Effective action for the high energy scattering in gravity

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1 Gluon reggeization in QCD

QCD Born amplitude at high energies $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,, \;\; \alpha_s \ln s \sim 1\,, \;\; \alpha_s = \frac{g^2}{4\pi} \ll 1$$

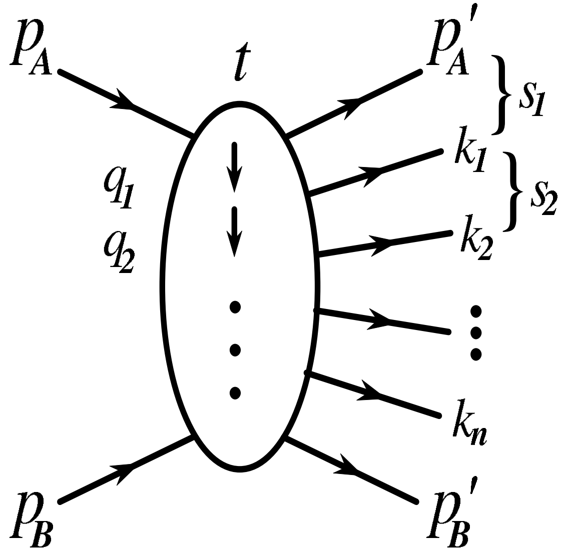
Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Mandelstum cuts

$$A(s,-|q|^2) \sim s \int d^2k\, \Phi^2(k,q-k)\, s^{\omega(-|k|^2)}\, s^{\omega(|q-k|^2)}$$

2 Amplitudes in multi-Regge kinematics



$$M^{BFKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\, \ln\frac{|q_r^2|}{\lambda^2}\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\,,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\,,\;\; \rho_r = x_r + iy_r$$

Möbius invariance and Pomeron intercept

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\; m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu\,,$$

$$E = \epsilon_m + \epsilon_{\widetilde{m}}\,,\;\; \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\;\; \Delta = \frac{g^2 N_c}{\pi^2}\ln 2 > 0$$

4 Integrability of the BFKL dynamics

Holomorphic separability of BKP hamiltonian at $N_c \to \infty$ (L.)

$$H = \frac{1}{2}(h+h^*)\,,\;\; h = \sum_{k=1}^{n} \left(\ln(p_k p_{k+1}) + \frac{1}{p_k} \ln \rho_{k,k+1}\, p_k + \frac{1}{p_{k+1}} \ln \rho_{k,k+1}\, p_{k+1} - 2\psi(1)\right)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

Monodromy matrix and Yang-Baxter equation (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}\,,\;\; L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix},$$

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)\,,\;\; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

5 Reggeon effective action in QCD

Locality in the rapidity space

$$y = \frac{1}{2}\ln\frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta << \ln\, s$$

Gluon and reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x)\,, \;\; A_\pm(x) = -iT^a A_\pm^a(x)\,, \, \delta A_\pm(x) = 0$$

Effective action for the reggeon interactions (L., 1995)

$$S = \int d^4x \left(L_{QCD} + Tr(V_+\partial_\mu^2 A_- + V_-\partial_\mu^2 A_+)\right)\;,$$

$$V_+ = -\frac{1}{g}\partial_+ \, P\exp\left(-g\int_{-\infty}^{x^+} v_+(x')d(x')^+\right) = v_+ - gv_+\frac{1}{\partial_+}v_+ + \ldots$$

6 Pomeron and graviton in N=4 SUSY

BFKL equation kernel in a diffusion approximation

$$j = 2 - \Delta - \Delta\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics for γ and Δ (KLOV, BPST)

$$\gamma = 1 - \sqrt{1 + (j-2)/\Delta}\,,\ \Delta = \frac{1}{\sqrt{\lambda}}\,,\ \lambda = g^2 N_c$$

Exact expression for the slope of γ at $j = 2$ (KLOV, Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

7 Perturbation theory in gravity

Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, R\,, \;\; R = R_{\mu\nu}\, g^{\mu\nu}$$

