Precise Charm- and Bottom-Quark Masses: Theoretical and Experimental Uncertainties*†

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Abstract

Recent theoretical and experimental improvements in the determination of charm and bottom quark masses are discussed. A new and improved evaluation of the contribution from the gluon condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ to the charm mass determination and a detailed study of potential uncertainties in the continuum cross section for $b\bar{b}$ production is presented, together with a study of the parametric uncertainty from the α_s-dependence of our results. The final results, $m_c(3\,\mathrm{GeV}) = 986(13)\,\mathrm{MeV}$ and $m_b(m_b) = 4163(16)\,\mathrm{MeV}$, represent, together with a closely related lattice determination $m_c(3\text{ GeV}) = 986(6)\,\mathrm{MeV}$, the presently most precise determinations of these two fundamental Standard Model parameters. A critical analysis of the theoretical and experimental uncertainties is presented.

The past years have witnessed significant improvement in the determination of charm and bottom quark masses as a consequence of improvements in experimental techniques as well as theoretical calculations. Quark mass determinations can be based on a variety of observables and theoretical calculations. The one presently most precise follows an idea advocated by the ITEP group more than thirty years ago [1]. Members of the ITEP group are in the audience and already for this reason it is appropriate to present the most recent advances based on this method. Indeed, this approach has gained renewed interest after significant advances in higher order perturbative calculations have been achieved. In particular the four-loop results (i.e. the coefficients $\bar{C}_n$ discussed below) are now available for the Taylor coefficients of the vacuum polarization, analytically up to $n = 3$ and numerically up to $n = 10$. The method exploits the fact that the vacuum polarization function $\Pi(q^2)$ and its derivatives, evaluated at $q^2 = 0$, can be considered as short distance quantities with an inverse scale characterized by the distance between the reference point $q^2 = 0$ and the location of the threshold $q^2 = (3\text{ GeV})^2$ and $q^2 = (10\text{ GeV})^2$ for charm and bottom, respectively. This idea has been taken up in [2] after the first three-loop evaluation of the moments became available [3, 4, 5] and has been further improved in [6] using four-loop results [7, 8] for the lowest moment. An analysis which included additional theoretical [9, 10, 11] and experimental results has been performed in [12]. In the following we present an improved treatment of the contribution from the gluon condensate to the moments of the charm correlator and a critical discussion of R_b in the transition region from the threshold to the perturbative continuum and the resulting impact on the m_b determination.

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Let us recall some basic notation and definitions. The vacuum polarization $\Pi_Q(q^2)$ induced by a heavy quark Q with charge Q_Q (ignoring in this short note the so-called singlet contributions), is an analytic function with poles and a branch cut at and above $q^2 = M^2_{J/\psi}$ for charm (or M^2_Υ for bottom). Its Taylor coefficients $\bar{C}_n$, defined through

$$\Pi_Q(q^2) \equiv Q_Q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n z^n \,, \tag{1}$$

can be evaluated in perturbative QCD (pQCD), presently up to order α_s^3. Here $z \equiv q^2/(4m_Q^2)$, where $m_Q = m_Q(\mu)$ is the running $\overline{\text{MS}}$ mass at scale μ. Using a once-subtracted dispersion relation

$$\Pi_Q(q^2) = \frac{1}{12\pi^2} \int_0^\infty \mathrm{d}s \frac{R_Q(s)}{s(s-q^2)} \,, \tag{2}$$

(with R_Q denoting the familiar R-ratio for the production of heavy quarks with flavor Q), the Taylor coefficients can be expressed through moments of R_Q. Equating perturbatively calculated and experimentally measured moments,

$$\mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_Q(s) \,, \tag{3}$$

leads to an (n-dependent) determination of the quark mass

$$m_Q = \frac{1}{2} \left(\frac{9Q_Q^2}{4} \frac{\bar{C}_n}{\mathcal{M}_n^{\text{exp}}} \right)^{\frac{1}{2n}} \,. \tag{4}$$

Significant progress has been made in the perturbative evaluation of the moments since the first analysis of the ITEP group. The $\mathcal{O}(\alpha_s^2)$ contribution (three loops) has been evaluated more than 13 years ago [3, 4, 5], as far as the terms up to $n = 8$ are concerned, recently even up to $n = 30$ [13, 14]. About ten years later the lowest two moments ($n = 0, 1$) of the vector correlator were evaluated in $\mathcal{O}(\alpha_s^3)$, i. e. in four-loop approximation [7, 8]. The corresponding two lowest moments for the pseudoscalar correlator were obtained in [15] in order to derive the charmed quark mass from lattice simulations [16]. In [9, 10] the second and third moments were evaluated for vector, axial and pseudoscalar correlators. Combining, finally, these results with information about the threshold and high-energy behavior in the form of a Padé approximation, the full q^2-dependence of all four correlators was reconstructed and the next moments, from four up to ten, were obtained with adequate accuracy [11].

