From stretched horizon to gluon-plasma observables

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Abstract

According to the holographic models of Yang-Mills theories properties of the liquid living on the stretched horizon are directly related to the properties of the gluon plasma. There are important reservations to this statement, however. In particular, the duality applies only to the non-perturbative component of the plasma, and this notion is not absolutely well defined. Also, the duality assumes measurements on the plasma made with poor resolution, which is, again, a somewhat ambiguous notion. Nevertheless, it is amusing that the duality does explain some results of the lattice simulations which look very exotic otherwise.

1 Introduction

Soon after the beginning of the era of the holographic models (for review and references see, e.g., [1]) a model was found [2, 3] which belongs *in the infrared* to the same universality class as the large-N_c Yang-Mills theories. The geometry of the model at temperature $T = 0$ is fixed as:

$$ds^2 \;=\; \Big(\frac{u}{R}\Big)^{3/2}\Big(-dt^2+\delta_{ij}dx^i dx^j+f(u)dx_4^2\Big)+\Big(\frac{u}{R}\Big)^{3/2}\Big(\frac{du^2}{f(u)}+u^2 d\Omega_4^2\Big), \qquad (1)$$

$$where\;\; f(u) \;=\; 1-\Big(\frac{u_\Lambda}{u}\Big)^3\;,\; x_4 \;\sim\; x_4+\beta_4\;,\; \beta_4=\frac{4\pi}{3}\Big(\frac{R^3}{u_\Lambda}\Big)\;.$$

Here u is an extra coordinate, common to all the holographic models. It is conjugate to the momentum transfer, or resolution of measurements in the standard 4d space. The 4d unit sphere, $d\Omega_4$, is holographically related to the physics of baryons and will not concern us here. The limit $u \rightarrow \infty$ corresponds to the physics in the ultraviolet, or measurements with perfect resolution. Commonly, we would expect to find the standard 4d space on this boundary, $u = \infty$. This is not true, however, in the case considered. Namely, we have instead a 5d space, with an extra compact coordinate, x_4, with periodicity, denoted β_4, of order of the hadronic scale, $\beta_4 \sim \Lambda^{-1}_{QCD}$. The presence of this, so to say, "not-needed" coordinate pushes the limits of the applicability of the model to large distances, r_{hydro},

$$r_{hydro} \;\gg\; \Lambda^{-1}_{QCD}\;\;. \qquad (2)$$

In other words, only the pion physics can be consistently considered at small temperatures [3]. Another application of the model (1) is the theory of vacuum defects, or condensates, for review see [6, 7]. In the both cases (of the pion physics and of the vacuum condensates) the model

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turns to be a success. Also, due to the presence of the horizon at $u = u_\Lambda$ the model incorporates the confinement.

There is one more crucial ingredient in the construction discussed. Namely, the compact x_4-direction is related to the topological charge of the Yang-Mills fields in the Euclidean space:

$$Q_{top} \;=\; \frac{g^2}{16\pi^2}\int d^4x (G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}) \quad , \qquad (3)$$

where $G^a_{\mu\nu}$ is the field-strength tensor of the gluon field. Wrapping n_{x_4} times around the x_4-circle implies a non-trivial topological charge associated with the defect,

$$Q_{top} \;=\; \pm\, n_{x4} \;,$$

where the sign depends on the direction of the wrapping.

At the temperatures above the deconfinement phase transition, $T > T_c$, going to the limit (2) becomes the hydrodynamic limit. This is the main focus of the present notes. Note that the geometry (1) is to be modifies at high temperatures. First, usually one is considering the Euclidean version of the model, with the compact time direction τ:

$$\tau \;\sim\; \tau \,+\, \frac{1}{T} \;. \qquad (4)$$

Moreover–and this point turns to be crucial–the position of the horizon is no longer fixed at $u_h = u_\Lambda$ but is rather related to the inverse temperature, $u_h \sim 1/T$. In the geometric language, the deconfinement phase transition happens at the point $\beta_4 = 1/T$ [4] and is in fact the Hawking-Page transition [5] of general relativity.

After these preliminary remarks we are in position to specify the problems considered in the present notes. The limitation (2) implies that we are to consider rather the infrared physics of large distances. Moreover we have to impose the condition of poor resolution as well. These two requirements are not necessarily the same. For example, to evaluate, say, viscosity in holographic models one studies graviton exchange, through the extra dimensions, between two points, (x, y) on the boundary $u = \infty$. Then the condition (2) can be readily satisfied. However, to apply the model (1) one should ensure also poor resolution of the points (x, y), i.e., $\Delta x, \Delta y \;\sim\; \Lambda^{-1}_{QCD}$. On the dual side, poor resolution implies going closer to the horizon, $u \sim u_h$. In the language of the black-hole physics this means going to the stretched horizon, for review and references see [12, 8]. Moreover, the properties of the liquid living on the stretched horizon can be found in all the generality, for the large black holes working with the Rindler space [9]. Thus, we will try to compare the properties of the liquid living on the stretched horizon with the results of the lattice simulations of the gluon plasma in the far infrared.