Riemann tensor

$$R_{\mu\nu} = R^{\sigma}_{\mu,\sigma\nu}\,, \;\; R^{\sigma}_{\mu,\alpha\beta} = \partial_\beta \Gamma^{\sigma}_{\mu\alpha} - \partial_\alpha \Gamma^{\sigma}_{\mu\beta} + \Gamma^{\rho}_{\mu\alpha}\, \Gamma^{\sigma}_{\rho\beta} - \Gamma^{\rho}_{\mu\beta}\, \Gamma^{\sigma}_{\rho\alpha}$$

Christophel symbol and gravity field

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma}\, \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\,, \;\; g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

General coordinate transformation

$$\delta h_{\mu\nu} = D_\mu \chi_\nu + D_\nu \chi_\mu\,, \;\; D_\mu \chi_\nu = \partial_\mu \chi_\nu - \Gamma^{\rho}_{\mu\nu} \chi_\rho$$

8 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2\to n} = -s^2 \Gamma^{\mu'\nu'}_{\mu\nu} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} \dots \Gamma^{\rho'\sigma'}_{\rho\sigma}$$

Graviton-graviton-reggeon vertex

$$\Gamma^{\mu'\nu'}_{\mu\nu} = \frac{\kappa}{4} \left(\Gamma_{\mu\mu'} \Gamma_{\nu\nu'} + \Gamma_{\mu\nu'} \Gamma_{\nu\mu'} \right)$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p^A_{\mu'} p^B_{\mu} + p^{A'}_{\mu} p^B_{\mu'}}{p^A p^B} + \frac{q^2}{2} \frac{p^B_{\mu} p^B_{\mu'}}{(p^A p^B)^2}$$

Reggeon–reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} \left(C_\rho C_\sigma - N_\rho N_\sigma \right) \,, \; N = \sqrt{q_1^2 q_2^2} \left(\frac{p^A}{kp^A} - \frac{p^B}{kp^B} \right)$$

9 Graviton trajectory in supergravity

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,, \;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2}(k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}_\mu^r \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma^r$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha|q|^2 \left(\ln\frac{|q|^2}{\lambda^2} + \frac{N-4}{2}\, \ln\frac{|\Lambda|^2}{|q|^2} \right)$$

10 Effective action for gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|} \,, \;\; |y - y_0| < \eta \,, \;\; \eta << \ln \, s$$

Reggeized graviton fields

$$\delta A^{++}(x) = \delta A^{--}(x) = 0 \,, \;\; \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left(\sqrt{-g}\, R + \frac{1}{2} \left(\partial_+ j^- \, \partial_\mu^2 A^{++} + \partial_- j_+ \, \partial_\mu^2 A^{--} \right) \right)$$

Hamilton-Jacobi equation for effective currents $j^{\pm} = 2x^{\pm} - \omega^{\pm}$

$$g^{\mu\nu} \partial_\mu \omega^{\pm} \partial_\nu \omega^{\pm} = 0 \,, \;\; \partial_{\pm} j^{\mp} = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_{\pm}} h_{\pm\pm} \right)^2 + ...$$

11 Effective currents for shock waves

Aichelburg - Sexl metric

$$(ds)^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu + a\, \ln|\overrightarrow{x}|\delta(x^-)\,(dx^-)^2\,,\;\; a = \frac{8}{\sqrt{2}}\, G\,\mu\,,\;\; z = a\,\frac{x^-}{|\overrightarrow{x}|^2}$$

Effective current for the shock wave

$$j^+ = -a\,\mu\left(\ln|\overrightarrow{x}| + \ln f(z) - \frac{1}{4}\,\frac{z}{f^2(z)}\right)\,,\;\; f(z) = \frac{1}{2} + \frac{\sqrt{1+2z}}{2}$$

Perturbative expansion

$$j^+ = -a\ln|\overrightarrow{x}| + \frac{a^2}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2 - \frac{a^3}{\partial_-}\,\frac{x_\mu}{2|\overrightarrow{x}|}\,\frac{\partial_\mu}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2 + \ldots$$