Most of the experimental input had already been compiled and exploited in [6] both for the charm and the bottom case. In [6, 12] an estimate for the gluon condensate was included which gives a tiny contribution for the charm case. This estimate for the moments was based on the results from [17] plus next-to-leading order (NLO) terms from [18]

$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m^2_{c,\text{pole}})^{n+2}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n (1 + \frac{\alpha_s}{\pi} b_n), \tag{5}$$

with

$$a_n = -\frac{2n+2}{15} \frac{\Gamma(4+n)\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)\Gamma\left(\frac{7}{2}+n\right)} \,,$$
$$b_1 = \frac{135779}{12960} \,, \quad b_2 = \frac{1969}{168} \,, \quad b_3 = \frac{546421}{42525} \,, \quad b_4 = \frac{661687433}{47628000} \,. \tag{6}$$

Using the $\mathcal{O}(\alpha_s)$ pole-$\overline{\text{MS}}$-mass conversion , $m_Q(\mu)/m_{Q,\text{pole}} = 1 - \alpha_s/\pi \times [4/3 + \ln(\mu^2/m_Q^2)]$, Eq. (5) was expressed in terms of the $\overline{\text{MS}}$ mass at the scale $\mu = 3\,\text{GeV}$ with new coefficients

n	$\mathcal{M}_n^{\rm res}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm thresh}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm cont}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm exp}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm np}$(LO) $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm np}$(NLO)
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0001(3)	−0.0002(5)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	−0.0002(5)	−0.0005(10)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	−0.0004(8)	−0.0008(16)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	−0.0006(12)	−0.0013(25)

Table 1: Experimental moments in $(\mathrm{GeV})^{-2n}$ as defined in Eq. (3), separated according to the contributions from the narrow resonances, the charm threshold region and the continuum region above $\sqrt{s} = 4.8$ GeV. In the last two columns the contribution from the gluon condensate in LO and NLO are shown.

n	m_c(3 GeV) [np$_{\rm LO}$]	m_c(3 GeV) [np$_{\rm NLO}$]	exp	α_s	μ	np$_{\rm LO}$	np$_{\rm NLO}$	total [np$_{\rm LO}$]	total [np$_{\rm NLO}$]
1	0.986	0.986	0.009	0.009	0.002	0.001	0.001	0.013	0.013
2	0.976	0.975	0.006	0.014	0.005	0.001	0.002	0.016	0.016
3	0.976	0.975	0.005	0.015	0.007	0.001	0.003	0.017	0.017
4	1.000	0.999	0.003	0.009	0.031	0.001	0.003	0.032	0.032

Table 2: Results for $m_c(3\,\mathrm{GeV})$ in GeV including the LO or NLO order gluon condensate contribution. The errors are from experiment, α_s, variation of μ and the different options for the gluon condensate.

$\bar{b}_n = b_n - (2n+4)(4/3 + \ln(\mu^2/m_Q^2))$. However, as a consequence of the high inverse power of the mass appearing in Eq. (5) this conversion becomes unstable.[1] (Although the effect obtained in [6, 12] was small, nevertheless.) For this reason we prefer to use directly the on-shell formulation, Eq. (5) with the parameters $m_{c,\rm pole} = 1.5$ GeV, $\alpha_s^{(4)}(3\text{ GeV}) = 0.258$ (obtained from $\alpha_s^{(5)}(M_Z) = 0.1189$) and $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.006 \pm 0.012\,\mathrm{GeV}^4$ [19]. The contributions to the moments are displayed in Tab. 1 both for the LO and the NLO predictions, together with the contribution from the narrow resonances, threshold and continuum, which are directly copied from [6]. The results for the charm mass, $m_c(3\,\mathrm{GeV})$, are shown in Tab. 2 again for the two choices of the condensate contribution.

As is evident from Tabs. 1 and 2, the effect of the gluon condensate remains small, in particular for the lowest three moments. The final result

$$m_c(3\,\mathrm{GeV}) = 986(13)\,\mathrm{MeV}\,, \tag{7}$$

remains unchanged when compared to [6, 12]. The consistency of this result with the ones for $n = 2, 3$ and 4 can be considered as additional confirmation. Transforming this to the scale-invariant mass $m_c(m_c)$ [20], including the four-loop coefficients of the renormalization group functions one finds [12] $m_c(m_c) = 1279(13)$ MeV. Let us recall at this point that a recent study [21], combining a lattice simulation for the pseudoscalar correlator with the perturbative three- and four-loop result [5, 15, 10] has led to $m_c(3\text{ GeV}) = 986(6)$ MeV in remarkable agreement with [6, 12]. Fair agreement is also found with a recent analysis based on finite energy sum rules using results up to four-loop order in perturbation theory [22] which leads to the result $m_c(3\text{ GeV}) = 1008(26)$ MeV.

Until about two years ago the only measurement of the cross section above but still close to the B-meson threshold, i.e. in the region between 10.6 and 11.2 GeV, had been performed by the CLEO collaboration in the mid-eighties [23]. Its large systematic uncertainty was responsible for a sizable fraction of the final error on m_b in the analysis of [6]. This measurement had been superseded by a measurement of BABAR [24] with a systematic error around 3%. In [12] the

[1] We would like to thank S. Bodenstein for drawing our attention to this point.