2 Exotic liquid living on the stretched horizon

It is known since long, see, e.g., [10, 11, 12], that the results of experimentation with black holes can be described by a distant observer in terms of a fictitious liquid living on the stretched horizon, that is a surface positioned at a certain (small) distance from the actual black-hole horizon. Formally, the stretched horizon is defined through the boundary condition that the general-relativity equations hold in the bulk, outside the horizon [11] The boundary action $S_{boundary}$ is fixed by this condition. Moreover, one can show that this action corresponds to a liquid described by the non-relativistic Navier-Stokes equations, for a recent derivation and further references see [9].

In Ref. [9] a similar program was realized in the simpler case of the Rindler space. Note that the near-the-horizon region of a large black holes can be approximated by the Rindler space (see, e.g., [12]) and therefore the results apply in case of the stretched horizon as well. A remarkable observation made in Ref. [9] is that the non-relativistic series relevant to the Navier-Stokes equation and its generalizations in the dissipation-less approximation can be summed up to a relativistically invariant action. This helps us to compare the results with the brane picture of the holographic models.

The action associated with a d-dimensional hypersurface in the (d+1) dimensional Rindler space takes the form [9]:

$$S_{liquid} = T \int d^d x \sqrt{-\gamma} \ , \tag{5}$$

where T is a constant ((generalized tension) and action refers to the following metric tensor γ_{ab} $(a, b = 0, 1, .., (d-1))$

$$\gamma_{ab} dx^a dx^b \equiv - r_c d\tau^2 + dx_i dx^i \ , \tag{6}$$

where r_c is a constant. The relation to the liquid is provided by the identification of the normalized gradient of the scalar field φ with the four-velocity of an ideal liquid u_a:

$$u_a \equiv \partial_a \varphi / \sqrt{X} \ , \ \ X \equiv -(\partial\varphi)^2 \equiv (\partial_0 \varphi)^2 - (\partial_i \varphi)^2 \ . \tag{7}$$

Finally, the equilibrium solution is given by:

$$\varphi_{equilibrium} = t \ . \tag{8}$$

Hydrodynamics of the ideal liquid arises as an expansion in derivatives from the field φ around the equilibrium.

Note that by tending $r_c \to 0$ we get the space reduced from $(d+1)$ to d dimensions, as it should be at the actual horizon of a black hole. A small r_c correspond to a stretched horizon. In the Rindler case the analytic form (5) can be found at any r_c. The results approximate the black hole physics only for a small ratio r_c/r_g where r_g is the Schwarzschild radius of the black hole.

In the limit $r_c \to 0$ some quantities become singular, as it should be at the horizon. In particular, for the equilibrium energy-momentum tensor one readily finds:

$$\Big(T_{ab}\Big)_{equilibrium} - (0, p.., p)), \ \ p = 1/\sqrt{r_c} \ , \tag{9}$$

and the pressure p is singular in the limit $r_c \to 0$ while the energy density ϵ is vanishing, $\epsilon = 0$. Note that such a behaviour of the equilibrium energy-momentum tensor for the liquid on the horizon is well known in the membrane paradigm [10].

We will come back to discuss further properties of the exotic liquid (5) later, while proceeding now to compare the action (5) with the holographic model (1).

3 On the sign of the brane action in holography

Although the energy-momentum tensor (9) looks exotic and it is indeed exotic from the point of view of the $(d+1)$-dimensional theory, its interpretation within the reduced, d-dimensional theory is very straightforward. Indeed, the d-dimensional space incorporated spatial, or Euclidean coordinates alone. Then, if there exists a d-dimensional uniform brane, the energy-momentum tensor is proportional to:

$$< T_{ab} >_{d-brane} = (const)\delta_{ab} \ , \quad a, b = 1, 2, .., d \ < \tag{10}$$

where we use only the symmetry considerations. Thus, the Eq. (9) corresponds to a d-dimensional brane living in $(d+1)$ dimensional space.

