-(p_3 - Gn)\chi_1 - \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l}{r} + \frac{e_0 B}{2} r\right)\chi_2 + m\chi_3 = \tilde{p}_0\chi_1, \tag{10}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l-1}{r} - \frac{e_0 B}{2} r\right)\chi_1 + (p_3 + Gn)\chi_2 + m\chi_4 = \tilde{p}_0\chi_2, \tag{11}$$

$$m\chi_1 + p_3\chi_3 + \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l}{r} + \frac{e_0 B}{2} r\right)\chi_4 = \tilde{p}_0\chi_3, \tag{12}$$

$$m\chi_2 - \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l-1}{r} - \frac{e_0 B}{2} r\right)\chi_3 - p_3\chi_4 = \tilde{p}_0\chi_4. \tag{13}$$

Now we define the creation and annihilation operators

$$R^+ = \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l-1}{r} - \frac{e_0 B}{2} r, \qquad R^- = \frac{\mathrm{d}}{\mathrm{d}r} + \frac{l}{r} + \frac{e_0 B}{2} r. \tag{14}$$

and get the system (15) - (18) in the following form

$$-(p_3 - Gn)\chi_1 - R^-\chi_2 + m\chi_3 = \tilde{p}_0\chi_1, \tag{15}$$

$$R^+\chi_1 + (p_3 + Gn)\chi_2 + m\chi_4 = \tilde{p}_0\chi_2, \tag{16}$$

$$m\chi_1 + p_3\chi_3 + R^-\chi_4 = \tilde{p}_0\chi_3, \tag{17}$$

$$m\chi_2 - R^+\chi_3 - p_3\chi_4 = \tilde{p}_0\chi_4. \tag{18}$$

To get the eigenvalues $p_0 = \tilde{p}_0 - 2Gn \sin^2 \theta_W$ of the Hamiltonian (3) we take into consideration properties of operators R^+ and R^-:

$$R^+ \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right) = -\sqrt{2e_0 B(s+l)}\, \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right), \tag{19}$$

$$R^- \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right) = \sqrt{2e_0 B(s+l)}\, \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right), \tag{20}$$

where $\mathcal{L}_s^l$ are the Laguerre functions [1].

The solution of system (15) - (18) (the eigenvector of the Hamiltonian (4)) can be written in the form

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} = \sqrt{e_0 B} \begin{pmatrix} C_1 \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right) \\ C_2 \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right) \\ C_3 \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right) \\ C_4 \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right) \end{pmatrix} \tag{21}$$

Now we can get the energy spectrum and of the eigenvalues of spin operator $\hat{T}^0$:

$$p_0 = \frac{Gn}{2} - 2Gn \sin^2 \theta_W + \varepsilon \sqrt{\left(-\frac{Gn}{2} \pm \sqrt{p_3^2 + 2e_0 B(l+s)}\right)^2 + m^2}, \quad \varepsilon = \pm 1, \tag{22}$$

$$T^0 = \frac{s'}{m}\sqrt{p_3^2 + 2e_0 B(l+s)}, \quad s' = \pm 1. \tag{23}$$

It is easy to see, that the spectrum (7) obtained above is in agreement with expressions (22) and (23). From this energy spectrum, it is straightforward that the well-known energy spectrum in magnetic field (the Landau levels) is modified by interaction of the electron with matter. However the radius of the classical orbits corresponding to a certain level (22) doesn't depend on the matter density:

$$\langle R \rangle = \int_0^\infty \Psi^+ r \Psi \, d\boldsymbol{r} = \sqrt{\frac{2N}{e_0 B}}. \tag{24}$$

Note that this result is a simple consequence of the fact that the orbital part of the wave functions (21) is not altered by the matter potential. To conclude this section we would like to note, that the effect of electron trapping on circular orbits in magnetized matter exists, and this can be important for astrophysical applications.

To finalize the section we note the properties of creation and annihilation operators. The functions $F_s^l = \sqrt{e_0 B}\mathcal{L}_s^l\left(\frac{e_0 B}{2} r^2\right)$ constitute a basis in the Hilbert space with scalar product defined as

$$\langle F_{s'}^l, F_{s''}^l \rangle = \int_0^\infty F_{s'}^l F_{s''}^l r \, \mathrm{d}r. \tag{25}$$

So that, we get for each s and l

$$\langle F_s^{l-1}, R^- F_s^l \rangle = -\langle F_s^l, R^+ F_s^{l-1} \rangle. \tag{26}$$

Hence for operators (14) we obtain $(R^-)^* = -R^+$ and $(R^+)^* = -R^-$, where symbol $*$ implies Hermitian conjugation of operators.