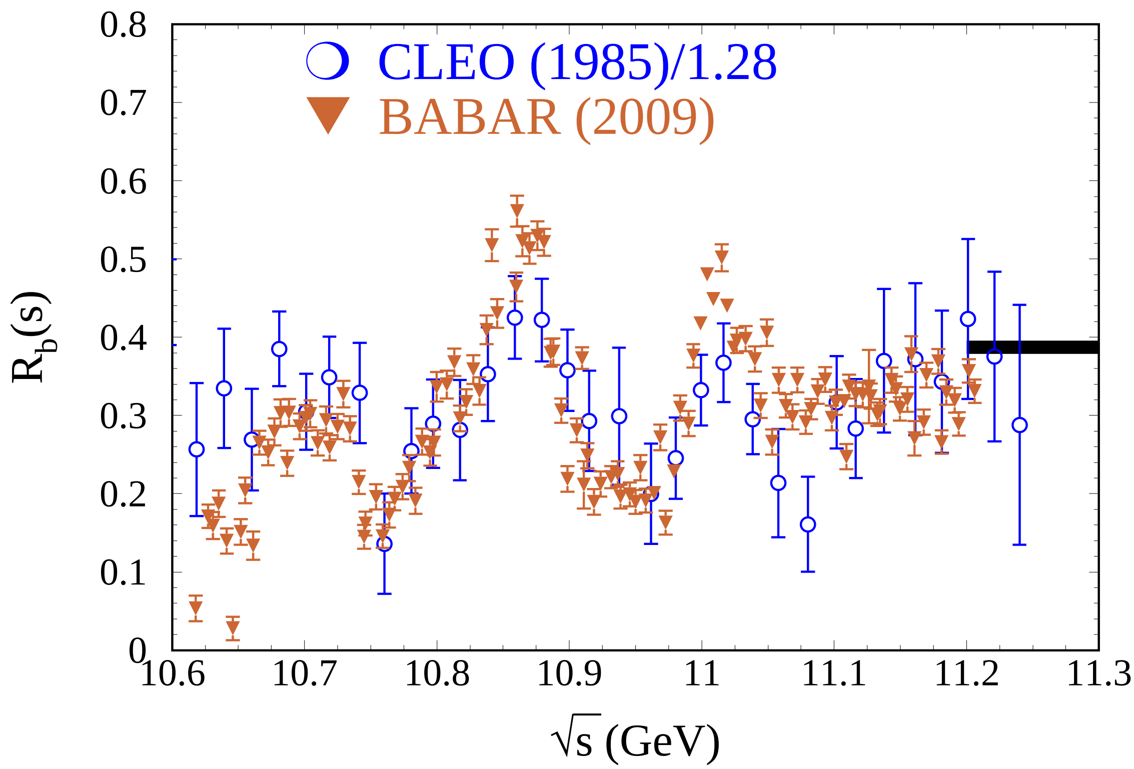

Figure 1: Comparison of rescaled CLEO data for R_b with BABAR data. [12, 24]. The black bar on the right corresponds to the theory prediction [25].

n	$\mathcal{M}_n^{\text{res},(1S\text{-}4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

Table 3: Moments for the bottom quark system in $(\text{GeV})^{-2n}$

radiative corrections were unfolded and used to obtain a significantly improved determination of the moments. These BABAR data are shown in Fig. 1 together with the theory prediction based on pQCD in $\mathcal{O}(\alpha_s^2)$. Observing that R_b flattens off above 11.1 GeV one should expect agreement between pQCD and experiment. The result for the region above 11.1 GeV is shown in Fig. 2, again with the theory prediction.

Taking the average of the data points above 11.1 GeV one finds $\overline{R}_b = 0.32$ with negligible statistical and uncorrelated systematical errors. The correlated systematical error is quoted to be 3.5%. In [6] and [12] data and pQCD were taken at face value for $\sqrt{s}$ below and above 11.2 GeV, respectively (with linear interpolation between the last data point $R_b(11.2062\,\text{GeV}) = 0.331$ and the pQCD prediction $R_b^{\text{pQCD}}(11.24\,\text{GeV}) = 0.387$). This leads to the moments and the quark masses as shown in Tabs. 3 and 4, respectively.

This procedure is based on the assumption that pQCD is valid in the region above ~ 11.2 GeV. where the relative momentum of b and $\bar{b}$ quarks has reached about 5 GeV. Indeed, considering the behaviour of R_b as displayed in Fig. 1 it is quite plausible that R_b quickly reaches the level