What is unusual, however, is the sign of the action(5). To make the point obvious let us use Euclidean time-space, that is change τ in (6) into $i\tau$, $\tau \rightarrow i\tau$. Then the time coordinate become periodic. The crucial point is that the Euclidean action corresponding to (5) is *negative*:

$$S_{liquid}^{Euclidean} \sim -T\sqrt{r_c}\int d^d x \quad , \tag{11}$$

where r_c is assumed to be small. The negative sign seems to be in contradiction with the general principles.

Turn now to the geometry (1), or more precisely to the geometry in the coordinates (t, x_i, x_4) near the horizon $u = u_\Lambda$. Mofreover, go the Euclidean time, $t \rightarrow it$. Then we have a cigar-shape geometry in these coordinates since the radius of the periodic x_4-coordinate vanishes at the horizon:

$$R_{x_4}(u = u_\Lambda) = 0 \ . \tag{12}$$

Thus, our four-dimensional space-time (Euclidean version) can be considered in the infrared as a brane imbedded into a 5d space with a small radius of the fifth coordinate. From the holographic point of view, the action of the brane is associated with the non-perturbative fluctuations of the Yang-Mills fields. Moreover, the vanishing radius (12) signals that the action of the defects with non-trivial topological charge vanishes in the infrared [13]. This phenomenon is well known of course within the QCD phenomenology and loosely can be described as condensation of the (anti)instantons. Quantitatively, the effect is described in terms of the so called gluon condensate, or the vacuum expectations value $< (G_{\mu\nu}^a)^2 >_{non-pert}$. The crucial point is then that $< (G_{\mu\nu}^a)^2 >_{non-pert}$ is negatiive. The physics is that non-perturbative fluctuations lower the energy of the vacuum. Somewhat more formally, one can argue [14] that the bag constant B is proportional to

$$B - \sim - < (G_{\mu\nu}^a)^2 >_{non-pert} \quad . \tag{13}$$

To summarize, the sign of the Euclidean action (11) is just the one which is expected on the basis of the duality with the non-perturbative Yang-Mills theory.

Consider now the confinement phase, $T > T_c$ the cigar-shape geometry is in coordinates (u, τ) where τ is now the Euclidean time:

$$R_\tau(u = u_h) = 0 \ . \tag{14}$$

This means that the action (5) in infrared refers now to 3d branes embedded into the Euclidean 4d space. As is explained in detail in Refs. [7, 15] this prediction of the holography is fully consistent with the lattice data. The negative sign of the action associated with the non-perturbative branes is again confirmed by the data.

4 Instantons and measurements with poor resolution

Consider again $T = 0$ and the geometry (1). Take into account now vibrations of the 4d branes in the x_4 direction. Then the Euclidean action associated with the branes would be equal to

$$S_{4d}^{Euclidean} = -T\int d^4x\sqrt{1 + (\partial\phi)^2}, \tag{15}$$

where the scalar field ϕ has now the meaning of the value of x_4 as function of the four other coordinates. Expanding in the derivatives we see that there exists a 4d massless mode. However, the sign of the kinetic energy corresponds rather to a ghost, not to an actual massless field. Such a prediction might look discouraging at first sight. Remember, however, that the x_4 coordinate is associated with topological charge (see Introduction). And, indeed, in this channel one can speak about a ghost, more precisely, about the famous Kogut-Susskind ghost.

To be more systematic, let us recall the reader the basic facts about the topological susceptibility of the Yang-Mills vacuum, for more detail and further references see, e.g., [16]. We begin with an obvious observation

$$< Q_{top}^2 > \quad > 0 \ , \tag{16}$$

where the topological charge, Q_{top} is defined in (3). Since $< Q_{top} = 0$ Eq. (16) can be rewritten as the positivity condition for the topological susceptibility:

$$\int d^4x < G\tilde{G}(x), G\tilde{G}(0) > \quad > \ 0 \ . \tag{17}$$

In fact, the condition (17) is far from being trivial since for any $x \neq 0$ the sign is just opposite:

$$< G\tilde{G}(x), G\tilde{G}(0) >_{exact} \quad < \ 0, \quad x \neq 0 \ . \tag{18}$$

This is a consequence of the unitarity (and refers to the Euclidean case). In particular, one can check that perturbatively at small x:

$$< G\tilde{G}(x), G\tilde{G}(0) >_{perturbative} \quad \sim \ \frac{\alpha_s^2}{x^8} \ , \tag{19}$$

where α_s is the Yang-Mills coupling entering the definition of the topological charge. Thus, to satisfy (17) we have to assume that the topological susceptibility contains a local, or singular term of the form:

$$< G\tilde{G}(x), G\tilde{G}(0) >_{local} \quad = \ \delta^4(x)\Big(\frac{(const)\alpha_s^2}{a^4} + (const)\Lambda_{QCD}^4\Big) \ , \tag{20}$$

where a is a cut off, or lattice spacing and the first term in the r.h.s. is to cancel the perturbative contribution (19) while the second term is reproducing the instanton contribution (or, more generally, non-perturbative topological fluctuations).