Variational principle for j^+

$$j^+ = \int_{-\infty}^{x^-}\left(g^{++}(y^-,\overrightarrow{\rho}(y^-)) + (\partial_-\overrightarrow{\rho})^2\right)\,,\;\; \frac{\delta j^+}{\delta\overrightarrow{\rho}(y^+)} = 0$$

12 Double-logarithmic asymptotics

Mellin representation for the graviton scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha |q|^2 \, \ln \frac{|q|^2}{\lambda^2}} \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega\,,\;\; a>0$$

Infrared evolution equation for $N=4$ supergravity (BLS (2012))

$$f_\omega = 1 + b \frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - b\,\frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}$$

Solution in terms of parabolic cylinder function

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln d^{(N)}(x)\,,\;\; d^{(N)}(x) = e^{\frac{x^2}{4}}\,D_{\frac{6-N}{2}}(x)\,,\;\; x = \frac{\omega}{\sqrt{b}}$$

Perturbative expansion of scattering amplitudes in $\xi = \alpha\,|t|\,\ln^2 \frac{s}{|q|^2}$

$$A(s,t) = A_{Born}\, s^{-\alpha |q|^2 \, \ln \frac{|q|^2}{\lambda^2}} \left(1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} + \ldots\right)$$

13 Production amplitudes in DLA

Multi-Regge kinematics

$$s \gg s_i = (k_{i-1}+k_i)^2 \approx s\alpha_i\beta_{i-1} \gg |k_r^{\perp}|^2\,, \quad \prod_{r=1}^{n+1} s_r = s\prod_{r=1}^{n}|k_r|^2 .$$

DL asymptotics of one graviton production amplitude (BLS)

$$A^{(N)}_{2\to 3} = A^{Born}_{2\to 3}\, s_1^{-\alpha |q_1|^2 \ln \frac{|q_1^2|}{\mu^2}}\, s_2^{-\alpha |q_2|^2 \ln \frac{|q_2^2|}{\mu^2}}\, r^{(N)}(\rho_1,\rho_2)\,, \;\; \rho_r - \sqrt{\alpha}\ln s_r/|q_r|^2\,,$$

$$r^{(N)} = \int_{-i\infty}^{+i\infty}\frac{dx_1 dx_2}{(2\pi i)^2}\, e^{x_1\rho_1+x_2\rho_2}\,\frac{d^{(N)}(\sqrt{\frac{|q_2|}{|q_1|}}x_1)\, d^{(N)}(\sqrt{\frac{|q_1|}{|q_2|}}x_2)}{d^{(N)}(x_1)\, d^{(N)}(x_2)}\,\phi^{(N)}(x_1,x_2)\,,$$

$$\phi^{(N)}(x_1,x_2) = \frac{2}{6-N}\,\frac{\dfrac{\left(d^{(N)}\left(\sqrt{\frac{|q_2|}{|q_1|}}x_1\right)\right)'}{d^{(N)}\left(\sqrt{\frac{|q_2|}{|q_1|}}x_1\right)} - \dfrac{\left(d^{(N)}\left(\sqrt{\frac{|q_1|}{|q_2|}}x_2\right)\right)'}{d^{(N)}\left(\sqrt{\frac{|q_1|}{|q_2|}}x_2\right)}}{\sqrt{\frac{|q_1|}{|q_2|}}x_2 - \sqrt{\frac{|q_2|}{|q_1|}}x_1}\,,$$

14 Discussion

1. Locality of reggeon interactions in the rapidity space.
2. BFKL pomeron as a composite state of two reggeized gluons.
3. High energy effective action for reggeized gluons in QCD.
4. Integrability of the multi-gluon interactions at large N_c.
5. Pomeron-graviton duality in $N = 4$ SUSY.
6. Multi-regge effective vertices in gravity.
7. Divergencies of graviton Regge trajectory.
8. Effective action for the high energy gravity.
9. Hamilton-Jacobi equation for effective currents.
10. Infrared evolution equations in DL approximation.