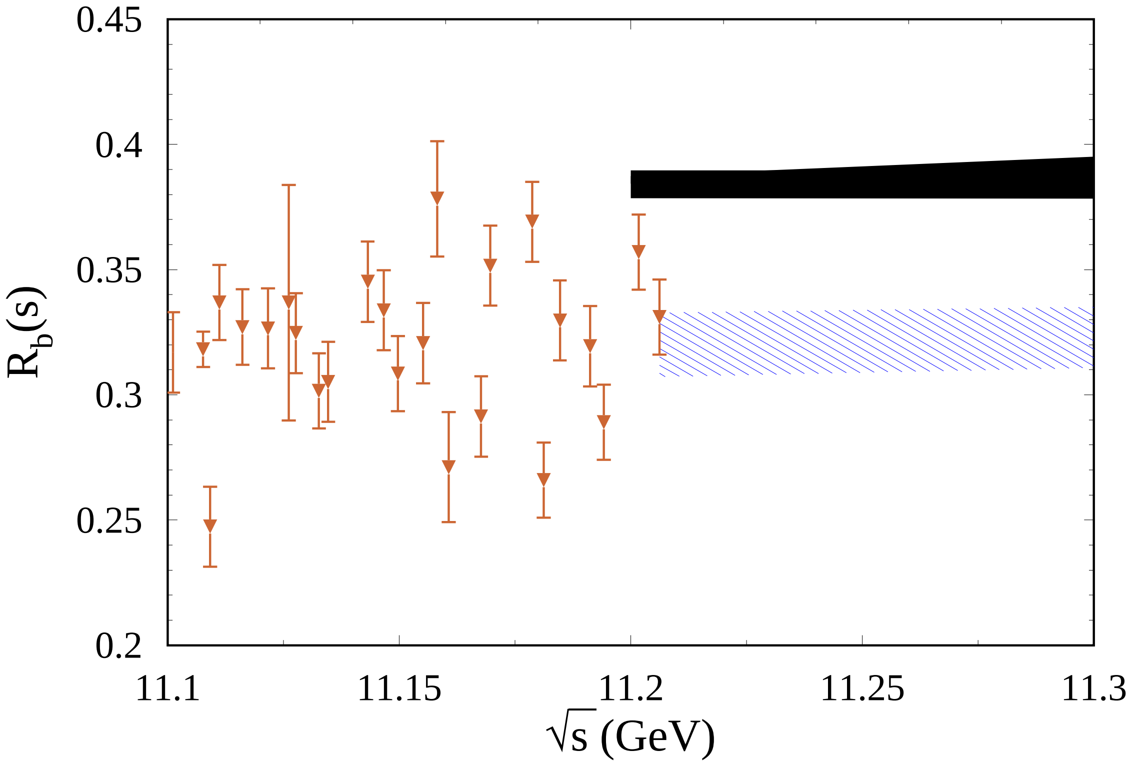

Figure 2: The region around $\sqrt{s} = 11.2$ GeV of Fig. 1 is magnified. Only the BABAR data is shown. The shaded band corresponds to a linear interpolation between $R(11.2062 \text{ GeV})$ and $R(13 \text{ GeV})$ as described in the text, see option A. The black bar on the right corresponds to the theory prediction [25].

n	$m_b(10 \text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3.597	0.014	0.007	0.002	0.016	4.151
2	3.610	0.010	0.012	0.003	0.016	4.163
3	3.619	0.008	0.014	0.006	0.018	4.172
4	3.631	0.006	0.015	0.020	0.026	4.183

Table 4: Results for $m_b(10 \text{ GeV})$ and $m_b(m_b)$ in GeV obtained from Eq. (4). The errors are from experiment, α_s and the variation of μ.

anticipated by pQCD. However, considering the 20% deviation between data and pQCD around 11.2 GeV one may consider the possibility that either pQCD is valid only at significantly higher energies (option A), or that the systematic error of the BABAR data is significantly underestimated, requiring a shift of the data by a sizeable amount (option B). These options should be considered to be "worst case" scenarios. Nevertheless we shall demonstrate that the resulting shifts are only slightly larger than the error quoted in Ref. [12]. Let us now explore these two options in detail:

Option A: pQCD is only valid at higher energies, say above 13 GeV with linear interpolation between $R_b(11.2 \text{ GeV}) = 0.32$ and $R_b^{\text{pQCD}}(13 \text{ GeV}) = 0.377$. The results for the moments and $m_b(10 \text{ GeV})$ are shown in Tabs. 5 and 6, respectively, assuming a 4% uncertainty at $\sqrt{s} = 11.2062$ GeV and no uncertainty for $R(\sqrt{s} = 13 \text{ GeV})$. A remarkable stability is

n	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.270(11)	2.854(17)	4.518(31)
2	1.459(23)	0.226(9)	1.133(11)	2.819(27)
3	1.538(24)	0.190(8)	0.596(8)	2.324(26)
4	1.630(25)	0.159(6)	0.353(5)	2.142(26)

Table 5: Moments for the bottom quark system in $(\text{GeV})^{-2n}$ obtained using option A. The contribution from the linear interpolation is contained in $\mathcal{M}_n^{\text{cont}}$.

n	$m_b(10\text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3.631	0.014	0.007	0.002	0.016	4.183
2	3.630	0.010	0.012	0.003	0.016	4.182
3	3.631	0.008	0.014	0.006	0.018	4.183
4	3.637	0.007	0.015	0.020	0.026	4.189

Table 6: Bottom quark mass in GeV obtained using the option A. The uncertainties are from experiment, α_s and the variation of μ.

observed for the bottom quark mass as can be seen from Tab. 6. For $n = 2$ one obtains $m_b(10\text{ GeV}) = 3.630$ GeV.