Turning to the isntantons, we observe

$$< G\tilde{G}(x), G\tilde{G}(0) >_{instantons} \quad > \ 0 \ . \tag{21}$$

Moreover, as far as we think about the instantons as of objects with the size of order Λ_{QCD}^{-1} the correlation length in Eq. (21) is of the same order, Λ_{QCD}^{-1}. We see, however, that (210 is in contradiction with the unitarity, see Eq. (18). This means that, no matter how far in the infrared region we go tending $x \rightarrow \infty$ instantons cannot dominate the correlator of the densities of the topological charge. One can never neglect perturbative contributions while evaluating the correlator at any finite x and one can neglect perturbative physics while evaluating the topological susceptibility, or the integral over x.

Note that if the measurements of the correlator of the topological densities are made with poor resolution then the delta-function $\delta^4(x)$ is poorly resolved and smeared over finite range of x. Then the "wrong sign" of Eq. (21) can be observed. If the resolution is improved then the positive contribution is becoming more and more squeezed to a local term. This is just the picture explicitly seen in the lattice simulations [16].

What we learn new from the holography, is that the positive, or ghost-like sign of the correlator of the topological densities is predicted by the duality of the non-perturbative Yang-Mills to the stretched horizon:

$$< G\tilde{G}(x), G\tilde{G}(0) >_{holography} \quad \sim \ \frac{1}{x^2} \ , \tag{22}$$

where $1/x^2$ corresponds to the massless ghost exchange. And the prediction is qualitatively confirmed by lattice simulations ad far as the resolution is poor.

5 "Euclidean superfluidity"

As the next step, we can consider $T > T_c$, or non-perturbative contributions to the gluon plasma. As is explained above we are then dealing in the infrared with 3d branes. Taking into account vibrations of the brane in the (Euclidean) time direction we get for the action of the branes:

$$S_{3d}^{Euclidean} \quad = \quad -T \int d^3x \sqrt{1 + (\partial_i \phi)^2} \quad , \tag{23}$$

where ϕ has now the meaning of the time coordinate as function of the three spatial coordinates. Continuing to the Minkowski space,

$$S_{3d^{Minkowski}} \quad = \quad +T \int d^3x \sqrt{1 - (\partial_i \phi)^2} \quad . \tag{24}$$

Integrating over the time coordinate, to obtain the four dimensional action, we reproduce the action (5) of the exotic liquid living on the horizon. Which is–we believe– is a direct demonstration of the relevance of the holographic models to the non-perturbative Yang-Mills physics.

Expanding Eq (24) in derivatives we find out a massless 3d mode. This time, unlike the $T = 0$ case considered in the preceding section, the sign of the kinetic energy of the massless mode is physical. The violation of the unitarity is still there, however. Indeed, a simple calculation reveals the superluminal propagation of this mode. This is known to be a common feature of liquids described by the non-linear actions of the type (5), see, e.g., Ref. [17]. Thus, we cannot expect to observe the effects associated with exchange of this mode, unless we have poor resolution.

Usually, three dimensional massless modes might signal superfluidity. In more detail, appearance of the massless mode in the correlator of momentum densities is a criterion of superfluidity:

$$\int d^3x \exp(iq_i r_i) < T_{0i}(r), T_{0k}(0) > \quad \sim \quad \frac{q_i q_k}{q^2} \quad . \tag{25}$$

To decide whether there arises such a term in the model (5) we turn to the expression for the momentum density in this model:

$$T_{0i} \quad = \quad \partial_0 \varphi \partial_i \varphi \ . \tag{26}$$

The value of the time derivative, $\partial_0 \varphi$ is a constant in the equilibrium, see Eq. (equilibrium).

From this observation we conclude that there is a correlator of the momentum densities of the form (25) according to the model (5). However, there is a superluminal signal behind this prediction. In other words, the model (24) is not actually consistent with the unitarity, the same as in the case of the vanishing temperature, see the preceding section. Optimistically, we can expect to observe correlator of the form (25) in measurements with poor resolution. One should look again into a wrong-sign contribution of the local form into the correlator of the momentum densities in the Euclidean space. Such a phenomenon, if observed, could be called "Euclidean superfluidity".

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