Spin coefficients and full wave function. Using of Hamiltonian and spin operator properties it is easy to obtain the spin coefficients C_i after some easy steps (more details see in [8]). Finally, we obtain the wave function:

$$\Psi(t,x,y,z) = \mathrm{e}^{-\mathrm{i}p_0 t} \frac{1}{\sqrt{L}} \mathrm{e}^{\mathrm{i}p_3 z} \sqrt{\frac{e_0 B}{2\pi}} \begin{pmatrix} C_1 \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right) \mathrm{e}^{\mathrm{i}(l-1)\phi} \\ \mathrm{i}C_2 \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right) \mathrm{e}^{\mathrm{i}l\phi} \\ C_3 \mathcal{L}_s^{l-1}\left(\frac{e_0 B}{2} r^2\right) \mathrm{e}^{\mathrm{i}(l-1)\phi} \\ \mathrm{i}C_4 \mathcal{L}_s^{l}\left(\frac{e_0 B}{2} r^2\right) \mathrm{e}^{\mathrm{i}l\phi} \end{pmatrix}, \tag{27}$$

where

$$C_1 = \frac{1}{2}\sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{\tilde{p}_0 - \frac{Gn}{2}}}\sqrt{1 + \frac{p_3}{mT^0}}, \qquad C_2 = \frac{s'}{2}\sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{\tilde{p}_0 - \frac{Gn}{2}}}\sqrt{1 - \frac{p_3}{mT^0}}, \tag{28}$$

$$C_3 = \frac{\varepsilon}{2}\sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{\tilde{p}_0 - \frac{Gn}{2}}}\sqrt{1 + \frac{p_3}{mT^0}}, \qquad C_4 = \frac{s'\varepsilon}{2}\sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{\tilde{p}_0 - \frac{Gn}{2}}}\sqrt{1 - \frac{p_3}{mT^0}} \tag{29}$$

and L is a normalizing factor.

Equations (27)-(29) represent the exact solution of (3) with the Hamiltonian (4) that describes the electron moving in matter and magnetic field. Note that in the case $n = 0$, these formulas are reduced to well-known solutions for the electron wave functions in a constant homogenous magnetic field [1]. The limit $B \to 0$ gives the result obtained in the paper [7].

Application. Let us consider the problem of an electron (or another charged particle) motion in a rotating matter with magnetic field can be solved. This problem is of interest in different astrophysical contexts.

If the angular velocity ω is small compare to the magnetic field, we can calculate the spectrum using a standard perturbation theory with the small parameter $\frac{Gn\omega}{e_0 B} \ll 1$. If, for example, we choose for the matter density, angular velocity and magnetic field the values peculiar for a rotating neutron star ($n = 10^{37} sm^{-3}$, $\omega = 2\pi \cdot 10^3 s^{-1}$, $B = 10^{10} Gs$), then the parameter is really small,

$$\delta = \frac{Gn\omega}{e_0 B} \sim 10^{-20} \ll 1. \tag{30}$$

Now we can take the spectrum and wave functions found above (i.e. without rotation) as the lowest order of perturbation series and find the correction term.

We consider the particular case of constant magnetic field and rotating uniform matter so that the electromagnetic field and effective matter potential are given by

$$A^{\mu} = (0, -\frac{yB}{2}, \frac{xB}{2}, 0), \quad f^{\mu} = -Gn(1, -\omega y, \omega x, 0). \tag{31}$$

For this case we obtain the problem (3) with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_0$ describes the electron in magnetic field and matter without rotation and is given by formula (4), and

$$\hat{H}_1 = \frac{Gn}{2}\gamma^0\gamma^1(1 - 4\sin^2\theta_W + \gamma^5)\omega y - \frac{Gn}{2}\gamma^0\gamma^2(1 - 4\sin^2\theta_W + \gamma^5)\omega x.$$

Using again the chiral representation of the γ-matrixes we obtain approximately taking into account that $4\sin^2\theta_W \simeq 1$ in the polar coordinates

$$\hat{H}_1 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i}\rho r e^{-\mathrm{i}\phi} & 0 & 0 \\ \mathrm{i}\rho r e^{\mathrm{i}\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\rho r e^{-\mathrm{i}\phi} \\ 0 & 0 & \mathrm{i}\rho r e^{\mathrm{i}\phi} & 0 \end{pmatrix}, \quad \rho = Gn\omega. \tag{32}$$

So, we get for the first correction to the energy spectrum (7) of the electron

$$\Delta p_0^N = \int \Psi_N^* H_1 \Psi_N \, dV = \sqrt{2e_0 B}(C_1 C_2 + C_3 C_4)\frac{Gn\omega}{e_0 B}\int_0^{\infty} \mathcal{L}_{N-l}^{l-1}(\xi)\sqrt{\xi}\mathcal{L}_{N-l}^{l}(\xi)\, d\xi, \tag{33}$$

where C_i are the spin coefficients from eq. (28), (29). After calculations and simplifying we get (we put $p_3 = 0$):

$$\Delta p_0^N = \delta\sqrt{2e_0 BN}. \tag{34}$$

This shift of levels in the energy spectrum depending on the energy quantum number $N = 0, 1, 2...$ leads to a corresponding shift in a frequency of synchrotron radiation of electron inside of dense rotating matter, that can be registered. Calculation of the next perturbation terms (higher order) is an interesting task that could reveal new effects. However, we pointed out that the small parameter (30) is really very small.

Conclusion. In this paper we found a class of exact solutions of the modified Dirac equation for the electron in matter and strong magnetic field by using of increasing and decreasing operators. In paper [8] we generalize this approach to a definite class of Dirac Hamiltonians. One can use obtained solutions in studies of complicated models of particle interactions in different astrophysical environments, for example, in studies of gamma-rays production during collapse or coalescence processes of neutron stars (the one predicted within the fireball models of GRBs [18]), as well as during a neutron star being "eaten up" by the black hole, or in investigations of other processes in environments discussed in [13]. Consideration of electrically millicharged neutrino in external fields performed in paper [19] provides another interesting application of obtained results.

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