Option B: Perturbative QCD is valid at 11.2 GeV, and the systematic error of [24] is assumed to be underestimated. Therefore the data in the threshold region are rescaled by a factor $R_b^{\text{pQCD}}/\overline{R}_b = 0.387/0.32 \approx 1.21$ corresponding to a shift of about 7 σ. The results for the moments are shown in Tab. 7, the corresponding predictions for $m_b(10\,\text{GeV})$ in Tab. 8, with $m_b(10\text{ GeV}) = 3.592$ GeV for $n = 2$. As expected, the trend of increasing m_b with increasing n already present in Tab. 4 is emphasized even more.

Similar to the charm case, the result for the bottom mass based on the lower moments is more stable than the one from moments $n = 4$ and above. In order to suppress the theoretically evaluated input above 11.2 GeV (which corresponds to roughly 60% for the lowest, 40% for the second and 26% for the third moment), the result from the second moment has been adopted as our final result,

$$m_b(10\text{ GeV}) = 3610(16)\text{ MeV}, \tag{8}$$

corresponding to $m_b(m_b) = 4163(16)$MeV. Note that options A and B are considered as "worst case" scenarios, nevertheless, the shift (for $n = 2$) is $+20$ MeV and -18 MeV, respectively, and thus only slightly higher than the uncertainty of 16 MeV. For this reason we stick to the original result of Eq. (8). The explicit α_s dependence of m_c and m_b can be found in [12]. When considering the ratio of charm and bottom quark masses, part of the α_s and of the μ dependence cancels

$$\frac{m_c(3\text{ GeV})}{m_b(10\text{ GeV})} = 0.2732 - \frac{\alpha_s(M_Z) - 0.1189}{0.002} \cdot 0.0014 \pm 0.0028\,. \tag{9}$$

This combination might be a useful input in ongoing analyses of bottom decays.

In Fig. 3 the results of this analysis are compared to others based on completely different methods. The m_c value is well within the range suggested by other determinations. In case of m_b our result is somewhat towards the low side, although still consistent with most other results.

The results presented in [12] constitute the most precise values for the charm- and bottom-quark masses available to date.[2] Nevertheless it is tempting to point to the dominant errors and

[2]For the charm quark a slightly more precise result has been obtained in Ref. [21].

n	$\mathcal{M}_n^{\rm res,(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\rm thresh}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\rm cont}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\rm exp}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.347(14)	2.911(18)	4.651(32)
2	1.459(23)	0.290(12)	1.173(11)	2.921(28)
3	1.538(24)	0.242(10)	0.624(7)	2.404(27)
4	1.630(25)	0.203(8)	0.372(5)	2.205(27)

Table 7: Moments for the bottom quark system in $(\mathrm{GeV})^{-2n}$ obtained using option B.

n	$m_b(10\text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3.570	0.015	0.008	0.002	0.017	4.124
2	3.592	0.010	0.012	0.003	0.016	4.146
3	3.607	0.008	0.014	0.006	0.018	4.160
4	3.622	0.006	0.015	0.020	0.026	4.175

Table 8: Bottom quark mass in GeV obtained using the option B. The uncertainties are from experiment, α_s and the variation of μ.

thus identify potential improvements. In the case of the charmed quark the error is dominated by the parametric uncertainty in the strong coupling $\alpha_s(M_Z) = 0.1189 \pm 0.002$. Experimental and theoretical errors are comparable, the former being dominated by the electronic width of the narrow resonances. In principle this error could be further reduced by the high luminosity measurements at BESS III. A further reduction of the (already tiny) theory error, e.g. through a five-loop calculation looks difficult. Further confidence in our result can be obtained from the comparison with the aforementioned lattice evaluation.

Improvements in the bottom quark mass determination could originate from the experimental input, e.g. through an improved determination of the electronic widths of the narrow Υ resonances or through a second, independent measurement of the R-ratio in the region from the $\Upsilon(4S)$ up to 11.2 GeV. The slight mismatch between the theory prediction above 11.2 GeV and the data in the region below with their systematic error of about 3% has been discussed above. An independent measurement in the continuum region, e.g. by the BELLE collaboration, would be extremely important.

In this connection it may be useful to collect the most important pieces of evidence supporting this remarkably small error. Part of the discussion is applicable to both charm and bottom, part is specific to only one of them. In particular for charm, but to some extent also for bottom, the μ dependence of the result increases for the higher moments, starting with $n = 4$, and dominates the total error. We will therefore concentrate on the moments $n = 1$, 2 and 3 which were used for the mass determination, results for $n = 4$ will only be mentioned for illustration.

Let us start with charm. Right at the beginning it should be emphasized that the primary quantity to be determined is the running mass at the scale of 3 GeV, the scale characteristic for the production threshold and thus for the process. Furthermore, at this scale the strong coupling $\alpha_s^{(4)}(3\text{ GeV}) = 0.258$ is already sufficiently small such that the higher order terms in the perturbative series decrease rapidly. Last not least, for many other observables of interest, like B-meson decays into charm, or processes involving virtual charm quarks like $B \to X_s\gamma$ or $K \to \pi\nu\bar{\nu}$, the characteristic scale is of order 3 GeV or higher. Artificially running the mass first down to $\mathcal{O}(1\text{ GeV})$ and then back to a higher scale thus leads to an unnecessary inflation of the error.

The quark mass determination is affected by the theory uncertainty, resulting in particular from our ignorance of yet uncalculated higher orders, and by the error in the evaluation of the experimental moments. The former has been estimated [6] by evaluating $m_c(\mu)$ at different

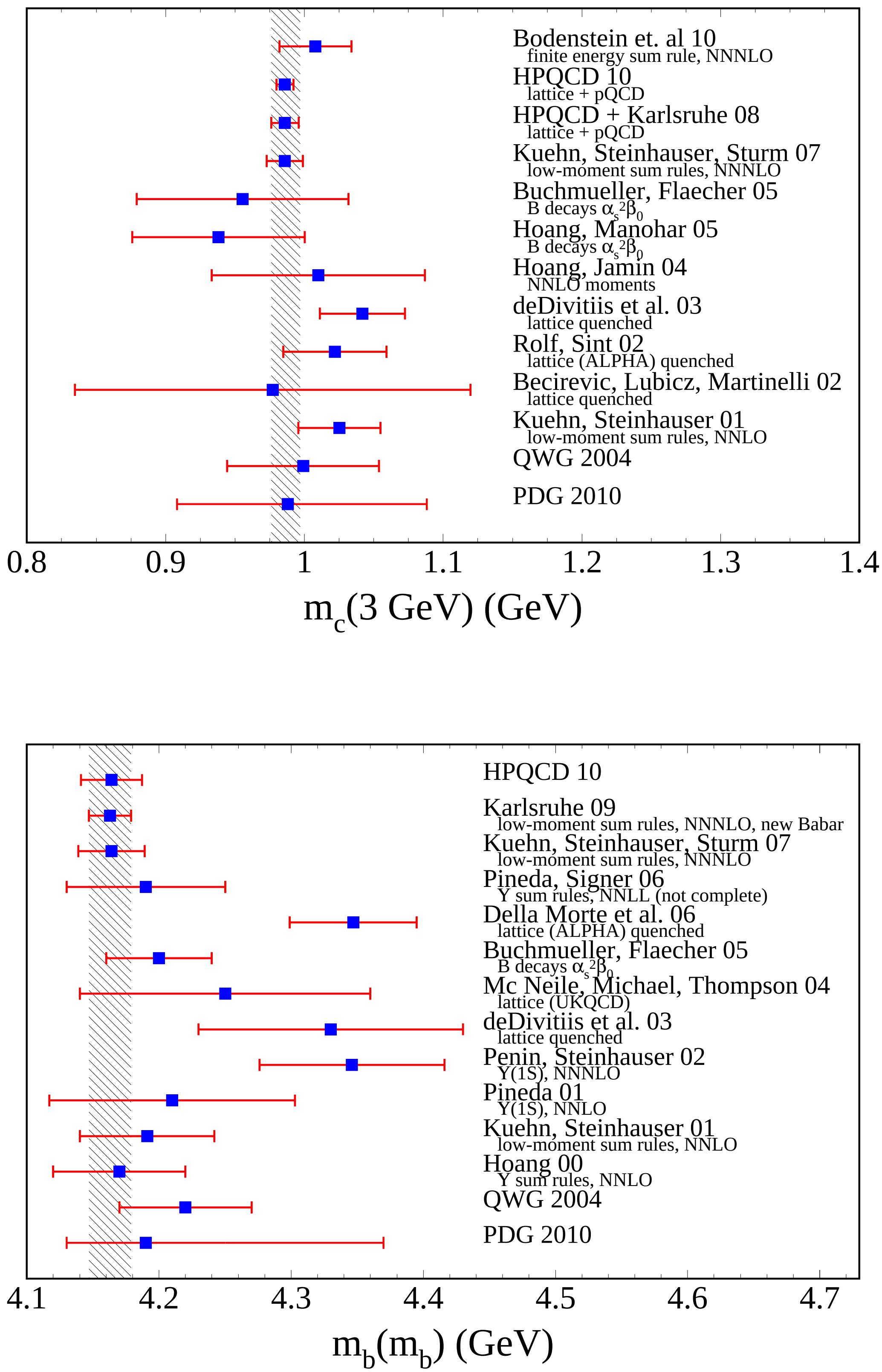

Figure 3: Comparison of recent determinations of $m_c(3\text{ GeV})$ and $m_b(m_b)$. Shaded band: [12].

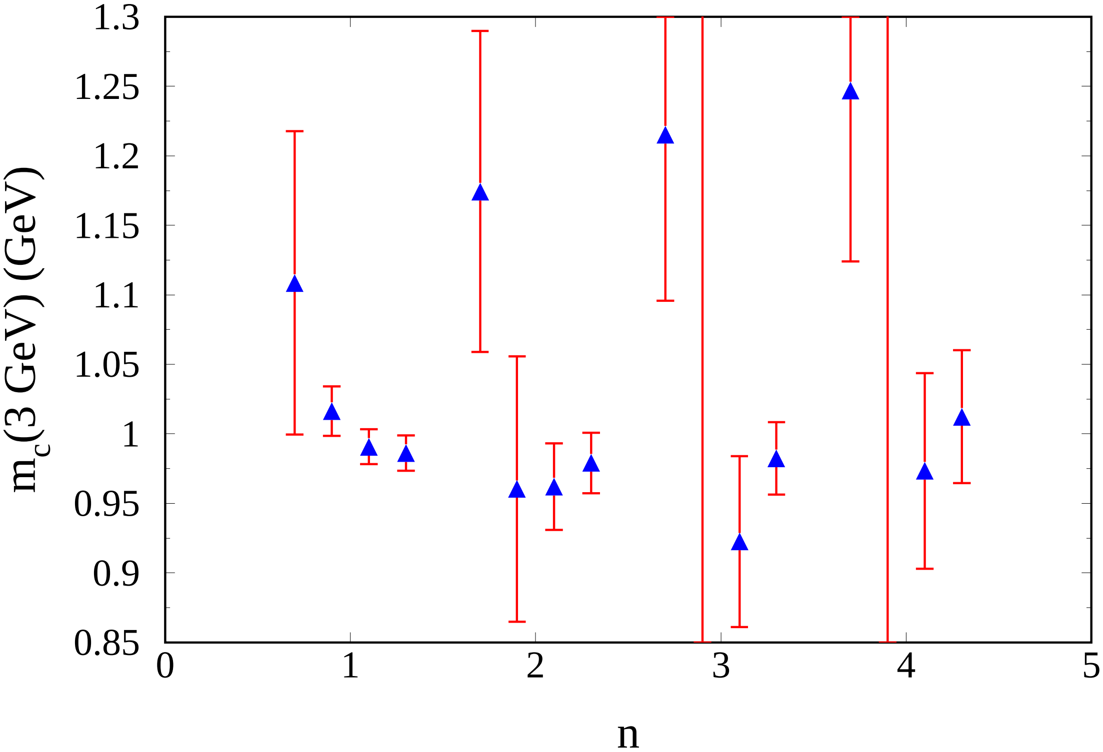

Figure 4: $m_c(3\text{ GeV})$ for $n = 1, 2, 3$ and 4. For each value of n the results from left to right correspond the inclusion of terms of order α_s^0, α_s^1, α_s^2 and α_s^3.

renormalization scales between 2 and 4 GeV (changing, of course, the coefficients $\bar{C}_n$ appropriately) and subsequently evolving $m_c(\mu)$ to $m_c(3\text{ GeV})$. The error estimates based on these considerations are listed in Tab. 2.

The stability of the result upon inclusion of higher orders is also evident from Fig. 4 where the results from different values of n are displayed separately in order α_s^i with $i = 0$, 1, 2 and 3. This argument can be made more quantitatively by rewriting eq. (4) in the form[3]

$$\begin{aligned} m_c &= \frac{1}{2}\left(\frac{9Q_c^2}{4}\frac{\bar{C}_n^{\text{Born}}}{\mathcal{M}_n^{\text{exp}}}\right)^{\frac{1}{2n}}\left(1 + r_n^{(1)}\alpha_s + r_n^{(2)}\alpha_s^2 + r_n^{(3)}\alpha_s^3\right) \\ &\propto 1 - \begin{pmatrix} 0.328 \\ 0.524 \\ 0.618 \\ 0.662 \end{pmatrix}\alpha_s - \begin{pmatrix} 0.306 \\ 0.409 \\ 0.510 \\ 0.575 \end{pmatrix}\alpha_s^2 - \begin{pmatrix} 0.262 \\ 0.230 \\ 0.299 \\ 0.396 \end{pmatrix}\alpha_s^3, \end{aligned} \tag{10}$$

where the entries correspond to the moments with $n = 1$, 2, 3 and 4. Note, that the coefficients are decreasing with increasing order of α_s. Estimating the relative error through $r_n^{\max}(\alpha_s(3\text{ GeV}))^4$ leads to 1.4 / 2.3 / 2.7 / 2.9 per mille and thus to an estimate clearly smaller than the one based on the μ dependence.

The consistency between the results for different values of n is another piece of evidence (Fig. 4 and Tab. 2). For the lowest three moments the variation between the maximal and the minimal value amounts to 10 MeV only. This, in addition, points to the selfconsistency of our data set. Let us illustrate this aspect by a critical discussion of the "continuum contribution", i.e. the region above 4.8 GeV, where data points are available at widely separated points

[3]QED corrections are contained in C_n^{Born}.

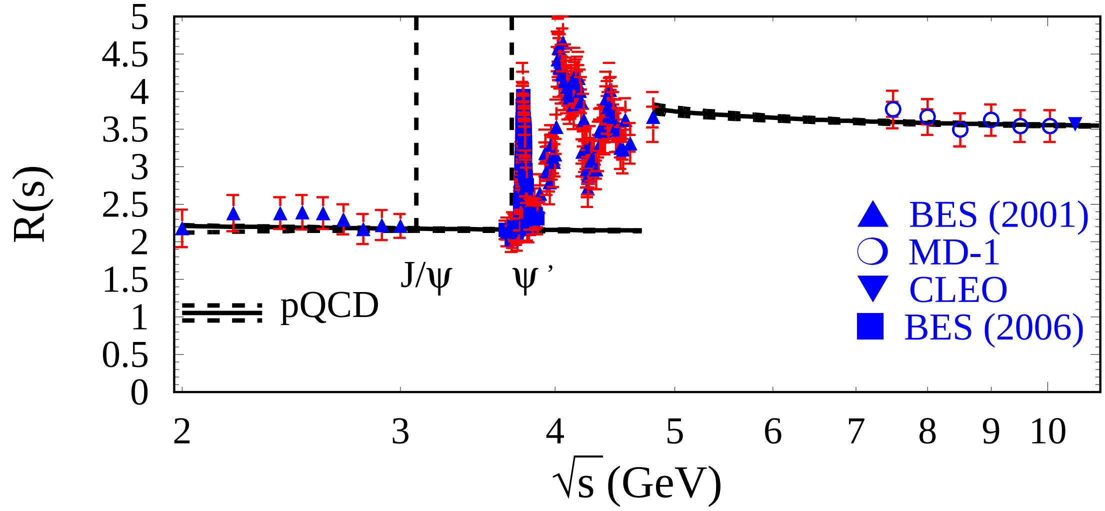

Figure 5: $R(s)$ for different energy intervals around the charm threshold region. The solid line corresponds to the theoretical prediction.

only. Instead of experimental data the theory prediction for $R(s)$ has been employed for the evaluation of the contribution to the moments. If the true contribution from this region would be shifted down by, say, 10%, this would move m_c, as derived from $n = 1$, up by about 20 MeV. However, this same shift would lead to a small increase by 3 MeV for $n = 2$ and leave the results for higher n practically unchanged. Furthermore, theory predictions and measurements in the region from 4.8 GeV up to the bottom-meson threshold, wherever available, are in excellent agreement, as shown in Fig. 5, with deviations well within the statistical and systematical error of 3 to 5%. Last not least, the result described above is in perfect agreement with the recent lattice determination mentioned above.

Let us now discuss beauty, with m_b evaluated at $\mu = 10$ GeV. Again we first study the stability of the perturbative expansion, subsequently the consistency of the experimental input. With $\alpha_s(10\text{ GeV}) = 0.180$ the higher order corrections decrease even more rapidly. Varying the scale μ between 5 and 15 GeV leads to a shift between 2 and 6 MeV (Tab. 4) which is completely negligible. Alternatively we may consider the analogue of eq. (10) with the correction factor

$$\frac{m_b}{m_b^{\text{Born}}} = 1 - \begin{pmatrix} 0.270 \\ 0.456 \\ 0.546 \\ 0.603 \end{pmatrix} \alpha_s - \begin{pmatrix} 0.206 \\ 0.272 \\ 0.348 \\ 0.410 \end{pmatrix} \alpha_s^2 + \begin{pmatrix} -0.064 \\ 0.048 \\ 0.051 \\ 0.012 \end{pmatrix} \alpha_s^3. \tag{11}$$

Taking $r_n^{\max}(\alpha_s(10\text{ GeV}))^4$ for an error estimate leads to a relative error of 0.28 / 0.48 / 0.57 / 0.63 per mille for $n = 1$, 2, 3 and 4, respectively, which is again smaller than our previous estimate. Let us now move to a critical discussion of the experimental input. The contribution from the lowest four Υ resonances has been taken directly from PDG [26] with systematic errors of the lowest three added linearly. The analysis [12] of a recent measurement [24] of R_b in the threshold region up to 11.20 GeV has provided results consistent with the earlier analysis [6] but has lead to a significant reduction of the error in m_b.

In comparison with the charm analysis a larger contribution arises from the region where data are substituted by the theoretically predicted R_b with relative contributions of 63, 41, 26 and 17 percent for $n = 1$, 2, 3 and 4 respectively. This is particularly valid for the lowest moment. For this reason we prefer to use the result from $n = 2$, alternatively we could have also taken the one from $n = 3$. Let us now collect the arguments in favour of this approach:

i) For light and charmed quarks the prediction for R based on pQCD works extremely well already two to three GeV above threshold. No systematic shift has been observed between theory and experiment, in the case of massless quarks, starting from around 2 GeV, and for the cross section including charm at and above 5 GeV up to the bottom threshold (Fig. 5). It is thus highly unplausible that the same approach should fail for bottom production.

ii) If the true R_b in the continuum above 11.2 GeV would differ from the theory prediction by a sizable amount, the results for $n = 1$, 2 and 3 would be mutually inconsistent. Specifically, a shift of the continuum term by 5% would move m_b, derived from $n = 1$, 2 and 3 by about 64 MeV, 21 MeV and 9 MeV respectively.

To summarize: Charm and bottom quark mass determinations have made significant progress during the past years. A further reduction of the theoretical and experimental error seems difficult at present. However, independent experimental results on the R-ratio would help to further consolidate the present situation. The confirmation by a recent lattice analysis with similarly small uncertainty gives additional confidence in the result for m_c.

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